

Quantum transport theory for neutrinos with flavor and particle-antiparticle mixing

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Theory of neutrino oscillations

- Fascinating features, e.g. flavor, particle-antiparticle and helicity coherence
- Multiple approaches: wave packets, density matrices, mean field equations, spin projection formalism ...
- From quantum field theoretical viewpoint: How to include
 - Coherence effects?
 - Collision term for coherent neutrino states?
 - Heavy neutrinos?
- Conceptual problems: decoherence effects (quantum entanglement), separation of oscillations scales, external magnetic fields...

Quantum kinetic equations

- Derive transport equations for mixing neutrinos from fundamental field theory formalism
 - Include all **flavor and particle-antiparticle coherences**
 - Only adiabatic background fields are assumed
 - Equations valid for **heavy and light neutrinos** (UR-limit is not assumed)
 - **Generalized Feynman rules** to compute collision terms which include coherently mixing species

Real time Kadanoff-Baym equations in Wigner space

$$\hat{k} S^p(k, x) - e^{-\frac{i}{2} \partial_x^\Sigma \cdot \partial_k} [\Sigma_{\text{out}}^p(\hat{K}, x) S^p(k, x)] = 1, \quad (\text{pole})$$

$$\hat{k} S^s(k, x) - e^{-\frac{i}{2} \partial_x^\Sigma \cdot \partial_k} [\Sigma_{\text{out}}^r(\hat{K}, x) S^s(k, x)] = e^{-\frac{i}{2} \partial_x^\Sigma \cdot \partial_k} [\Sigma_{\text{out}}^s(\hat{K}, x) S^a(k, x)] \quad (\text{statistical})$$

- Exact to a given approximation for the self-energy function
 $\Rightarrow$ information about flavor and all particle-antiparticle coherences
- Need for **an approximation scheme** which does not lose information about coherence

$$iS^<(u, v) \equiv \langle \bar{\psi}(v) \psi(u) \rangle$$

$$iS^>(u, v) \equiv \langle \psi(u) \bar{\psi}(v) \rangle$$

$$S^r(u, v) \equiv -\theta(u_0 - v_0)(S^> + S^<)$$

$$S^a(u, v) \equiv -\theta(v_0 - u_0)(S^> + S^<)$$

$$\hat{k} = k + \frac{i}{2} \partial_x \quad p = <, >, \quad s = r, a$$

$$g(k, x) \equiv \int d^4 r e^{ik \cdot r} g(x + \frac{1}{2} r, x - \frac{1}{2} r)$$

$$\Sigma_{\text{out}}(k, x) \equiv e^{\frac{i}{2} \partial_x^\Sigma \cdot \partial_k} \Sigma(k, x)$$

Reduction of the KB equations

Decoupling

- Statistical functions splitted into background and perturbation parts

Localization

- Reduction of the infinite order gradient expansion: adiabatic background fields and local limit

Density matrix equations

- Projective basis

General transport equations

- Describes flavor and all particle-antiparticle coherence effects
- Holds for light and heavy neutrinos

$$\underbrace{\partial_t f_{khij}^{<ee'} + (\mathcal{V}_{khij}^{e'e})_{aa'} \hat{\mathbf{k}} \cdot \nabla f_{khij}^{<aa'}}_{\text{Liouville term}} = \underbrace{-2i\Delta\omega_{kij}^{ee'} f_{khij}^{<ee'}}_{\text{Leading oscillation scale}} + \underbrace{i[(\mathcal{W}_{khji}^{\text{He}'e})_a^I]^* f_{khl}^{<ea} - i(\mathcal{W}_{khij}^{\text{Hee}'})_a^I f_{khlj}^{<ae'}}_{\text{Generalized forward scattering terms}} + \underbrace{\bar{\mathcal{C}}_{khij}^{<hee'}}_{\text{Collision term}}$$

(Integrating particle-antiparticle coherence out)



$$\partial_t f_{khij}^e + \bar{v}_{kij} \hat{\mathbf{k}} \cdot \nabla f_{khij}^e = -2ie\Delta\omega_{kij} f_{khij}^e + f_{khl}^e i\sigma_{khjl}^{eee} V_{khlj}(e\omega_{kl}) - i\sigma_{khlji}^{eee} V_{khl}(e\omega_{kl}) f_{khlj}^e + \bar{\mathcal{C}}_{khij}^{<hee'}$$

(UR-limit)



$$\partial_t f_{khij}^e + \bar{v}_{kij} \hat{\mathbf{k}} \cdot \nabla f_{khij}^e = -i[H_{kh}^e, f_{kh}^e]_{ij} + \bar{\mathcal{C}}_{khij}^{<ee'} \quad \text{with} \quad \bar{v}_{kij} \equiv \frac{1}{2}(v_{ki} + v_{kj})$$

Collision terms

- How to compute collision terms?
- Cannot use the usual non-coherent Feynman rules
 - Neglect collisions between coherent particle species
- Necessary to understand neutrino evolution in dense environments
 - Role of particle-antiparticle coherence?
 - Role of flavor coherence (collisions between flavor off-diagonal particle species)?

Collision terms

- Can be written as

$$\bar{\mathcal{C}}_{H,khij}^{<hee'} = \sum_Y \frac{1}{2\bar{\omega}_{klj}^{aa'}} \int d\text{PS}_3 \left[\frac{1}{2} (\mathcal{M}^2)^{ee'}_{khij\{\mathbf{p}_i, Y\}} \Lambda_{khj\{\mathbf{p}_i, Y\}, x} + (h.c.)_{ji}^{e'e} \right].$$

- Here we defined

$$\int d\text{PS}_3 \equiv \int \left[\prod_{i=1,3} \frac{d^3 \mathbf{p}_i}{(2\pi)^3 2\bar{\omega}_{\mathbf{p}_i l_i l'_i}} \right] (2\pi)^4 \delta^4(k_l^a + p_{2l_2}^{a_2} - p_{1l'_1}^{a'_1} - p_{3l'_3}^{a'_3}),$$

$$\Lambda_{khj\{\mathbf{p}_i, Y\}}(x) = f_{khij}^{<aa'}(x) f_{X_1 \mathbf{p}_1}^{>}(x) f_{X_2 \mathbf{p}_2}^{<}(x) f_{X_3 \mathbf{p}_3}^{>}(x) - (>\leftrightarrow<).$$

- All summed indices are in curly brackets $Y \equiv \{X_i, h', a, a', l\}$
- Shorthand notation $f_{\mathbf{p}_i h_i l_i l'_i}^{s a_i a'_i}(x) \equiv f_{\mathbf{p}_i X_i}^s(x)$, where h_i is the helicity, a_i energy sign index, and l_i flavor index

Feynman rules for the matrix element

$$\text{---} \xrightarrow{ai} \text{---} \text{---} \xrightarrow{bh} \text{---} \text{---} \xrightarrow{bj} \text{---} \sim D_{k hij}^{ab}$$

$$\text{---} \xrightarrow{al} \text{---} \text{---} \xrightarrow{a'j} \text{---} \text{---} \xrightarrow{e'j} \text{---} \text{---} \xrightarrow{ei} \text{---} \sim \frac{1}{2\bar{\omega}_{kij}^{e'e}} D_{khlj}^{aa'} \gamma^0 D_{k h j i}^{e'e}$$

$$\text{---} \xrightarrow{k} \text{---} \xrightarrow{ai} \text{---} \text{---} \xrightarrow{bj} \text{---} \text{---} \xrightarrow{p} \text{---} \sim \frac{ig}{2c_w} \gamma^\mu P_L \bar{U}_{ij}$$

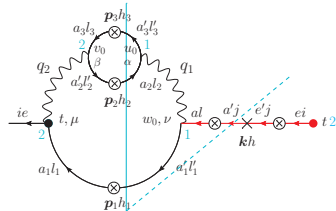
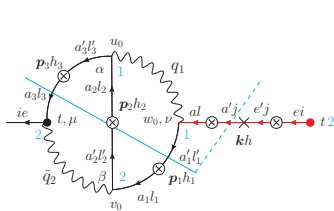
Z $(q_0, \mathbf{q})$

$$\text{---} \xrightarrow{k} \text{---} \xrightarrow{ai} \text{---} \text{---} \xrightarrow{b\alpha} \text{---} \text{---} \xrightarrow{p} \text{---} \sim \frac{ig}{\sqrt{2}} \gamma^\mu P_L U_{\alpha i}^*$$

W $(q_0, \mathbf{q})$

- Simple and straightforward rules to compute the matrix element

- $D_{k hij}^{ab} \equiv ab \hat{N}_{kij}^{ab} P_{kh} (\not{k}_i^a + m_i)(\not{k}_j^b + m_j)$



Results and Applications

- Particle-antiparticle coherence negligible
- Usual density matrix equations and collision terms reproduced at the UR-limit
- Useful to study decoherence effects, quantum entanglement ...
 - Flavor dependence in the velocity term
 - Controlled gradient expansion
- Quantitative way to study how observer-system interference affects the evolution of the system

Summary

- Developed general formalism to model neutrino evolution with all local coherences and the full collision term for coherent neutrino states
- Only flavor coherence is needed for most of the neutrino physics problems

Projective representation

- $\bar{S}_{kij}^{<}$ can be parametrized as

$$\bar{S}_{ij}^{<}(k, \mathbf{x}) = \sum_{haa'} f_{khij}^{<aa'}(t, \mathbf{x}) P_{khij}^{aa'}$$

where

$$P_{khij}^{ab} = N_{kij}^{ab} P_{kh} P_{ki}^a \gamma^0 P_{kj}^b$$

and the normalization factors are chosen as

$$N_{kij}^{ab} \equiv \left(\text{Tr} [P_{kh} P_{ki}^a \gamma^0 P_{kj}^b \gamma^0] \right)^{-1/2} = \left[\frac{2\omega_{ki}\omega_{kj}}{\omega_{ki}\omega_{kj} + ab(m_i m_j - |\mathbf{k}|^2)} \right]^{1/2}$$

- The energy and helicity projection operators are defined as

$$P_{kh} \equiv \frac{1}{2} (\mathbb{1} + h\alpha \cdot \hat{\mathbf{k}} \gamma^5) \quad \text{and} \quad P_{ki}^a \equiv \frac{1}{2} \left(\mathbb{1} + a \frac{\mathcal{H}_{ki}}{\omega_{ki}} \right)$$

with

$$\mathcal{H}_{kij} = (\alpha \cdot \mathbf{k})_i \delta_{ij} - \gamma^0 m_i \delta_{ij}, \quad \hat{h}_{\mathbf{k}} \equiv \alpha \cdot \hat{\mathbf{k}} \gamma^5$$

Projective representation

- Normalization factors can be chosen freely, but the above ones simplify the dynamical equations the most
- Apparent singularities at $\omega_{\mathbf{k}_i}\omega_{\mathbf{k}_j} = ee'(|\mathbf{k}|^2 - m_i m_j)$ do not cause problems, since they cancel at the end

Transport equations

$$\partial_t f_{\mathbf{k}hij}^{<ee'} + (\mathcal{V}_{\mathbf{k}hij}^{e'e})_{aa'} \hat{\mathbf{k}} \cdot \nabla f_{\mathbf{k}hij}^{<aa'} = -2i\Delta\omega_{\mathbf{k}ij}^{ee'} f_{\mathbf{k}hij}^{<ee'} + i[(\mathcal{W}_{\mathbf{k}hji}^{\text{He}'e})_a]^* f_{\mathbf{k}hil}^{<ea} - i(\mathcal{W}_{\mathbf{k}hij}^{\text{Hee}'})_a^l f_{\mathbf{k}hlj}^{<ae'} + \bar{\mathcal{C}}_{\mathbf{k}hij}^{<hee'}$$

where

$$(\mathcal{V}_{\mathbf{k}hij}^{e'e})_{aa'} = \delta_{a'e'} \mathcal{V}_{\mathbf{k}hij}^{eae'} + \delta_{ae} \mathcal{V}_{\mathbf{k}hji}^{a'e'e},$$

$$\mathcal{V}_{\mathbf{k}hij}^{abc} \equiv \frac{1}{2} N_{\mathbf{k}ij}^{ac} N_{\mathbf{k}ij}^{bc} \left(\frac{1}{\omega_{\mathbf{k}i}} \left[\frac{a}{(N_{\mathbf{k}ij}^{bc})^2} + \frac{b}{(N_{\mathbf{k}ij}^{ac})^2} \right] - \frac{c}{\omega_{\mathbf{k}j}} \delta_{a-b} \right),$$

$$2\Delta\omega_{\mathbf{k}ij}^{ee'} \equiv e\omega_{\mathbf{k}i} - e'\omega_{\mathbf{k}j},$$

$$(\mathcal{W}_{\mathbf{k}hij}^{ee'})_a^l \equiv \text{Tr} \left[P_{\mathbf{k}hji}^{e'e} \bar{\Sigma}_{\text{eff}kil}^{\text{H}} P_{\mathbf{k}hlj}^{ae'} \right].$$

Transport equations

- We consider vector-like gauge interactions (a_{ij} and b_{ij} are flavor matrices, and u is the plasma 4-velocity):

$$\begin{aligned}\bar{\Sigma}_{\text{H},ij}(k, x) &= \gamma^0(a_{ij}\not{k} + b_{ij}\not{u})P_L \\ &= (k_0 a_{ij} + b_{ij})P_L - a_{ij}\boldsymbol{\alpha} \cdot \mathbf{k}P_L, \\ &\rightarrow \left((k_0 + h|\mathbf{k}|)a_{ij} + b_{ij}\right)P_L \equiv V_{khij}(k_0, x)P_L.\end{aligned}$$

- In this case the forward scattering tensor coefficient reads

$$\begin{aligned}(\mathcal{W}_{khij}^{e'e})_a^l &= \sigma_{khlji}^{eae'} V_{khil}(a\omega_{\mathbf{k}l}, x), \\ \sigma_{khlji}^{abc} &\equiv \frac{1}{2} N_{kil}^{ca} N_{kji}^{bc} \left(\frac{\hat{P}_{khl}^a}{(N_{kji}^{bc})^2} + \frac{\hat{P}_{khj}^b}{(N_{kil}^{ca})^2} - \hat{P}_{khi}^c \left(\frac{1}{(N_{klj}^{ab})^2} - ab \frac{m_l m_j}{\omega_{\mathbf{k}l} \omega_{\mathbf{k}j}} \right) \right)\end{aligned}\tag{1}$$

Transport equations in the UR-limit

- In the UR-limit tensors $\sigma_{kijl}^{haee'}$ and $\mathcal{V}_{kij}^{haee'}$ become very simple:

$$\mathcal{V}_{kij}^{haee'} = \frac{e'}{|\mathbf{k}|} \delta_{a,e'} (\delta_{e,e'} + \delta_{e,-e'})$$

$$\sigma_{kijl}^{haee'} = \delta_{a,e'} (\delta_{e,e'} \delta_{h,-e} + \delta_{e,-e'} \delta_{h,e})$$

- Particle-antiparticle coherence averages out at timescales relevant for neutrino oscillations
- According to above, the transport equations can be written in the UR-limit as

$$\partial_t f_{kh}^e + \frac{1}{2} \{v_k, \hat{\mathbf{k}} \cdot \nabla f_{kh}^e\} = -i[H_{kh}^e, f_{kh}^e] + \bar{\mathcal{C}}_{kh}^e$$

Integrating out the particle-antiparticle coherence

- Leading time-dependence of the particle-antiparticle coherence functions:

$$f_{khij}^{e-e}(t) \sim \exp(-2ie\bar{\omega}_{kij}t) \quad \text{with} \quad 2\bar{\omega}_{kij} = \omega_{ki} + \omega_{kj},$$

- Leading time-dependence of the flavour coherence functions:

$$f_{khij}^{ee}(t) \sim \exp(-2ie\Delta\omega_{kij}t) \quad \text{with} \quad 2\Delta\omega_{kij} = \omega_{ki} - \omega_{kj}$$

- Weierstrass transform of equation: $\int dt' W(t, t')[\text{e.o.m.}(t')]$
- $W(t, t') \sim \exp(-(t - t')^2)/2\sigma^2$
- Given hierarchy $\Delta\omega_k \ll \bar{\omega}_k$, we can choose a σ such that $1/\Delta\omega_k \gg \sigma \gg 1/\bar{\omega}_k$

Integrating out the particle-antiparticle coherence

- Terms proportional to the coherence functions get exponentially suppressed:

$$\int dt' W(t, t') c_k^e(t') \delta f_{kh}^{e-e}(t') \sim c_k^e(t) \delta f_{kh}^{e-e}(t) \exp(-2(\bar{\omega}_k \sigma)^2)$$

- c_{kh}^e stands for any coefficient of δf_{kh}^{e-e} in the equation of motion

- For a generic 2-2 scattering process:

$$\phi_\Lambda(t) = - \left(2\Delta\omega_{kij}^{ee'} + 2\Delta\omega_{p_1 l_1 l'_1}^{a_1 a'_1} + 2\Delta\omega_{p_2 l_2 l'_2}^{a_2 a'_2} + 2\Delta\omega_{p_3 l_3 l'_3}^{a_3 a'_3} \right) t$$

- In the flavor oscillation scales $\phi(t)$ causes fast oscillations if any of the terms corresponds to coherence terms ($a \neq a'$)

$\Rightarrow$ Collision term averages out in the flavor oscillation scales