Recollections of happy days with Andrea in Saclay. . .

. . . and elsewhere

Jean-Bernard Zuber (LPTHE, Sorbonne Université)

Firenze, 2 February 2024

Andrea in Saclay : March 1986 – October 1988

8 papers on CFT's, including two on a joint work with Claude Itzykson and myself
on the classification of modular invariant partition functions
in minimal models and SU(2) WZW models,

but also on superconfl theories, energy-momentum tensor and c-theorem in higher dims [A. Coste, D. Friedan, J. Latorre]

Fall-winter 85–86: Claude and I start to work on 2D CFT's (one year after visit of Ian Affleck and Natan Andrei). Exercises with free field (bosonic or fermionic), modular invariance etc.

Then, *three unexpected fortunate events* (serendipity ?)

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Then, *three unexpected fortunate events* (serendipity ?)

1st happy circumstance: John Cardy's paper on modular invariance in minimal models (communicated to us by Bernard Derrida !).

Content of that crucial paper: Consider a "rational" CFT with Hilbert space $\mathcal{H} = \oplus_{i,\bar{i}} Z_{i\bar{i}} V_i \otimes \bar{V}_{\bar{i}}$ (operator content). Its partition function on a torus of modulus τ (aka aspect ratio $\tau = \omega_2/\omega_1$)

(i) may be written in terms of characters $\chi_i(q) = \mathrm{tr}_{V_i} q^{L_0 - c/24}$

$$Z(q = e^{2\pi \mathrm{i}\tau}) = \sum_{i\bar{\imath}} Z_{i\bar{i}} \chi_i(q) \chi_{\bar{i}}(\bar{q})\,;$$

(ii) is modular invariant, *i.e.*, invariant under $PSL(2,\mathbb{Z})$ transformations of τ;

(iii) must satisfy $Z_{i\bar{i}} \in \mathbb{N}$, $Z_{11} = 1$ (unicity of vacuum).

(iv) Characters of minimal models of CFT, $\mathcal{M}(p,p')$, form a finite-dimensional, unitary, representation of $PSL(2,\mathbb{Z})$.

Thus "diagonal" partition functions $\sum_i |\chi_i(q)|^2$ are modular invariant;

(v) but there also exist "non-diagonal" p.f., for example, that of the critical 3-state Potts model.

Cardy's paper **opens the way to a classification of CFT's**!...

We write a paper generalizing John's observation: there exists an infinie series of non diagonal modular invariants (called now "D-even series") (submitted March 3)

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- Together, we realize that in minimal models $\mathcal{M}(p, p')$, p and p' coprimes, you may construct modular invariants associated with divisors of $p \cdot p'$.
- We find more and more *physical* modular invariants (now called E_6, E_7, D_{odd}, E_8) without seeing yet the pattern nor realizing that our list is complete!...

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3d happy circumstance: crucial discussion with Vincent Pasquier who explains his construction of integrable lattice models based on (simply-laced) A-D-E Dynkin diagrams! It matches our list which is thus complete (but still conjectural!).
[Prior letter of V. Kac, mentioning a "very exceptional case" related to E_6.]

In the mean time, (thanks to Andrea) we have extended our discussion to the case of $\widehat{su}(2)_k$ cft's (aka WZW) [D. Gepner–E. Witten,'86]

Competition ! Parallel works by D. Gepner, W. Nahm, D. Bernard, C. Imbimbo and A.Schwimmer,. . .

Frantic writing of a paper (Sept 86)

Nuclear Physics B280 [FS 18] (1987) 445–465
North-Holland, Amsterdam

MODULAR INVARIANT PARTITION FUNCTIONS IN TWO DIMENSIONS

A. CAPPELLI*, C. ITZYKSON and J.-B. ZUBER

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Received 3 September 1986

We present a systematic study of modular invariance of partition functions, relevant both for two-dimensional minimal conformal invariant theories and for string propagation on a SU(2) group manifold. We conjecture that all solutions are labelled by simply laced Lie algebras.

1. Introduction

The minimal two-dimensional conformal invariant field theories [1] carry a set of representations of two Virasoro algebras of common central charge

$$c = 1 - \frac{6(p-p')^2}{pp'}, \tag{1.1}$$

with (p, p') a pair of coprime positive integers. Belavin, Polyakov and Zamolodchikov have shown that it is consistent to retain only a finite number of primary fields $\phi_{h,\bar{h}}$, of conformal dimensions h and $\bar{h}$ chosen among the Kac values [2]

$$h_{rs} = \frac{(rp-sp')^2-(p-p')^2}{4pp'} = h_{p'-r,p-s}, \tag{1.2a}$$

with

$$1 \leqslant r \leqslant p'-1, \qquad 1 \leqslant s \leqslant p-1. \tag{1.2b}$$

An important subset of these minimal theories consists of the unitary $c<1$ conformal theories, for which p and p' must be consecutive integers: $|p-p'|=1$ [3].

Cardy [4] has shown that putting such a conformal theory in a finite box with periodic boundary conditions, i.e. on a torus, gives stringent constraints on its

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(North-Holland Physics Publishing Division)

Double conjecture:

– the “commutant": general form of Z_{ij} s.th. $[S, Z] = [T, Z] = 0$.

– the “physical" invariants $Z_{ij} \geq 0$, $Z_{11} = 1$: ADE

Table of modular invariants of $\widehat{sl}(2)_k$ cft’s

level	Z	diagram
$k \geq 0$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k = 4\rho \geq 4$	$\sum_{\lambda\ \mathrm{odd}=1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$D_{2\rho+2}$
$k = 4\rho - 2 \geq 6$	$\sum_{\lambda\ \mathrm{odd}=1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \mathrm{even}=2}^{2\rho-2} (\chi_\lambda \bar{\chi}_{4\rho-\lambda} + \mathrm{c.c.})$	$D_{2\rho+1}$
$k = 10$	$\lvert\chi_1 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_8\rvert^2 + \lvert\chi_5 + \chi_{11}\rvert^2$	E_6
$k = 16$	$\lvert\chi_1 + \chi_{17}\rvert^2 + \lvert\chi_5 + \chi_{13}\rvert^2 + \lvert\chi_7 + \chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3 + \chi_{15})\bar{\chi}_9 + \mathrm{c.c.}]$	E_7
$k = 28$	$\lvert\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}\rvert^2 + \lvert\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\rvert^2$	E_8

Table 1: List of modular invariant partition functions of $\widehat{sl}(2)$ RCFTs

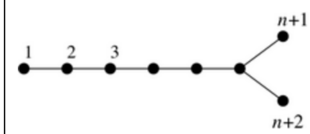
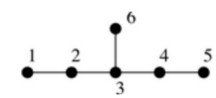
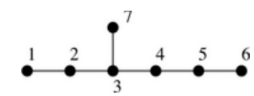
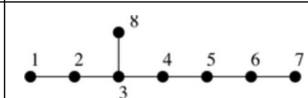
G	diagram	h	exponents ℓ_n
A_n		$n+1$	$1, 2, \cdots, n$
D_{n+2}		$2(n+1)$	$1, 3, \cdots, 2n+1, n+1$
E_6		12	$1, 4, 5, 7, 8, 11$
E_7		18	$1, 5, 7, 9, 11, 13, 17$
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$

Table 2: ADE Dynkin diagrams with Coxeter numbers h and exponents ℓ_n.

with an analogous table for Virasoro minimal models

Andrea immediately extends the classification to modular invariants of minimal superconformal theories

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MODULAR INVARIANT PARTITION FUNCTIONS OF SUPERCONFORMAL THEORIES

A. CAPPELLI [1]

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Received 29 October 1986

The modular invariant partition functions of two-dimensional minimal superconformal theories are obtained by extending a systematic method developed for conformal theories. They are classified in three infinite series and a few exceptional cases and labelled by simply laced Lie algebras.

In a recent paper [1] (hereafter referred as I), a systematic method has been developed which yields modular invariant partition functions of two-dimensional minimal conformal invariant theories with central charge $c<1$ [2,3]. There are strong indications that this is a complete classification, where each solution is labelled by a pair of simply laced Lie algebras. Superconformal minimal theories [4] are invariant under a larger class of local transformations satisfying a pair of superconformal Neveu–Schwarz–Ramond algebras and have $c<\frac{3}{2}$.

The tricritical Ising model in two dimensions is an example of the simplest superconformal theory [4]; since it has $c=\frac{7}{10}$ it is also a minimal conformal theory. Amazingly, it provides a realization of $N=1$ supersymmetry in nature.

In this letter we obtain modular invariant partition functions for superconformal theories by extending the methods in I. The two mathematical problems are very similar and we shall see that superconformal solutions are made by the same building blocks as the conformal one.

The modular invariance problem has been settled in ref. [5] and two simpler solutions have been obtained. The unitarity condition for representations of the superconformal algebra constrains the values of $c<\frac{3}{2}$ to the discrete series [4]

$$c=\tfrac{3}{2}[1-8/m(m+2)], \quad m=3,4,\ldots . \tag{1}$$

The values of the highest weights can be consistently constrained to a finite set (the Kac table):

$$h_{rs}=h_{m-r,m+2-s}=\{[(m+2)r-ms]^2-4\}/8m(m+2)+\tfrac{1}{32}[1-(-)^{r-s}],$$
$$1\leqslant r\leqslant m-1, \quad 1\leqslant s\leqslant m+1. \tag{2}$$

The superconformal algebra and its representations split into two sectors, the Neveu–Schwarz (NS) and Ramond (R) sectors, which are selected by antiperiodic or periodic boundary conditions on fermionic fields, respectively. In eq. (2), the NS states have $r-s$ even and the R states $r-s$ odd. The NS vacuum state has $h=0$, i.e. $r=s=1$, while the R "vacuum" [‡1] has $h=\frac{1}{24}c$ and appears for even m only, at the self-symmetric point of the Kac table $(r,s)=(\frac{1}{2}m, \frac{1}{2}m+1)$.

A fundamental domain Δ is a set of independent h values in each sector: $\Delta_{\mathrm{NS}}=\{h_{rs}\,|\,1\leqslant s\leqslant r\leqslant m-1\}$, $r-s$ even and $\Delta_{\mathrm{R}}=\{h_{rs}\,|\,1\leqslant s\leqslant r-1$ for $1\leqslant r\leqslant[\frac{1}{2}(m-1)]$ and $1\leqslant s\leqslant r+1$ for $[\frac{1}{2}(m+1)]\leqslant r\leqslant m-1\}$.

[1] On leave of absence from Dipartimento di Fisica and INFN, Sezione di Firenze, Largo E. Fermi 2, I-50125 Florence, Italy.
[‡1] This is the state of lowest energy in the R sector and it is globally supersymmetric invariant.

Then several months before full proof.

– D. Gepner and Z. Qiu (Jan. 87): commutant ✓

– CIZ 2 (also A. Kato) (May '87): ADE ✓

Commun. Math. Phys. 113, 1–26 (1987)

Communications in Mathematical Physics

The A-D-E Classification of Minimal and $A_1^{(1)}$ Conformal Invariant Theories

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Abstract. We present a detailed and complete proof of our earlier conjecture on the classification of minimal conformal invariant theories. This is based on an exhaustive construction of all modular invariant sesquilinear forms, with positive integral coefficients, in the characters of the Virasoro or of the $A_1^{(1)}$ Kac-Moody algebras, which describe the corresponding partition functions on a torus. A remarkable correspondence emerges with simply laced Lie algebras.

I. Introduction

1. The minimal conformal invariant models describe a class of massless two dimensional field theories, with known critical properties [1]. Their anomalous dimensions and operator content are encoded in the expression of the partition function on a torus. The sum over states decomposes into pairs of irreducible representations of the Virasoro algebra, with central charge c rational and smaller than 1, yielding a sesquilinear form in the characters χ_h,

$$Z(\tau)=\sum \mathcal{N}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}^*(\tau) \ .$$

In this formula τ is the ratio of the two periods on the torus, and the summation extends over a finite table of known $(h,\bar{h})$ values. The non-negative integral coefficients $\mathcal{N}_{h,\bar{h}}$ yield the multiplicities of primary scaling operators $\varphi_{h,\bar{h}}$, which are in one to one correspondence with the products $\chi_h\chi_{\bar{h}}^*$ of characters. Cardy [2] noticed that modular invariance is a consistency condition on these partition functions.

Our aim here is to present a detailed proof of the classification of these positive modular invariants, announced in [3]. As these theories describe statistical models at criticality, this classifies the universality classes of two dimensional critical phenomena, pertaining to $c<1$, with finitely many primary observables. They include for instance the Ising and three-state Potts models.

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Later (1999) simpler proof by T. Gannon

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January, 1999

The Cappelli-Itzykson-Zuber A-D-E Classification

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In 1986 Cappelli, Itzykson and Zuber classified all modular invariant partition functions for the conformal field theories associated to the affine A_1 algebra; they found they fall into an A-D-E pattern. Their proof was difficult and attempts to generalise it to the other affine algebras failed – in hindsight the reason is that their argument ignored most of the rich mathematical structure present. We give here the “modern” proof of their result; it is an order of magnitude simpler and shorter, and much of it has already been extended to all other affine algebras. We conclude with some remarks on the A-D-E pattern appearing in this and other RCFT classifications.

1. The problem

Why A-D-E ?

Several possible answers

– associated integrable lattice models [Pasquier]

built on graphs: largest e-value of adjacency matrix < 2

– $N = 2$ twisted superconf. theories [Lerche–Vafa–Warner, Martinec]

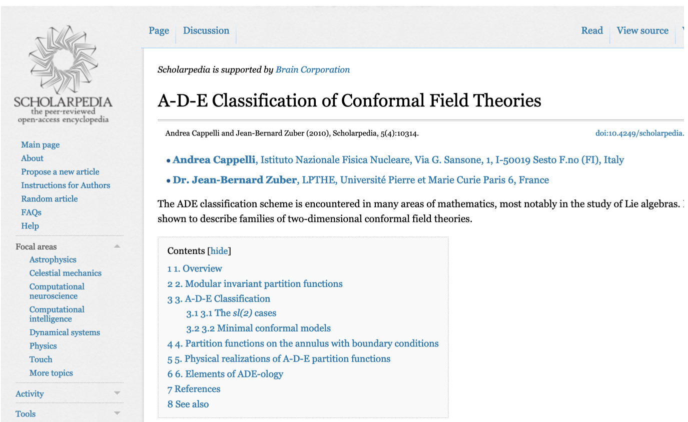
and their superpotential is a simple singularity, hence of ADE type [?]

– Boundary conds (Cardy's equations) [Behrend–Pearce–Petkova-Z '98]

coded by matrices ("nimreps" of fusion algebra) of eigenspectrum < 2

– . . .

Andrea and I : Scholarpedia . . .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A-D-E Classification of Conformal Field Theories

Andrea Cappelli and Jean-Bernard Zuber (2010), Scholarpedia, 5(4):10314. doi:10.4249/scholarpedia.

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- **Dr. Jean-Bernard Zuber**, LPTHE, Université Pierre et Marie Curie Paris 6, France

The ADE classification scheme is encountered in many areas of mathematics, most notably in the study of Lie algebras. shown to describe families of two-dimensional conformal field theories.

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4 4. Partition functions on the annulus with boundary conditions
5 5. Physical realizations of A-D-E partition functions
6 6. Elements of ADE-ology
7 References
8 See also

After that, Andrea goes on to

– c-theorem, stress-energy tensor in higher dims, [D. Friedan, A. Coste, J. Latorre ,...]

– QHE and W_∞ symmetry [C.Trugenberger, G.Zemba, G.Dunne, I.Todorov, . . .]

– topological insulators

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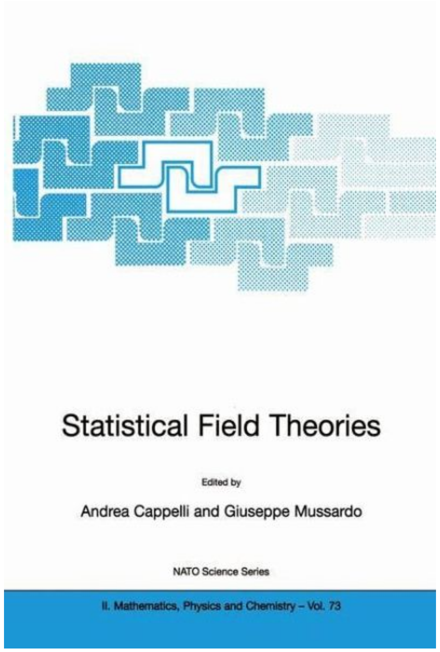
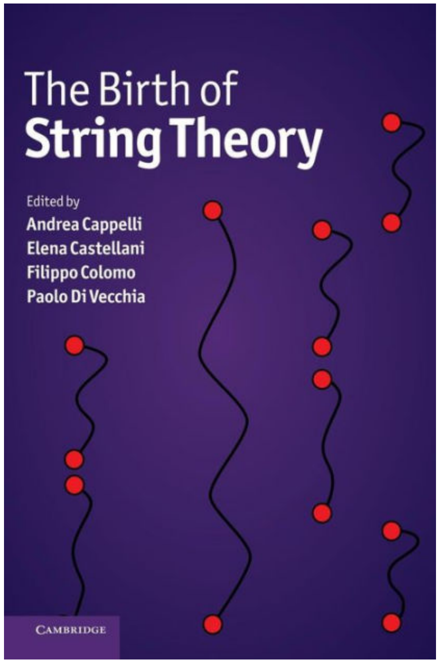
plus two books





Happy days in Saclay . . .

. . . and elsewhere:

Cortona, Les Houches, Firenze and Paris (xxx), Santa Barbara. . .

Andrea, I have always admired your versatility, your equanimity,

. . .

and your big and warm smile !

Congratulations, $\mathbb{A}$n$\mathbb{D}$r$\mathbb{E}$a

Many happy returns

and keep your curiosity

and your big smile . . .