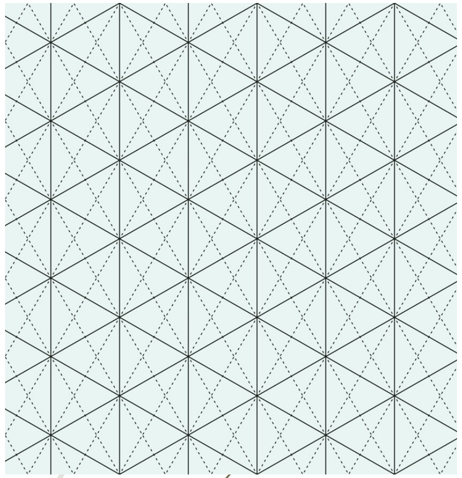


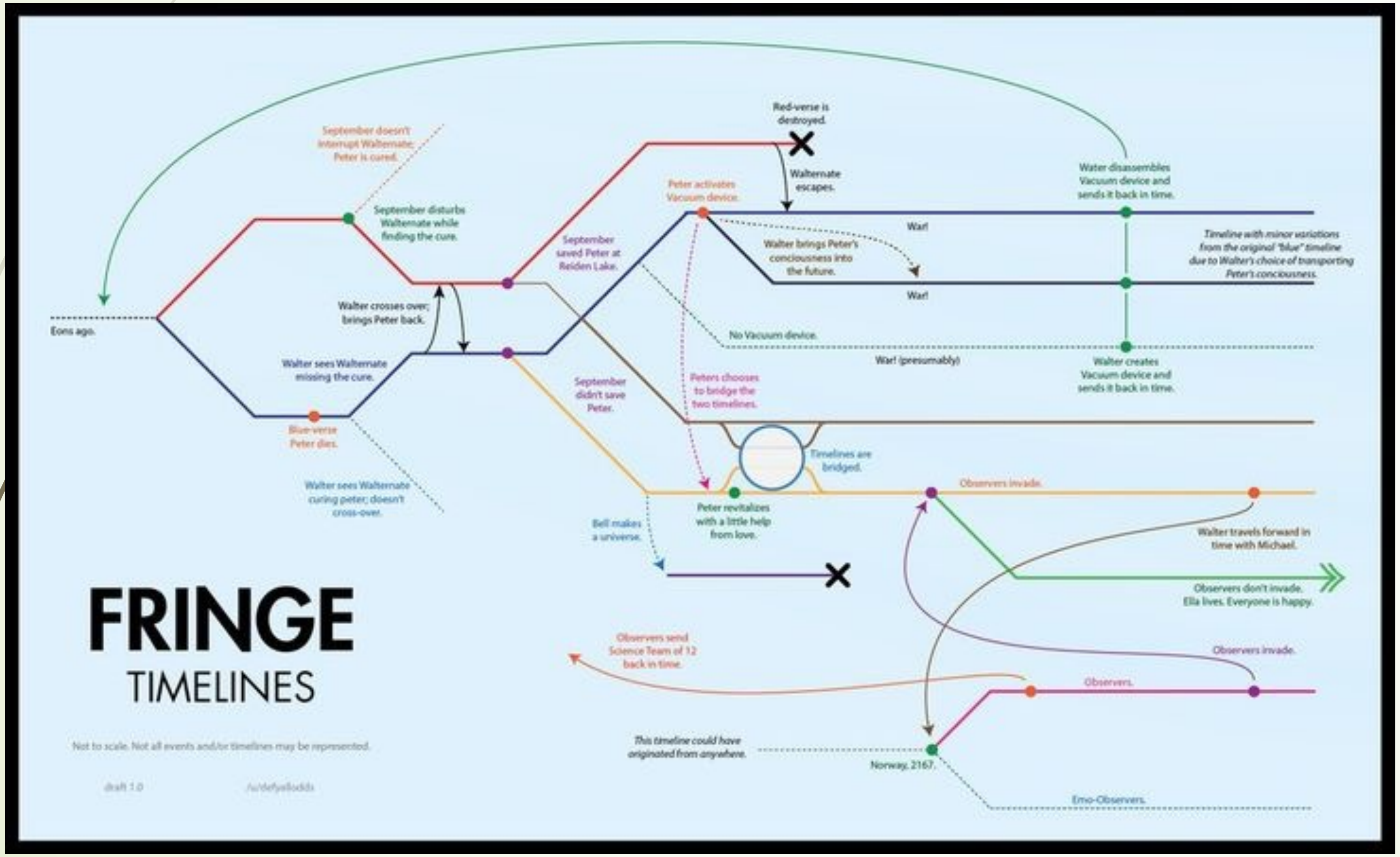
Andrea Cappelli 65

Conformal Field Theory 40

and...

Hagedorn transitions in exact $U_q(sl_2)$ S-matrix theories with arbitrary spins

Timelines intersecting...



Well... not so much!
but Andrea and I had some funny intersection in the first years of our careers...

Andrea & Francesco

Firenze

Firenze

Born in

Univ & PhD

Cortona

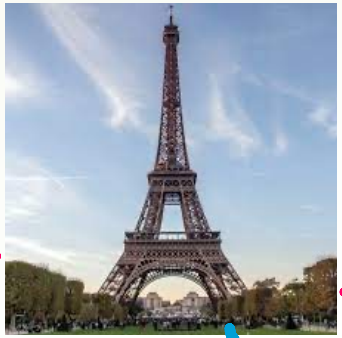
PhD discussed
The same day
In the same place
(EUR – Rome)

Then….

Novara

Torino

Paris
Saclay



In Bologna

Finally got our permanent
Positions at INFN

On the same
Day Nov 1, 1988

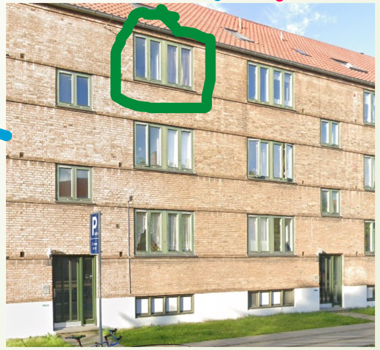
Postdoc

But then….

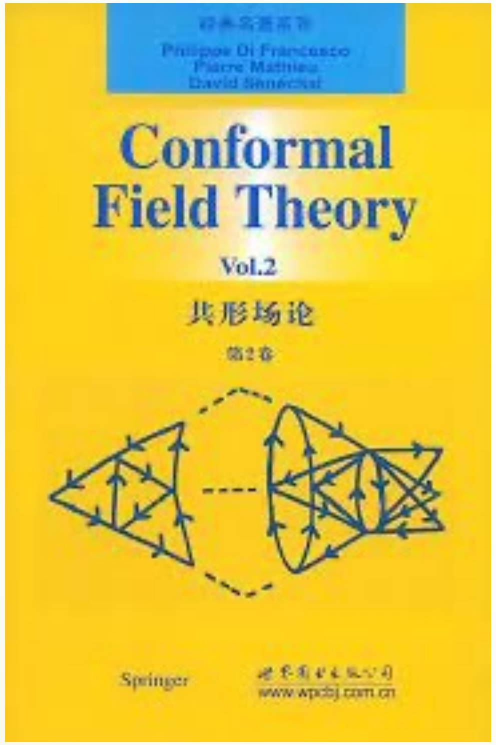
Housed in…
Hillerødgade 65

In Flornece

Copenhagen
NBI - NORDITA



But the main *intersection* we had was…



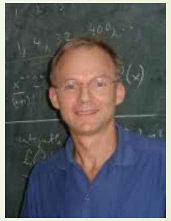
for which we both have to give a great

THANK YOU

to the mentors and friends we met during those years

Paolo

Claude



Jean-Bernard

and many others…

Hagedorn transitions in exact $U_q(sl_2)$ S-matrix theories with arbitrary spins

Scattering Theory in 2D

Integrability -> Factorised S-matrices that satisfy:

- YBE
- Unitarity
- Crossing symmetry

	Sine-Gordon → s = ½	**Sausage → s = 1**
Factorised scattering	Doublet soliton-antisoliton	Triplet of solitons
S-matrix prop. to	R-matrix of spin ½ x ½	R-matrix of spin 1 x 1

Why not try higher spin s ?

Factorised scattering of (iso-)spin s particles

$\mathcal{U}_{\mathsf{q}}(\mathfrak{su}_2)$ symmetry

$$[\lambda]_{\mathsf{q}} \equiv \frac{\mathsf{q}^{\lambda/2} - \mathsf{q}^{-\lambda/2}}{\mathsf{q}^{1/2} - \mathsf{q}^{-1/2}}$$

$$[\mathbb{J}_{\pm}, , \mathbb{J}_3] = \pm \mathbb{J}_{\pm} \quad , \quad [\mathbb{J}_{+}, , \mathbb{J}_{-}] = [2\mathbb{J}_3]_{\mathsf{q}}$$

2 particle S-matrix:

$$\mathsf{S}(\theta) = \sigma \left(\mathsf{P} \sum_{J=0}^{2s} f_{\mathsf{q}}^{[J]}(\theta)\, \mathbb{P}_{\mathsf{q}}^{[J]} \right) \sigma^{-1}$$

gauge permutator q-Projectors

Case q = 1 introduced in Aladim, Martins 1994

q-Projectors

$$\mathbb{P}_{\mathsf{q}}^{[J]}{}^{m_1' m_2'}_{m_1 m_2} = \sum_{M=-J}^{J} \langle s, m_1'; s, m_2' | J, M \rangle_{\mathsf{q}} \langle J, M | s, m_1; s, m_2 \rangle_{\mathsf{q}}$$

- q-Clebsch-Gordan

$$\langle s, m_1; s, m_2 | J, M \rangle_{\mathsf{q}} = f(J) \cdot \mathsf{q}^{(2s-J)(2s+J+1)/4 + s(m_2 - m_1)/2}$$

$$\times \quad \{[s+m_1]![s-m_1]![s+m_2]![s-m_2]![J+M]![J-M]!\}^{1/2} \sum_{\nu \geq 0} (-1)^{\nu} \frac{\mathsf{q}^{-\nu(2s+J+1)/2}}{\mathcal{D}_{\nu}}$$

$$\mathcal{D}_{\nu} \quad = \quad [\nu]![2s-J-\nu]![s-m_1-\nu]![s+m_2-\nu]![J-s+m_1+\nu]![J-s-m_2+\nu]!$$

$$f(J) \quad = \quad \left\{ \frac{[2J+1]_{\mathsf{q}}([J]!)^2[2s-J]!}{[2s+J+1]!} \right\}^{1/2}$$

Rapidity functions & prefactor

$$\mathsf{q} = e^{2\pi i\gamma}$$

$$f_{\mathsf{q}}^{[J]}(\theta) = S_0(\theta) \prod_{k=1}^{J} \frac{\sinh\left[\gamma(ik\pi - \theta)\right]}{\sinh\left[\gamma(ik\pi + \theta)\right]}, \qquad J = 0, 1, \cdots, 2s.$$

Unitarity: $S_0(\theta)S_0(-\theta) = 1$

Crossing: $S_0(i\pi - \theta) = \prod_{k=1}^{2s} \frac{\sinh\left[\gamma(i(k+1)\pi - \theta)\right]}{\sinh\left[\gamma(ik\pi + \theta)\right]} S_0(\theta)$

$$S_0(\theta) = \prod_{k=1}^{2s} \left[\frac{\sinh\left[\gamma(i\pi k + \theta)\right]}{\sinh\left[\gamma(i\pi k - \theta)\right]} \left(\prod_{\ell=1}^{\infty} \frac{\sinh\left[\gamma(i\pi(k+\ell) - \theta)\right] \sinh\left[\gamma(i\pi(k-\ell) - \theta)\right]}{\sinh\left[\gamma(i\pi(k+\ell) + \theta)\right] \sinh\left[\gamma(i\pi(k-\ell)k + \theta)\right]} \right) \right]$$

More on prefactor

When s is integer the prefactor greatly symplifies

$$S_0(\theta)=\prod_{m=1}^{s}\frac{\sinh\left[\gamma(\theta+i2m\pi)\right]}{\sinh\left[\gamma(\theta-i2m\pi)\right]}\quad\Longrightarrow\quad \mathsf{S}_{ss}^{ss}(\theta)=\prod_{m=1}^{s}\frac{\sinh\left[\gamma(\theta-i(2m-1)\pi)\right]}{\sinh\left[\gamma(\theta+i(2m-1)\pi)\right]}$$

When s is half-integer it gives rise to the infinite Γ product

$$\begin{aligned} S_0(\theta) \quad &= \prod_{m=1}^{2s}\Bigg\{\frac{1}{i\pi}\sinh\left[\gamma(\theta+im\pi)\right]\Gamma\left[1-\gamma(m-1)+\frac{i\gamma\theta}{\pi}\right]\Gamma\left[1-\gamma m-\frac{i\gamma\theta}{\pi}\right] \\ &\times\prod_{n=1}^{\infty}\left[\frac{R_n^{[s,m]}(\theta)R_n^{[s,m]}(i\pi-\theta)}{R_n^{[s,m]}(0)R_n^{[s,m]}(i\pi)}\right]\Bigg\}\end{aligned}$$

In both cases the integral representation holds

$$\mathsf{S}_{ss}^{ss}(\theta)=\exp\int_{-\infty}^{\infty}\frac{dk}{k}\frac{\sinh(\pi k s)\sinh\pi k(s-\frac{1}{2\gamma})}{\sinh\frac{\pi k}{2\gamma}\sinh\pi k}e^{ik\theta}$$

Examples

- s = ½ : **Sine-Gordon**
- s = 1 : **Sausage**
- s = 3/2 :

$$\mathsf{S}_{11}^{11}=1,\ \mathsf{S}_{12}^{12}=\frac{(0)}{(3)},\ \mathsf{S}_{12}^{21}=\frac{s_3}{(3)},\ \mathsf{S}_{13}^{13}=\frac{(0)(-1)}{(2)(3)},\ \mathsf{S}_{13}^{22}=\frac{s_2\sqrt{s_3/s_1}(0)}{(2)(3)},$$

$$\mathsf{S}_{13}^{31}=\frac{(s_1s_4+2s_2)(0)}{(2)(3)},\ \mathsf{S}_{22}^{22}=\frac{f_1}{(2)(3)},\ \mathsf{S}_{14}^{14}=\frac{(0)(-1)(-2)}{(1)(2)(3)},\ \mathsf{S}_{14}^{23}=\frac{s_3(0)(-1)}{(1)(2)(3)}$$

$$\mathsf{S}_{14}^{32}=\frac{s_2s_3(0)}{(1)(2)(3)},\ \mathsf{S}_{14}^{41}=\frac{s_1s_2s_3}{(1)(2)(3)},\ \mathsf{S}_{23}^{23}=\frac{(0)f_1}{(1)(2)(3)},\ \mathsf{S}_{23}^{32}=\frac{s_2f_2}{(1)(2)(3)},$$

$$(n)\equiv 2\sinh\left[\gamma(\theta-i\pi n)\right],\qquad s_n\equiv 2\sinh(in\pi\gamma),$$

$$f_1=2\cosh\left[\gamma(2\theta-i\pi)\right]+\frac{s_{10}}{s_5}-2\frac{s_2}{s_1},\ f_2=2\frac{s_2}{s_1}\cosh\left[\gamma(2\theta-i\pi)\right]+s_2^2-2s_1^2-4$$

Thermodynamic Bethe Ansatz

- Bethe Yang equation

$$e^{iR\mathsf{m}\sinh\theta_j}\mathbb{T}(\theta_j|\{\theta_i\}) = 1,$$

$$\mathbb{T}(\theta_j|\{\theta_i\})^{m_1',\cdots,m_{\mathcal{N}}'}_{m_1,\cdots,m_{\mathcal{N}}} = \sum_{n_1,\cdots,n_{\mathcal{N}}} \mathsf{S}^{n_2 m_1'}_{n_1 m_1}(\theta_1-\theta_j)\mathsf{S}^{n_3 m_2'}_{n_2 m_2}(\theta_2-\theta_j)\cdots\mathsf{S}^{n_1 m_{\mathcal{N}}'}_{n_{\mathcal{N}} m_{\mathcal{N}}}(\theta_{\mathcal{N}}-\theta_j)$$

- String hypothesis

$$\lambda^{(n)}_{j,\alpha} = \lambda^{(n)}_j + \frac{i\pi}{2}(n+1-2\alpha), \quad \alpha = 1,2,\cdots,n$$

we restrict to the simpler case $\gamma = \frac{1}{N}$

- Diagonalisation in terms of Bethe Ansatz of XXZ higher spin chains [Kulish Reshetikhin]
- Thermodynamic limit and intergal eqs. for densities of centers of strings σ and $\tilde{\sigma}$
- Minimisation of free energy and TBA

$$\sigma_n(\theta) + \tilde{\sigma}_n(\theta) = \delta_{n0}\mathsf{m}\cosh\theta - \nu_n \sum_{m=0}^{N} K_{nm} \star \sigma_n(\theta) \qquad K_{nm}(\theta) = \frac{1}{2\pi i}\frac{d}{d\theta}\ln S_{nm}(\theta)$$

TBA equations: free energy & scaling fct.

Pseudoenergies

$$\epsilon_0(\theta) = \log \frac{\tilde{\sigma}_0}{\sigma_0}, \qquad \epsilon_n(\theta) = \log \frac{\sigma_n}{\tilde{\sigma}_n}, \quad n = 1, \ldots, N-1, \qquad \epsilon_N(\theta) = \log \frac{\tilde{\sigma}_N}{\sigma_N}$$

Universal kernel $p(\theta) = \dfrac{1}{2\pi \cosh \theta}$

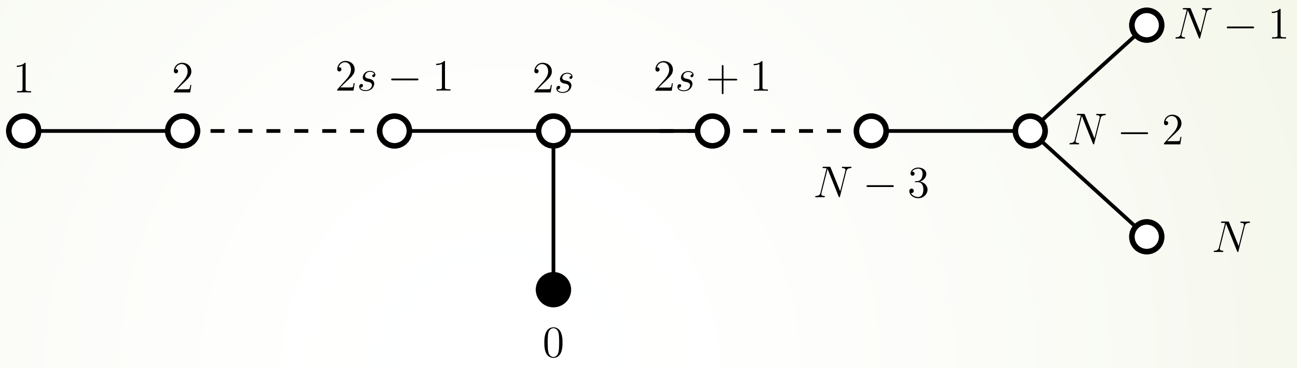
Incidence matrix

$$\epsilon_n(\theta) = \delta_{n,0} \mathsf{m} L \cosh \theta - \sum_{m=0}^{N} \mathbb{I}_{nm}\, p \star \log\left(1 + e^{-\epsilon_m}\right)(\theta)$$

Free energy

$$\frac{f(T)}{T} = -\int_{-\infty}^{\infty} \frac{\mathsf{m}}{2\pi} \cosh \theta \ln\left(1 + e^{-\epsilon_0(\theta)}\right) d\theta$$

TBA graph: not a Dynkin diagram



Finite size vacuum energy of the mirror theory $E_0(T) = \frac{f(T)}{T}$

Dimensionless parameter $r = \mathsf{m}/T$ measures the size

Scaling function

$$\widetilde{c}(r) = \frac{3}{\pi^2}\mathsf{m}\int_{-\infty}^{\infty} r\cosh(\theta)L_0(\theta)d\theta$$

$$\lim_{r\to 0}\widetilde{c}(r) = c - 24\Delta_{\min}$$

Plateaux or not plateaux?

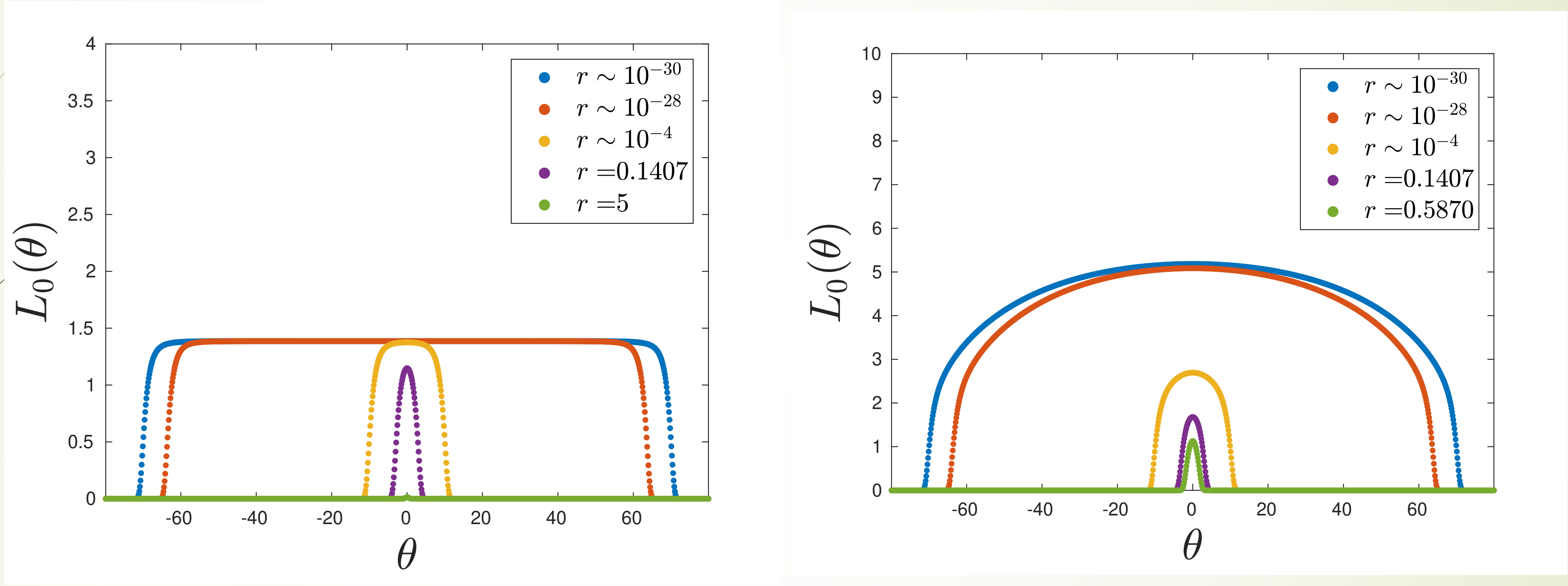


Figure 3: The functions $L_0(\theta)$ for spin $s = 1/2$ (left) and $s = 1$ (right) with $\gamma = 1/7$, for different values of r. One can see that for smaller values of r the plateau starts to form.

Behaviour for various s

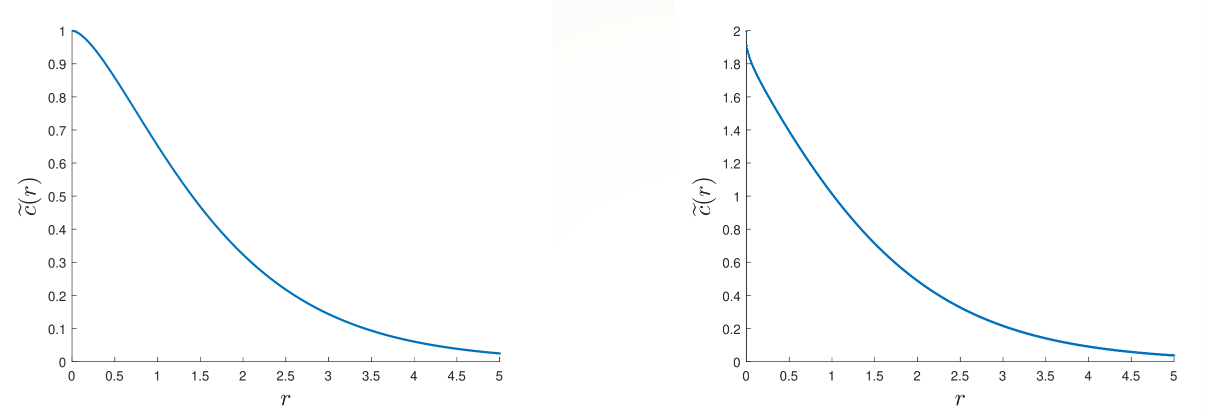


Figure 4: Scaling functions for spin $s = 1/2$ (left) and $s = 1$ (right).

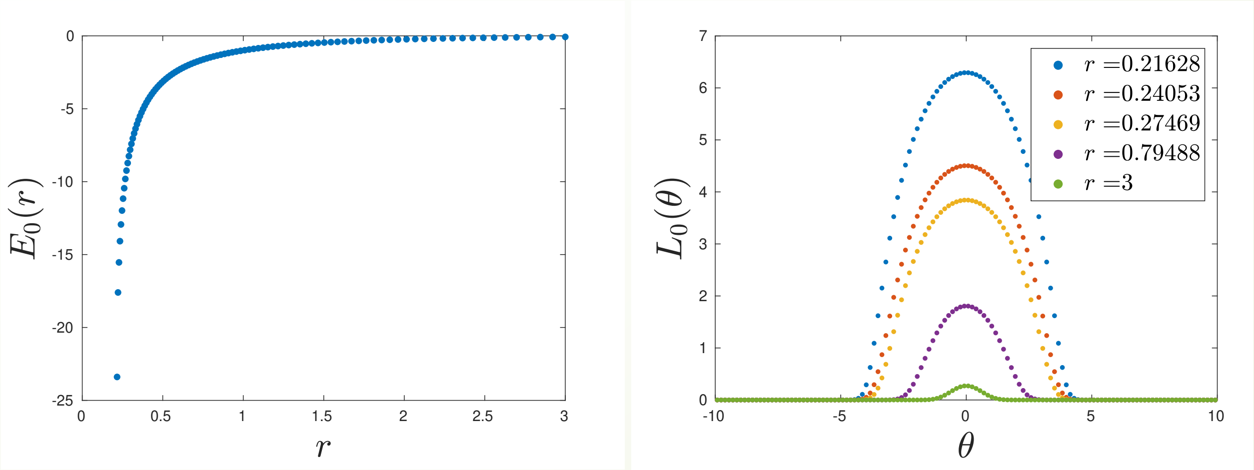


Figure 5: Left: the vacuum energy $E_0(r)$ as it approaches the singular point $r^* = 0.21628(2)$; right: the kernel $L_0(\theta)$ at different values of r. Both were obtained for $s = 5/2$ and $N = 12$.

Critical temperature

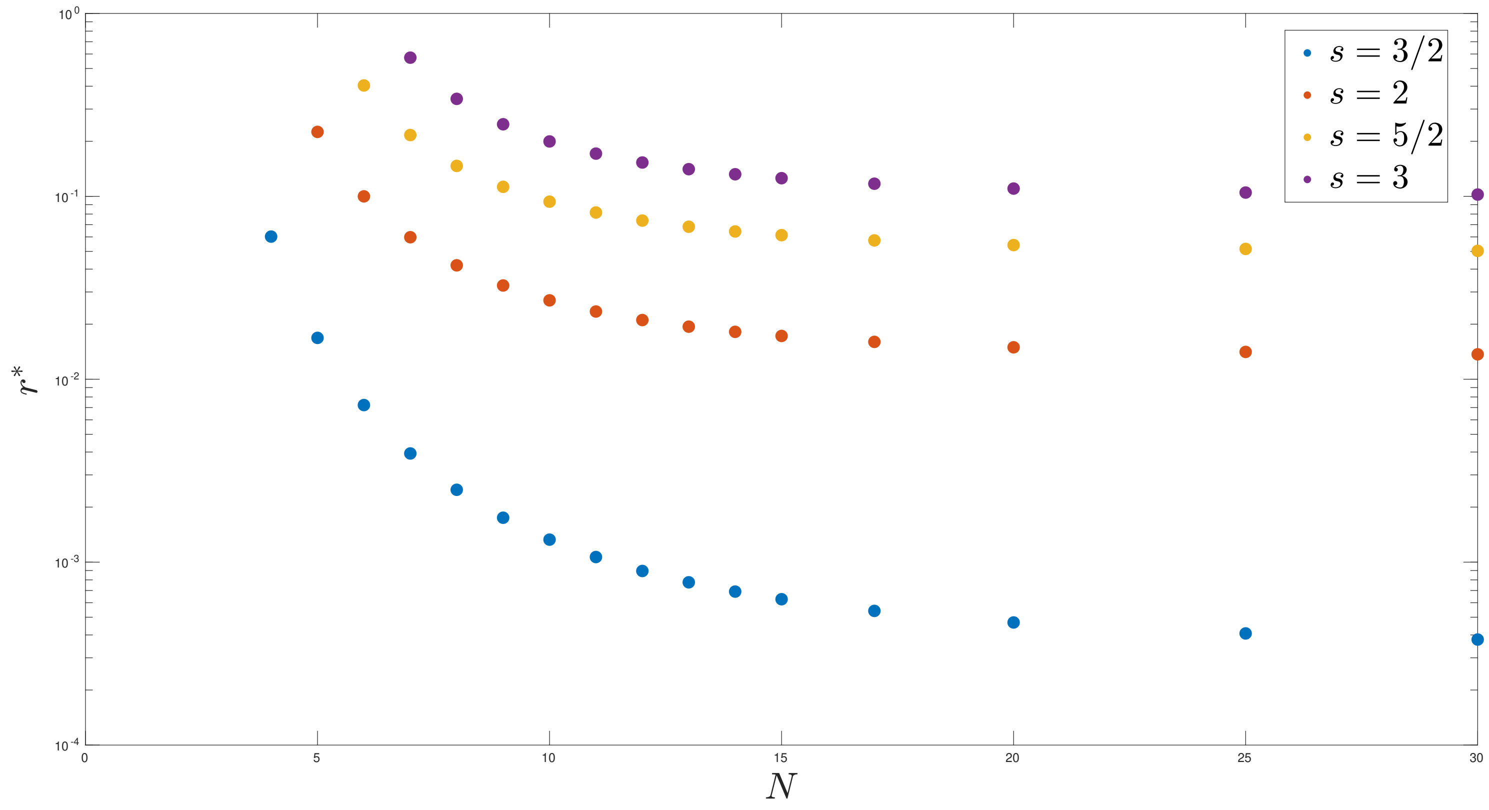


Figure 6: Value of the singular point r^*, for different values of spin and coupling constant $\gamma = 1/N$. The values are computed with precision to the 6th decimal digit. The r^*-axis is log-scaled.

Hagedorn transition ?

$$E_0(r) \sim_{r \to r^*} c_0 + c_{1/2}\sqrt{r - r^*}$$

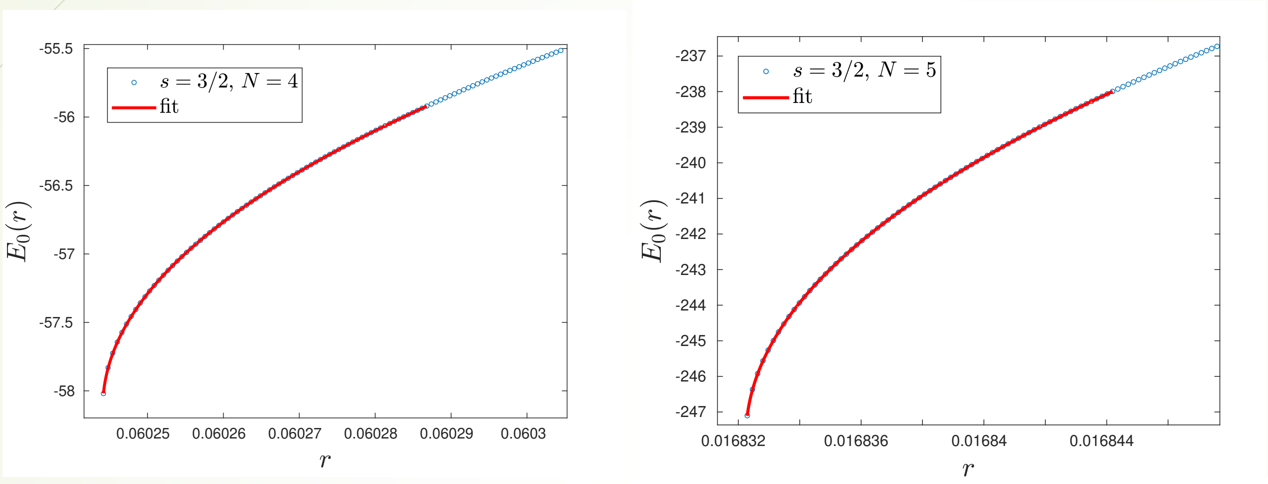
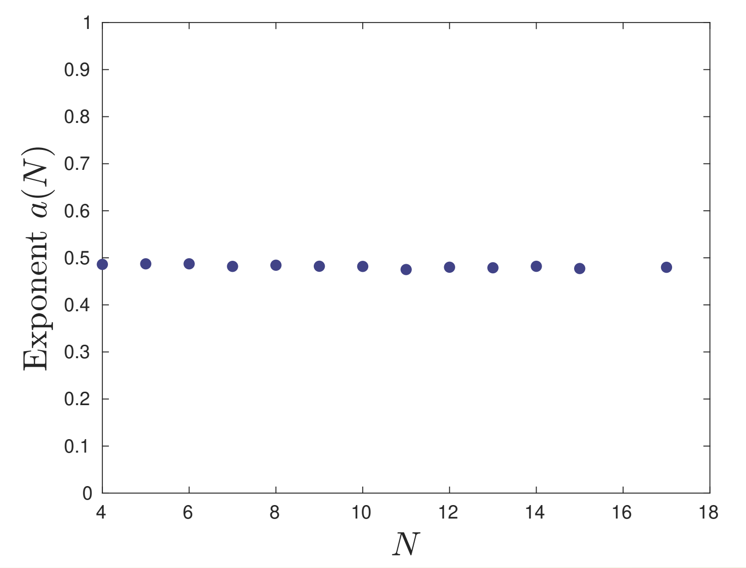


Figure 7: Examples of fitting for $s = 3/2$ and $N = 4$ (left) and $N = 5$ (right).

$$b(r - r^*)^a + c_0$$



Thank you