The Effective Field Theory of the Calogero-Sutherland model

Guillermo R. Zemba

Comisión Nacional de Energía Atómica

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Collaborators:

Bosonization:

Federico L. Botessi
Raffaele Cariacciolo
Marialuisa Frau
Alberto Lerda
Stefano Sciuto

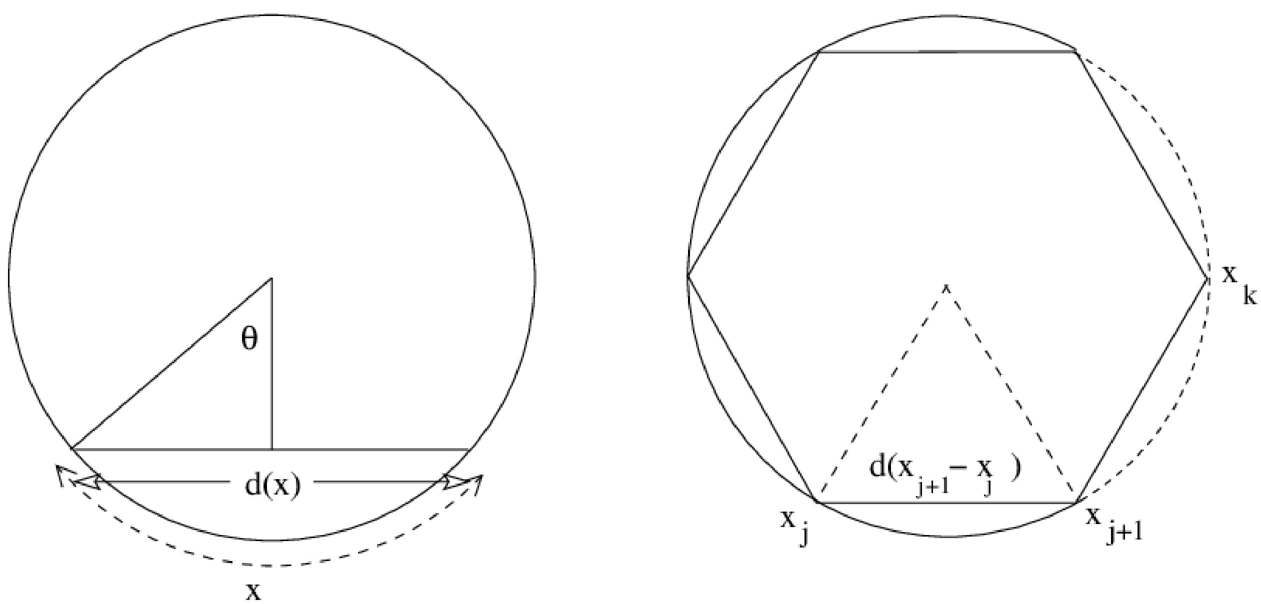
$W_{1+\infty}$ dynamics (QHE effect):

Andrea Cappelli
Carlo A. Trugenberger

The Calogero-Sutherland model

$$h_{CS} = \sum_{j=1}^{N} \left(\frac{1}{i} \frac{\partial}{\partial x_j} \right)^2 + g \frac{\pi^2}{L^2} \sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j)/L)}$$

$$\hbar = 1 = 2m$$



d is the distance along the chord

The free fermionic field of the C-S model

- $\phi_k(x) = \frac{1}{\sqrt{L}} \exp\left(i\frac{2\pi}{L}kx\right) \qquad \phi_k(x,t) = \phi_k(x)\exp\left(-i\epsilon_k t\right),\ \epsilon_k = (2\pi/L)^2 k^2$

- $\Psi(x,t) \equiv \sum_{k=-\infty}^{\infty} c_k\ \phi_k(x,t)\ , \qquad \{\ c_k, c_l^\dagger\ \} = \delta_{k,}$

- $|\ \Omega\ , N\ \rangle = a^\dagger_{-M} a^\dagger_{-M+1} \ldots a^\dagger_{M-1} a^\dagger_M |\ 0\ \rangle \qquad M \equiv (N-1)/2$

- $n(x,t) \equiv \langle\ \Omega\ , N\ |\Psi^\dagger(x,t)\Psi(x,t)|\ \Omega\ , N\ \rangle = \sum_{k=-M}^{M} |\phi_k(x,t)|^2 = \frac{N}{L}$

In the following, $N/L = n_0 = \rho_0 \qquad N \to \infty, L \to \infty$

Fermi sea: L  R $p_F = \pi(N-1)/L$

The second quantized Hamiltonian

- $$\mathcal{H} = (2\pi\rho_0)^2 \sum_{k=0}^{2} \frac{1}{N^k} \mathcal{H}_{(k)}$$ (because interaction is of $O(1/L^2)$)

- $$\mathcal{H}_{(0)} = \frac{1}{4}(1+g) \sum_{r=-\infty}^{\infty} \left(: a_r^\dagger \, a_r : + : b_r^\dagger \, b_r : \right)$$

- $$\begin{aligned} \mathcal{H}_{(1)} \; &= \; \left(1 + \frac{g}{2}\right) \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2}\right) \left(: a_r^\dagger \, a_r : + : b_r^\dagger \, b_r : \right) \\ &+ \frac{g}{2} \sum_{\ell,r,s=-\infty}^{\infty} : a_{r-\ell}^\dagger \, a_r :: b_{s-\ell}^\dagger \, b_s : \quad , \end{aligned}$$

- $$\begin{aligned} \mathcal{H}_{(2)} \; &= \; \sum_{r=-\infty}^{\infty} \left[\left(r - \frac{1}{2}\right)^2 + \frac{g}{4}\left(r^2 - r\right) \right] \left(: a_r^\dagger \, a_r : + : b_r^\dagger \, b_r : \right) \\ &- \frac{g}{4} \sum_{\ell,r,s=-\infty}^{\infty} |\ell| \left[: a_{r-\ell}^\dagger \, a_r :: a_{s+\ell}^\dagger \, a_s : + : b_{r+\ell}^\dagger \, b_r :: b_{s-\ell}^\dagger \, b_s : \right. \\ &\left. + \, 2 \; : a_{r-\ell}^\dagger \, a_r :: b_{s-\ell}^\dagger \, b_s : \right] \\ &+ \frac{g}{2} \sum_{\ell,r,s=-\infty}^{\infty} (r + s - \ell - 1) : a_{r-\ell}^\dagger \, a_r :: b_{s-\ell}^\dagger \, b_s : \quad . \end{aligned}$$

The W-infinity algebra

The $W_{1+\infty}$ algebra $\quad W_{1+\infty} \times \overline{W}_{1+\infty}$

(chiral) currents $V_n^i \qquad h = i+1 \geq 1 \qquad n \in \mathbf{Z}$

- $$[\ V_n^i, V_m^j\ ] = (jn - im)V_{n+m}^{i+j-1} + q(i,j,n,m)V_{n+m}^{i+j-3} + \cdots + \delta^{ij}\delta_{n+m,0}\ c\ d(i,n)$$

- $$\begin{aligned} [\ V_n^0, V_m^0\ ] &= c\ n\ \delta_{n+m,0} \quad , \\ [\ V_n^1, V_m^0\ ] &= -m\ V_{n+m}^0 \quad , \\ [\ V_n^1, V_m^1\ ] &= (n-m)V_{n+m}^1 + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \end{aligned}$$

- $$\begin{aligned} \left[\ V_n^2, V_m^0\ \right] &= -2m\ V_{n+m}^1 \quad , \\ \left[\ V_n^2, V_m^1\ \right] &= (n-2m)\ V_{n+m}^2 - \frac{1}{6}\left(m^3 - m\right) V_{n+m}^0 \quad , \\ \left[\ V_n^2, V_m^2\ \right] &= (2n-2m)\ V_{n+m}^3 + \frac{n-m}{15}\left(2n^2 + 2m^2 - nm - 8\right) V_{n+m}^1 \\ &\quad + c\ \frac{n(n^2-1)(n^2-4)}{180}\ \delta_{n+m,0} \quad . \end{aligned}$$

Fermionic and bosonic realizations

- $$V_n^0 = \sum_{r=-\infty}^{\infty} : a^\dagger_{r-n}\, a_r : \quad ,$$
  $$V_n^1 = \sum_{r=-\infty}^{\infty} \left(r - \frac{n+1}{2} \right) : a^\dagger_{r-n}\, a_r : \quad ,$$
  $$V_n^2 = \sum_{r=-\infty}^{\infty} \left(r^2 - (n+1)\, r + \frac{(n+1)(n+2)}{6} \right) : a^\dagger_{r-n}\, a_r :$$
  $$\{ a_k, a_l^\dagger \} = \delta_{k,l}$$

- $$W_\ell^0 = \alpha_\ell \quad ,$$
  $$W_\ell^1 = \frac{1}{2} \sum_{r=-\infty}^{\infty} : \alpha_r\, \alpha_{\ell-r} : \quad ,$$
  $$W_\ell^2 = \frac{1}{3} \sum_{r,s=-\infty}^{\infty} : \alpha_r\, \alpha_s\, \alpha_{\ell-r-s} :$$
  $$[\alpha_n, \alpha_m] = \xi n \delta_{n+m,0}$$

The EFT of the CS model

- $(c, \bar{c}) = (1, 1)\ CFT$

- $$\mathcal{H}_{CS} = \left(2\pi n_0 \sqrt{\xi}\right)^2 \left\{ \left[\frac{\sqrt{\xi}}{4} W^{\,0}_{\,0} + \frac{1}{N} W^{\,1}_{\,0} + \frac{1}{N^2} \left(\frac{1}{\sqrt{\xi}} W^{\,2}_{\,0} - \frac{\sqrt{\xi}}{12} W^{\,0}_{\,0} \right.\right.\right.$$
$$\left.\left.\left. - \frac{g}{2\xi^2} \sum_{\ell=1}^{\infty} \ell\, W^{\,0}_{-\ell} W^{\,0}_{\,\ell} \right) \right] + \left(W \leftrightarrow \overline{W} \right) \right\} ,$$

$$\xi = \left(1 + \sqrt{1+2g}\right)/2$$

(Note that g=0 ↔ ξ=1)

$$\left[W^{\,0}_{\,\ell}, W^{\,0}_{\,m} \right] = c\, \xi \ell\, \delta_{\ell+m,0} ,$$
$$\left[W^{\,1}_{\,\ell}, W^{\,0}_{\,m} \right] = -m\, W^{\,0}_{\,\ell+m} ,$$
$$\left[W^{\,1}_{\,\ell}, W^{\,1}_{\,m} \right] = (\ell - m) W^{\,1}_{\,\ell+m} + \frac{c}{12} \ell(\ell^2 - 1)\delta_{\ell+m,0} ,$$
$$\left[W^{\,2}_{\,\ell}, W^{\,0}_{\,m} \right] = -2m\, W^{\,1}_{\,\ell+m} ,$$
$$\left[W^{\,2}_{\,\ell}, W^{\,1}_{\,m} \right] = (\ell - 2m)\, W^{\,2}_{\,\ell+m} - \frac{1}{6}\left(m^3 - m\right) W^{\,0}_{\,\ell+m} ,$$
$$\left[W^{\,2}_{\,n}, W^{\,2}_{\,m} \right] = (2n - 2m)\, W^{\,3}_{\,n+m} + \frac{n-m}{15}\left(2n^2 + 2m^2 - nm - 8\right) W^{\,1}_{\,n+m}$$
$$+ c\, \frac{n(n^2-1)(n^2-4)}{180}\, \delta_{n+m,0} .$$

Charged and neutral excitations

- $$|\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 = V^{\,0}_{-k_1} \dots V^{\,0}_{-k_r} \overline{V}^{\,0}_{-\overline{k}_1} \dots \overline{V}^{\,0}_{-\overline{k}_s} |\Delta N, \Delta D\rangle_0$$

$$k_1 \geq k_2 \geq \dots \geq k_r > 0 \qquad \overline{k}_1 \geq \overline{k}_2 \geq \dots \geq \overline{k}_s > 0$$

$$V^{\,0}_{\,0} |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 = \left(\frac{\Delta N}{2} + \Delta D\right) |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0$$

$$\overline{V}^{\,0}_{\,0} |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0 = \left(\frac{\Delta N}{2} - \Delta D\right) |\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0$$

- $$W^{\,0}_{\,0} |\Delta N; \Delta D\rangle_W = \left(\sqrt{\xi}\,\frac{\Delta N}{2} + \frac{\Delta D}{\sqrt{\xi}}\right) |\Delta N; \Delta D\rangle_W$$

$$\overline{W}^{\,0}_{\,0} |\Delta N; \Delta D\rangle_W = \left(\sqrt{\xi}\,\frac{\Delta N}{2} - \frac{\Delta D}{\sqrt{\xi}}\right) |\Delta N; \Delta D\rangle_W$$

Energy spectrum of the EFT of CS model

$$\mathcal{E} = \left(2\pi n_0\sqrt{\xi}\right)^2 \left\{\left[\frac{\sqrt{\xi}}{4} Q + \frac{1}{N}\left(\frac{1}{2}Q^2 + k\right) + \frac{1}{N^2}\left(\frac{1}{3\sqrt{\xi}}Q^3 - \frac{\sqrt{\xi}}{12}Q \right.\right.\right.$$
$$\left.\left.\left. + \frac{2k}{\sqrt{\xi}}Q + \frac{\sum_j k_j^2}{\xi} - \sum_j (2j-1)\,k_j\right)\right] + \left(Q \leftrightarrow \overline{Q}\,,\ \{k_j\} \leftrightarrow \{\overline{k}_j\}\right)\right\}$$

$$Q = \sqrt{\xi}\,\frac{\Delta N}{2} + \frac{\Delta D}{\sqrt{\xi}} \quad , \quad \overline{Q} = \sqrt{\xi}\,\frac{\Delta N}{2} - \frac{\Delta D}{\sqrt{\xi}} \qquad k = \sum_j k_j \quad , \quad \overline{k} = \sum_j \overline{k}_j$$

Compactification radius $r = \frac{1}{\sqrt{\xi}}$

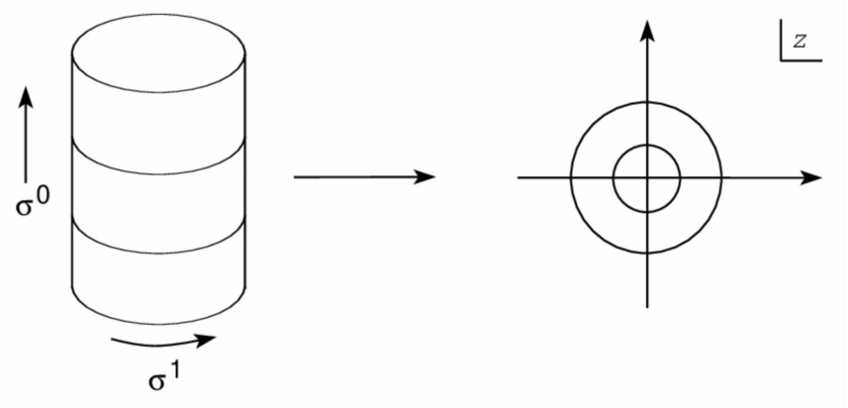
Fermi velocity $v = 2\pi n_0 \xi$

Relation to hydrodynamics

- Map cylinder to z-plane

$$z = \exp\left(\frac{u}{R}\right) = \exp\left(\frac{\tau}{R} - i\theta\right)$$

$$R = L/(2\pi),\ x = R\theta,$$



$$W^i(z) \equiv \sum_n W^i_n\, z^{-n-i-1}$$

$$W^0_R(u) = \frac{z}{R} W^0(z)$$

$$W^1_R(u) = \frac{z}{R^2}\left(z^2 W^1(z) - \frac{1}{24}\right),$$

$$W^2_R(u) = \frac{z}{R^3}\left(z^3 W^2(z) - \frac{z}{12} V^0(z)\right)$$

Relation to hydrodynamics

- Hamiltonian (reintroduce m)

$$H_{CS} = \left(\frac{\pi n_0 \xi}{mR}\right) \left\{ \left[\frac{N\sqrt{\xi}}{4} W^{\,0}_{\,0} + W^{\,1}_{\,0} + \frac{1}{N} \left(\frac{1}{\sqrt{\xi}} W^{\,2}_{\,0} - \frac{\sqrt{\xi}}{12} W^{\,0}_{\,0} \right.\right.\right.$$
$$\left.\left.\left. - \frac{(\xi - 1)}{\xi} \sum_{\ell=1}^{\infty} \ell\, W^{\,0}_{-\ell} W^{\,0}_{\ell} \right) \right] + \left(W \leftrightarrow \overline{W} \right) \right\} \quad ,$$

- Time evolution of the density field

$$\frac{\partial W^0_R(u)}{\partial t} = -i \left[W^0_R(u), H_{CS} \right] = -i \frac{z}{R} \left[W^0(z), H_{CS} \right]$$

$$n(x,t) = \frac{1}{\pi \xi^{3/2}} \left(W^{\,0}_{\,R}(x,t) + \pi n_0 \sqrt{\xi} \right)$$

Some calculations...

$$\frac{\partial n}{\partial t} = \left(\frac{\pi\xi^2}{2m}\right)\frac{\partial}{\partial x}\left(n^2\right) + $$
$$- \frac{iz(\xi-1)n_0}{m\sqrt{\xi}NR^2}\sum_{\ell=1}^{\infty}\left[\ell^2\, z^{-\ell-1}W^{\,0}_{\ell} - \ell^2\, z^{\ell-1}W^{\,0}_{-\ell}\right]$$

(normal ordered)

$$\sum_{\ell=1}^{\infty}\left[\ell^2\, z^{-\ell-1}W^{\,0}_{\ell} - \ell^2\, z^{\ell-1}W^{\,0}_{-\ell}\right] = -\frac{\partial}{\partial z}\left[z\frac{\partial}{\partial z}\left(zW^0_{+}(z)\right) - z\frac{\partial}{\partial z}\left(zW^0_{-}(z)\right)\right]$$

Fields with positive or negative modes only

$$W^0_{+}(z) = \sum_{n=1}^{\infty} W^0_n\, z^{-n-1}$$
$$W^0_{-}(z) = \sum_{n=-\infty}^{-1} W^0_n\, z^{-n-1}$$

Relation to hydrodynamics

$$\frac{\partial n}{\partial t} = \left(\frac{\pi\xi^2}{2m}\right)\frac{\partial}{\partial x}\left(n^2\right) + - \frac{i(\xi-1)}{2\pi m\sqrt{\xi}}\frac{\partial^2}{\partial x^2}\left(\left(W_R^0\right)_+ - \left(W_R^0\right)_-\right)$$

Hilbert transform

$$\left(W_R^0\right)_H = i\left[\left(W_R^0\right)_+ - \left(W_R^0\right)_-\right]$$? with $$\left(W_R^0\right)_H(x,t) = \frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{W_R^0(x,t')}{(t-t')}\,dt'$$

$$W_R^0(u) = \frac{z}{R}W^0(z) = \frac{1}{R}\sum_{n=-\infty}^{\infty} z^{-n}\,W_n^0$$

$$\left(W_R^0\right)_H(x,\tau) = -\frac{1}{\pi}\sum_{n=-\infty}^{\infty} C_n\,\exp(-n\tau/R)\,W_n^0$$

$$C_n = PV\int_0^{\infty}\frac{s^{-n-1}}{\ln s}\,ds$$

$$C_n = \begin{cases} -i\pi & (n>0) \\ +i\pi & (n<0)\end{cases}$$

Coefficients perform the projection on positive or negative modes

Quantum Benjamin-Ono equation

$$\frac{\partial n}{\partial t} = \left(\frac{\pi\xi^2}{2m}\right)\frac{\partial}{\partial x}\left(n^2\right) + \frac{(\xi-1)\xi}{2m}\frac{\partial^2 n_H}{\partial x^2}$$

$\alpha = \pi\xi^2/(2m)$ $\beta = \xi(\xi-1)/(2m)$ $\beta/\alpha = (\xi-1)/(\pi\xi)$

A. G. Abanov and P. B. Wiegmann, Phys. Rev. Lett. 95, 076402 (2005)

Classical B-O: Non-linear waves in deep water

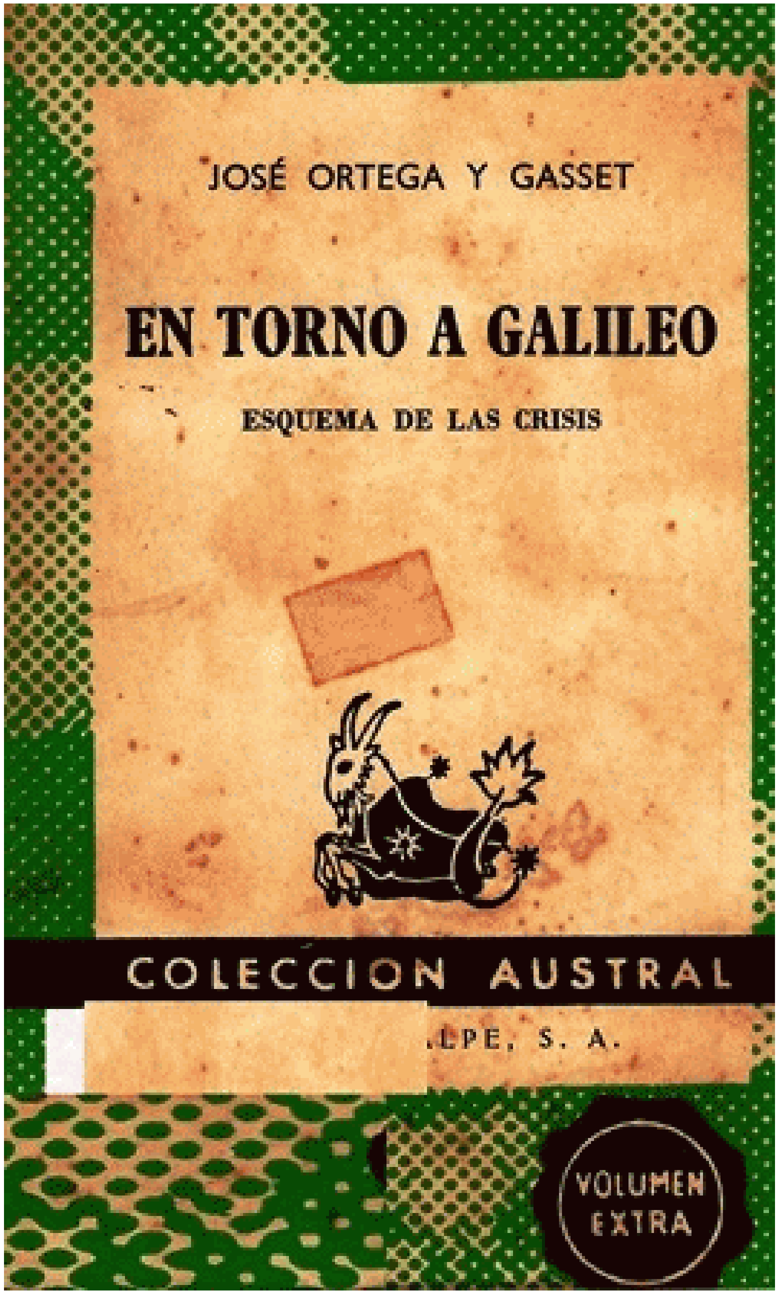
Feynman-Wheeler interpretation

$$\left(W_R^0\right)_H = i\left[\left(W_R^0\right)_+ - \left(W_R^0\right)_-\right]$$

The second term in the RHS can be interpreted as waves moving backwards in time from the distant future ("advanced waves")

Conclusions

- The long-distance, low-energy behavior of the Calogero-Sutherland admits both an EFT and hydrodynamics descriptions
- We have reviewed the EFT based on the W-infinity bosonization
- We have verified consistency with the predicted quantum Benjamin-Ono equation for the density field obtained with the hydrodynamics formulation using EFT methods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EN TORNO A GALILEO
ESQUEMA DE LAS CRISIS
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.LPE, S. A.
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Many thanks!

Thank you for your attention and thanks to the organizers!