Emergent Fermions in Hydrodynamics

P. Wiegmann

University of Chicago

Celebration of The Works of Andrea Cappelli

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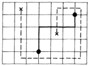
Emergent fermions: Onsager I

Onsager, 1944: Fermions in Ising model



Numerous examples in 1+1 dimensions, starting from kink-type configurations in spin chains (Lieb, Schultz, Mattis 1964)

Fermions in a semiclassical fluid

Fermions under certain physical conditions form fluids:

electrons in superconductors cooper, He^3, organic conductors, quark-gluon plasma, etc.

Can an individual fermion emerge from as a special low energy 'flow' without destroying a fluid?

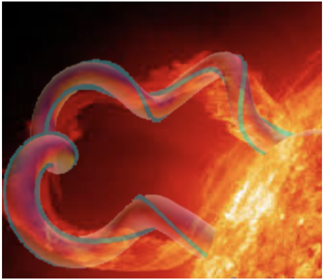
Not always, but some time, yes it can.

What hydrodynamic equations describe such situation?

The subject arisen from discussions with Andrea and Sasha Abanov.

SEMICLASSICAL FLUID

Euler equation: $n(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{p} + \nabla P = 0\,,$
$\mathbf{p} = m\mathbf{v}\,.$



Circulation: $\oint p_\mu dx^\mu = \oint p = 2\pi\hbar \times \text{integer},$ (Onsager 1949)

Helicity: $\int \epsilon^{\mu\nu\lambda\sigma} p_\nu \partial_\lambda p_\sigma d^3x = \int p \wedge dp = (2\pi\hbar)^2 \times$number of twists (torsion of the vortex filament

EULER EQUATIONS

Canonical formulation of fluid dynamics does not require a metric! (Andre Lichnerowitcz, 1941)

$$\text{Euler+continuity}: \begin{cases} n^\mu \partial_\mu p_\nu + \partial_\nu P = 0\,, \\ \\ \partial_\mu n^\mu = 0\,. \end{cases}$$

$$\text{Equations are written in terms of} \quad \begin{cases} n^\mu - \text{vector field (mass current)}\,, \\ p_\mu - \text{momentum (a contravariant vector), or 1-form } p_\mu dx^\mu\,, \\ n^\mu = \partial P/\partial p_\mu - \text{pressure}\,. \end{cases}$$

$$\text{Deformed continuity:} \quad \partial_\mu(n^\mu + \tfrac{k}{2}\hbar\epsilon^{\mu\nu\lambda\sigma}p_\nu\partial_\lambda p_\sigma) = 0, \quad k \in \mathbb{Z}\,,$$

$$\text{In terms of diff. forms:} \quad d(n + k\hbar\, p \wedge dp) = 0\,.$$

Cohomology and Fermions

Fermion are spatially localized (particle-like) field configuration whose adiabatic process changes the action as

$$\text{Fermions spin } k/2: \quad \begin{cases} \text{spatial rotation:} & S \to S + \pi i k\hbar\,, \\ \text{exchange of positions of 2 particles:} & S \to S + 2\pi i k\hbar \end{cases}$$

$$\Psi \sim e^{\frac{i}{\hbar}S} \to (-1)^k \Psi$$

Fermions is a semiclassical phenomenon: $\quad S = S_0 + k\hbar\Gamma, \quad \Gamma \to \Gamma + \pi$

Adiabatic phase is metric independent (a topological phase), hence is expressed in terms of differential 1-forms, which requires a homological condition

$$H_{d+1}(\mathscr{M}) \neq 0, \quad d = \text{spacetime dimension, even}$$

$H(\mathscr{M}) = \mathbb{Z}$ (Polyakov, P. W. 1983, Witten 1983, Polyakov 1989);
$H(\mathscr{M}) = \mathbb{Z}_2$ (Dzyaloshinskii, Polyakov, P. W. 1988).

MULTIVALUED ACTION

Field theory on Lie group manifold deformed by a multivalued Novikov's, functional:

$$\boxed{\begin{aligned} &S = \tfrac{1}{2\lambda}\int_{S^D} \mathrm{tr}\,(g^{-1}dg)^2 + k\hbar\Gamma\,, \\ &\Gamma = \tfrac{1}{d+1}\int_{D^{d+1}} \mathrm{tr}\,(g^{-1}dg)^{d+1}\,, \\ &\partial D^{d+1} = S^d \text{ (spacetime)} \quad g \in G, \quad H_{d+1}(G) = \mathbb{Z}\,. \end{aligned}}$$

$d = 2$: Witten 1983, Polyakov & P. W. 1983; $d = 4$:Witten 1983, $G = SU(N)$.

$$\frac{1}{6}\int_{D^3} \mathrm{tr}(g^{-1}dg)^3, \quad g \in SU(2), \quad \partial D^3 = S^2$$

The integrand is a closed form;
in terms of Euler angles it is

$$\tfrac{1}{6}\mathrm{tr}(g^{-1}dg)^3 = d\psi \wedge d\cos\theta \wedge d\varphi = d\ \mathrm{vol}(S^3)$$

the value of the integral depends on boundary values of the angle ψ

$$\psi\ d\cos\theta \wedge d\varphi = \psi\ d\ \mathrm{vol}(S^2)$$

The action changes under rotation $\psi \to \psi + 2\pi$

$$S \to S + \pi k\hbar$$

At the same time EOM depend on $d\psi$

Polyakov 1989

$$S = \int_{S^2\times\mathbb{R}^1} A\,\dot{x} + k\hbar\Gamma, \quad \Gamma = \int_{S^2\times\mathbb{R}^1} A\wedge dA\,.$$

The added term provides a 'geometric' interaction by identifying the particle word trajectory with a magnetic flux

$$\dot{x} = kdA\,.$$

The added term is not gauge invariant $A_0 \to A_0 + \dot{\psi}$ on a punctured manifold (particles)

$$\int_{S^2\times\mathbb{R}^1} A\wedge dA \to \int_{D^3} A\wedge dA + \oint_{S^2} \psi\,\dot{x}\,.$$

Polyakov argued that trajectory of a particle becomes a framed curve, a ribbon, and that ψ is a physical degree of freedom identified with a frame angle. Under 2π-twist of the ribbon the the action changes by

$$S \to S + \pi k\hbar$$

FLUID DYNAMICS

According to Vladimir Arnold (1966), fluid dynamics could be viewed as Hamiltonian mechanics operating on a group manifold of diffeomorphisms of spacetime $\boxed{\mathrm{Diff}(\mathscr{M}^4)}$. Can we construct a topological (metric independent) functional on this manifold?

A natural guess is Novikov's [Wess, Zumino, Witten] multivalued functional on this group manifold is

$$S = S_0 + \Gamma, \quad \Gamma = \frac{1}{3}\int_{D^5} p \wedge dp \wedge dp, \quad \partial D^5 = S^4 .$$

Under a gauge transformation $p \to p + d\psi$ the action changes as $S \to S + \frac{k}{2}\int_{S^4} \psi dp \wedge dp$.

Under a 2π-twist

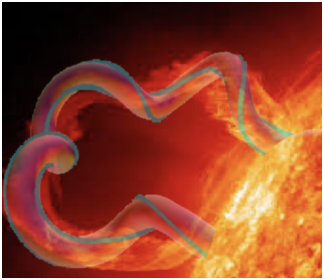
$$\Delta S = \pi k \hbar \int_{S^4} dp \wedge dp = \pi k[\text{Self-Linking number of vortex lines}] = \pi \hbar k[\text{Torsion of a framed curve}]$$

Fermions in hydrodynamics

$$n^{\mu}\partial_{\mu}p_{\nu} + \partial_{\nu}P = 0\,,$$

$$d(n + \tfrac{k}{2}p \wedge dp) = 0\,,$$

$$S = S_0 + \tfrac{k}{3}\hbar \int_{D^5} p \wedge dp \wedge dp$$



Celebrating Life with Andrea