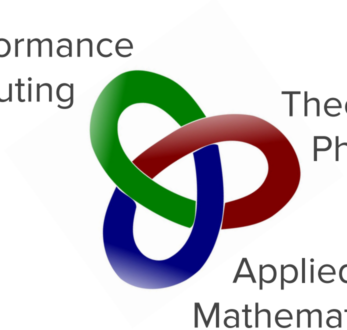
Introduction to Lattice QCD and Hadron Structure – Numerical Methods


High-Performance Computing

Theoretical Physics

Applied Mathematics

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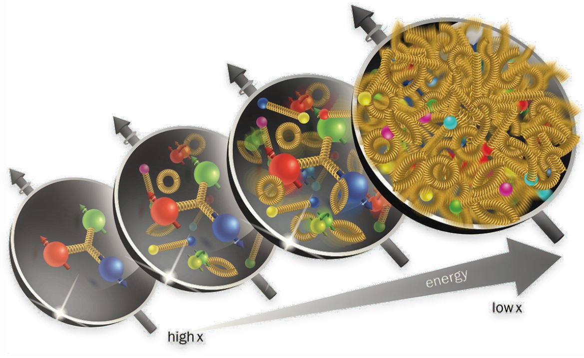
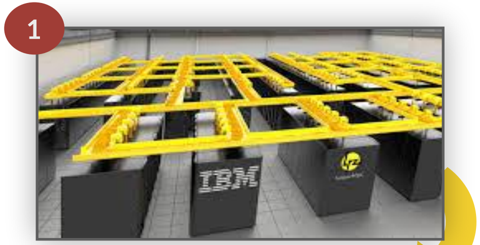
Slides available at `http://tiny.cc/ggi-lqcd`



{20,21}/02/25 - Galileo Galilei Institute

Lattice QCD Calculations in a Nutshell

$$\langle \mathcal{O}\rangle_{a,V,\vec{\mu}_f} = \frac{1}{\mathcal{Z}}\int \mathrm{D}[U]^V\, \mathrm{D}[\psi\bar{\psi}]^{Vn_f}\, \mathcal{O}(U,\psi,\bar{\psi})\, e^{-S(a,\vec{\mu}_f;\,U,\psi,\bar{\psi})}$$



Simulation

Part 1

- Markov chain Monte Carlo to generate ensembles of gluon-field configurations via importance sampling

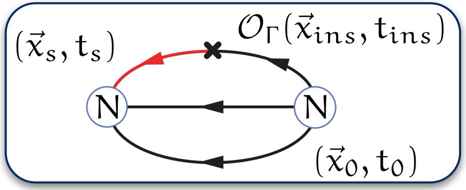
$$\langle\mathcal{O}\rangle = \frac{1}{N}\sum_i^N O(U_i)\ \ \textit{with}\ \ \ p(U) = \frac{1}{\mathcal{Z}}\exp(-S(U))$$

Production – quantity-specific

Part 2

- Construction of hadron correlation functions on background field configurations:

$$\langle N(p',s')|A_\mu|N(p,s)\rangle$$

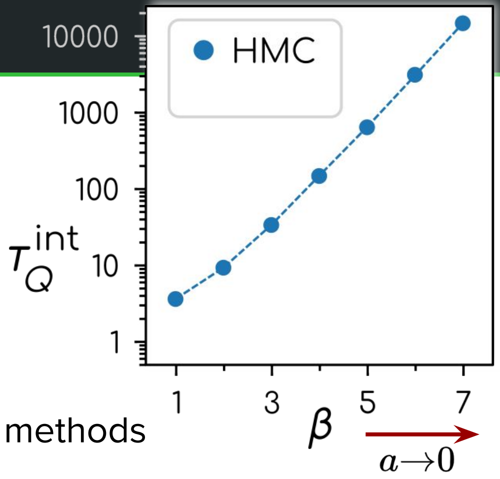


Data analysis – post-processing

- Statistical analysis, resampling, derived quantities
- Excited state contamination and stochastic errors
- Continuum and infinite volume extrapolation

Part 3

Part 1: Simulations summary

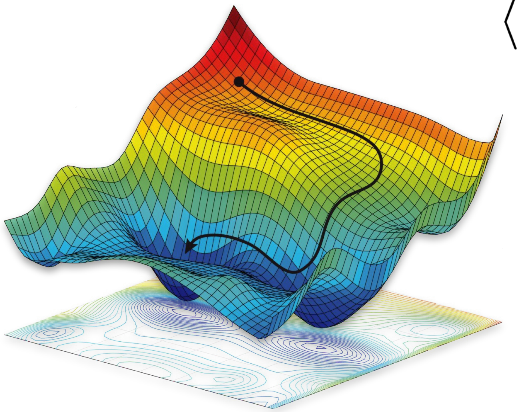


In Lattice Gauge Theories, the expectation value of an observable is defined as

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[U]\, \mathcal{O}(U)\ \exp(-S(U))$$

Computed via importance sampling and using Markov-chain Monte Carlo (MCMC) methods

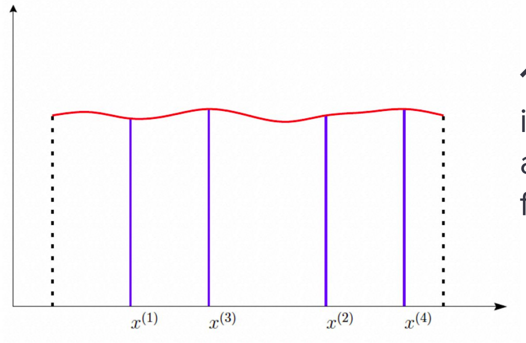
$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_i^N O(U_i) \quad \textit{with} \quad p(U) = \frac{1}{\mathcal{Z}} \exp(-S(U))$$



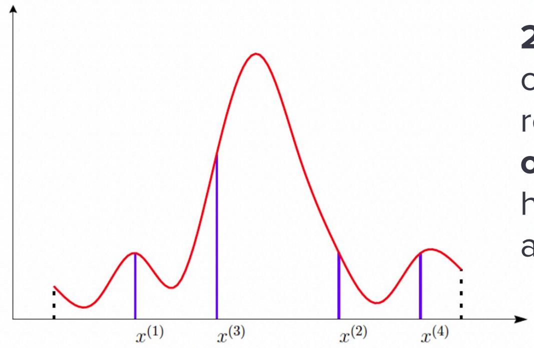
- **Requires independent and identically distributed (IID) samples**
- **State-of-the-art:** Hybrid Monte Carlo (HMC) algorithm

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[U]\, \mathrm{D}[p]\, \mathcal{O}(U)\ \exp(-p^2/2 - S(U))$$

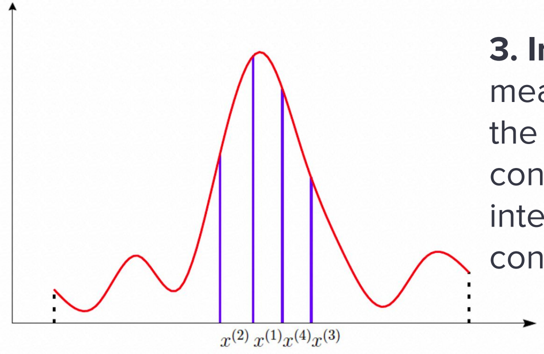
Importance Sampling



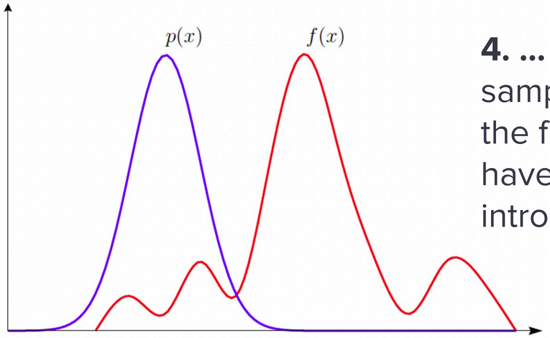
1. Uniformly sampling is effective for approximately constant functions…



2. … for more complicated functions it requires a **huge number of samples** to reach high precision and accuracy



3. Importance sampling measures more frequently the regions that give more contribution to the integral, achieving faster convergence



4. … however, when the sampling distribution and the function to integrate have little overlap, it may introduce bias!

Markov-Chain Monte Carlo

A Markov-chain is a sequence of samples $\phi^{(0)} \to \phi^{(1)} \to \cdots \to \phi^{(t)} \to \cdots\cdots \to \phi^{(N)}$ $\phi(t) \in \Omega \leftarrow \text{state space}$

whose probability of extraction is given by a transition probability $T(\phi \to \phi')$

Properties:

- **Markovian:** $T(\phi \to \phi')$ only depends on the current (ϕ) and future (ϕ') state
- **Time-homogeneous:** $T(\phi \to \phi')$ is constant along the chain, i.e. t-independent
- **Probability density:** $\int D\phi' \; T(\phi \to \phi') = 1 \quad \text{and} \quad T(\phi \to \phi') >= 0$
- **Ergodic & Irreducible:** $T(\phi \to \phi') > 0 \quad \forall \phi, \phi' \in \Omega$

A chain is completely specified by the starting distribution $P_0(\phi^{(0)})$ and the transition probability $T(\phi \to \phi')$

Equilibrium distribution and detailed balance

Theorem: Given an ergodic Markov chain with transition probability ***T***, the limit

$$\lim_{t\to\infty} P^{(t)} = \lim_{t\to\infty} T^t P^{(0)} = \Pi \in P_\Omega$$

exists, is unique, and is independent on $P^{(0)}$. Moreover, Π is the unique fixed point of the chain,

i.e. $$(TP) = P \Leftrightarrow P = \Pi$$

Detailed Balance:

A sufficient (but not necessary) condition to define ***T*** is detailed balance

$$\Pi(\phi')T(\phi' \to \phi) = \Pi(\phi)T(\phi \to \phi')$$

Proof: Integrate on both sides over ϕ $\Longrightarrow$ $\Pi(\phi') = (T\Pi)(\phi')$, which is the fixed point!

Metropolis-Hastings algorithm

A simple way to satisfy detailed balance is

$$T(\phi \to \phi') = P_C(\phi \to \phi') P_A(\phi \to \phi') \quad \text{with} \quad P_A(\phi \to \phi') = \min\left[1, \frac{\Pi(\phi') P_C(\phi' \to \phi)}{\Pi(\phi) P_C(\phi \to \phi')}\right]$$

Algorithm:

1. A candidate ϕ' is proposed from ϕ with probability P_C
2. ϕ' is accepted as the next step in the chain with probability P_A
3. If ϕ' is rejected, then ϕ remains the next element, i.e. it is repeated in the chain
4. Repeat 1.-3. for all points in the configuration (sweep) and iterate.

Note: if $P_C(\phi \to \phi') = P_C(\phi' \to \phi)$ (symmetrical proposal), then $P_A(\phi \to \phi') = \min\left[1, \frac{\Pi(\phi')}{\Pi(\phi)}\right]$

Remarks & Challenges

- Only the relative probabilities $\frac{\Pi(\phi')}{\Pi(\phi)}$ are needed to construct ***T***.

  No need to know the partition function.

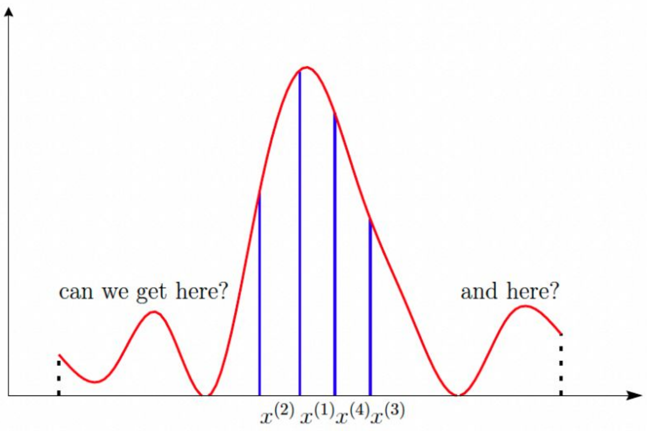
- Similarly, Markov chain can be used to compute integrals of ratios only, e.g.

$$\langle \square \rangle = \frac{\int D\phi \ \Pi(\phi) \square(\phi)}{\int D\phi \ \Pi(\phi)}$$

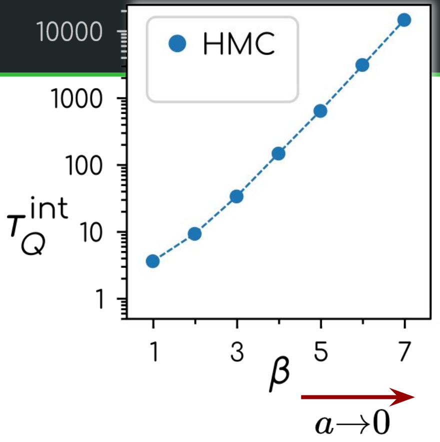
- The key challenge is to ensure ergodicity.

  A ***T*** not ergodic can lead to wrong results and biases!

- **Autocorrelation...**



Autocorrelation



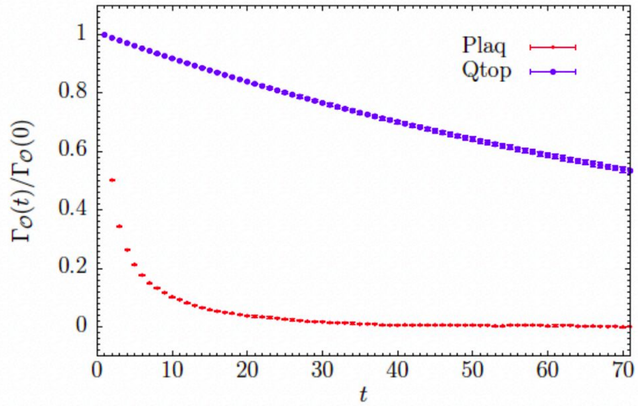
Subsequent states in a Markov chain are correlated (*autocorrelation*)

$$\sigma^2(\overline{\Box}) = \langle\langle(\overline{\Box -} \langle\Box\rangle)^2\rangle\rangle = \frac{1}{N^2}\sum_{k,l=1}^{N} \langle\langle\Box^{(k)}\Box^{(l)}\rangle\rangle - \langle\Box\rangle^2 \qquad [\langle\langle\Box^{(k)}\rangle\rangle = \langle\Box\rangle]$$

and this can be rewritten by introducing the *autocorrelation time τ*

$$\sigma^2(\overline{\Box}) = \frac{2\tau^{int,\Box}\mathrm{var}(\Box)}{N} \qquad \text{with} \quad \mathrm{var}(\Box) = \langle\Box^2\rangle - \langle\Box\rangle^2 \,,$$

$$\tau^{int,\Box} = \frac{1}{2}\left[1 + 2\sum_{t=1}^{N-1}\frac{\Gamma^{(\Box)}(t)}{\Gamma^{(\Box)}(0)}\right]$$, and $$\Gamma^{(\Box(t))} = \langle\langle\Box^{(t+i)}\Box^{(i)}\rangle\rangle - \langle\Box\rangle^2$$



Tutorial 1: Harmonic Oscillator

We now do a tutorial on Markov-chain Monte Carlo simulations of a 1-dimensional harmonic oscillator.

The continuum action of the theory is $$S[x] = \int_0^T dt \left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2\right]$$

A possible discretization is $$S[x] = \sum_{i=0}^{N-1} \left[\frac{m}{2dt}(x_{i+1} - x_i)^2 + \frac{m\omega^2 dt}{2} x_i^2\right]$$

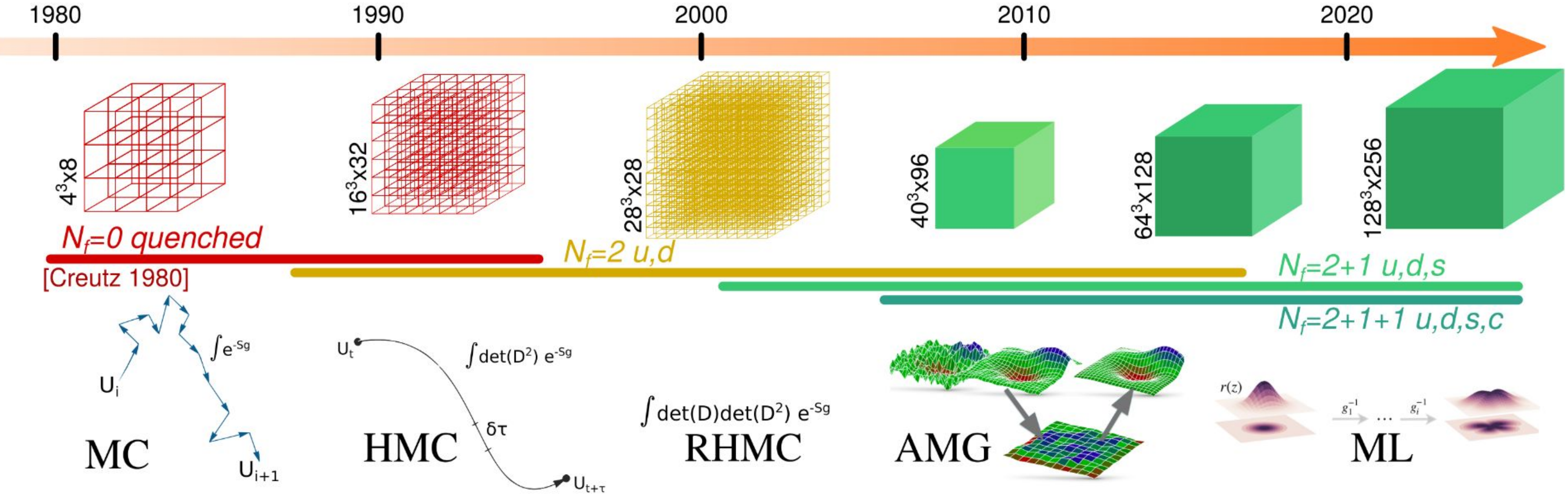
Our goal is to compute the expectation value of the correlator $C(t) = \langle x(t)x(0)\rangle$

whose continuum limit is the well known result $$C(t) = \frac{1}{2m\omega} e^{-\omega|t|}$$

For this, we will use the Jupyter Notebook you find at `http://tiny.cc/ggi-lqcd-nb`

50 year of Lattice QCD

K. G. Wilson
1974



N_f=0 quenched

[Creutz 1980]

N_f=2 u,d

N_f=2+1 u,d,s

N_f=2+1+1 u,d,s,c

Algorithms

$\int e^{-S_g}$ U_i U_{i+1}

MC

U_t $\int \det(D^2)\, e^{-S_g}$ $\delta\tau$ $U_{t+\tau}$

HMC

$\int \det(D)\det(D^2)\, e^{-S_g}$

RHMC

AMG

ML

Machines

APE100
6.4 GFlops

QCDPAX
12.5 GFlops

CP-PACS
600 GFlops

QCDOC
10 TFlops

JUQUEEN
10 PFlops

Summit
200 PFlops

LUMI
400 PFlops

Frontier
1.5 EFlops

IBM
400 Qubits

Part 2: Correlation Functions

The correlation function between two states ψ is equivalent to the matrix elements of the transport operator up to exponentially-suppressed thermal effects, due to the finite size.

$$C(t) \equiv \left\langle \psi(t)\psi^\dagger(0) \right\rangle = \left\langle \psi \middle| \underbrace{T^t}_{\hookrightarrow\, e^{-H}} \middle| \psi \right\rangle + \dots$$

By inserting a complete set of eigenstates of the transfer matrix $\sum_n |n\rangle \langle n|$, we obtain

$$C(t) = \sum_n \underbrace{\lambda_n^t}_{\langle n|\, T \,|n\rangle} \langle \psi|n\rangle \langle n|\psi\rangle \underset{\substack{\uparrow \\ T = e^{-H}}}{=} \sum_{n=0}^{\infty} |\underbrace{Z_n}_{\langle n|\psi\rangle}|^2 e^{-E_n t} \longrightarrow$$

Exponential suppression!

Correlation Functions

The correlation function between two states ψ is equivalent to the matrix elements of the

transport o

size.

In Lattice QCD we can easily extract ground-state matrix elements:

$$\lim_{t\to\infty} C(t) = |Z_0|^2 \; e^{-E_0 t}$$

By inserting a complete set of eigenstates of the transfer matrix $\sum_n |n\rangle\langle n|$, we obtain

$$C(t) = \sum_n \underbrace{\lambda_n^t}_{\langle n|T|n\rangle} \langle\psi|n\rangle\langle n|\psi\rangle \underset{T=e^{-H}}{=} \sum_{n=0}^{\infty} |\underbrace{Z_n}_{\langle n|\psi\rangle}|^2 e^{-E_n t} \longrightarrow$$

Exponential suppression!

Interpolating Fields: How to measure properties of hadrons?

Our goal is then to construct an interpolating field ψ, whose ground state is the desired hadron.

$\langle n|\psi\rangle = 0$, if wrong quantum numbers

To achieve this we can exploit quantum numbers and other properties:

- **Flavor structure:** Use the correct combination of quark fields to represent the desired flavor quantum numbers.
- **Spin and parity selection:** Choose the appropriate Dirac structure (Γ) to match the desired spin J and parity P.
- **Other symmetries:** Certain hadrons are e.g. *even under exchange of u↔d*. Implement these symmetries correctly.
- **Momentum projection:** Sum the interpolating operator over spatial points with a phase factor to project onto definite momentum states.

Additionally the interpolating field should preserve:

- **Gauge invariance:** Ensure the interpolating operator is gauge-invariant, using color indices and Wilson lines (if necessary).
- **Symmetry under the cubic group:** Since the lattice breaks continuous rotational symmetry, operators must transform according to irreps of the cubic group.

Meson Interpolating Fields

For **mesons**, the simplest interpolating operator is a bilinear quark-antiquark field of the form:

$$\psi_{\text{meson}}(x) = \sum_{a,\alpha,\beta} \bar{q}_1(x)^a_\alpha \, \Gamma_{\alpha,\beta} \, q_2(x)^a_\beta$$

- **Flavor structure:** The specific quarks q_1 and q_2 determine the flavor quantum numbers of the meson.
- **Spin and parity selection:** The choice of Γ determines the spin and parity of the meson. For example:
  - $\Gamma = \gamma_5$ gives a pseudoscalar meson (like the pion), with $J^P = 0^-$.
  - $\Gamma = \gamma_\mu$ gives a vector meson (like the ρ-meson), with $J^P = 1^-$.
  - $\Gamma = 1$ gives a scalar meson, with $J^P = 0^+$.
  - $\Gamma = \gamma_5\gamma_\mu$ gives an axial-vector meson (like the a_1-meson), with $J^P = 1^+$.
- **Gauge invariance:** The quark fields are in the same location *x* and color indices are traced.

- **Momentum projection:**

$$\psi(\vec{p}, t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \psi(\vec{x}, t)$$

Baryon Interpolating Fields

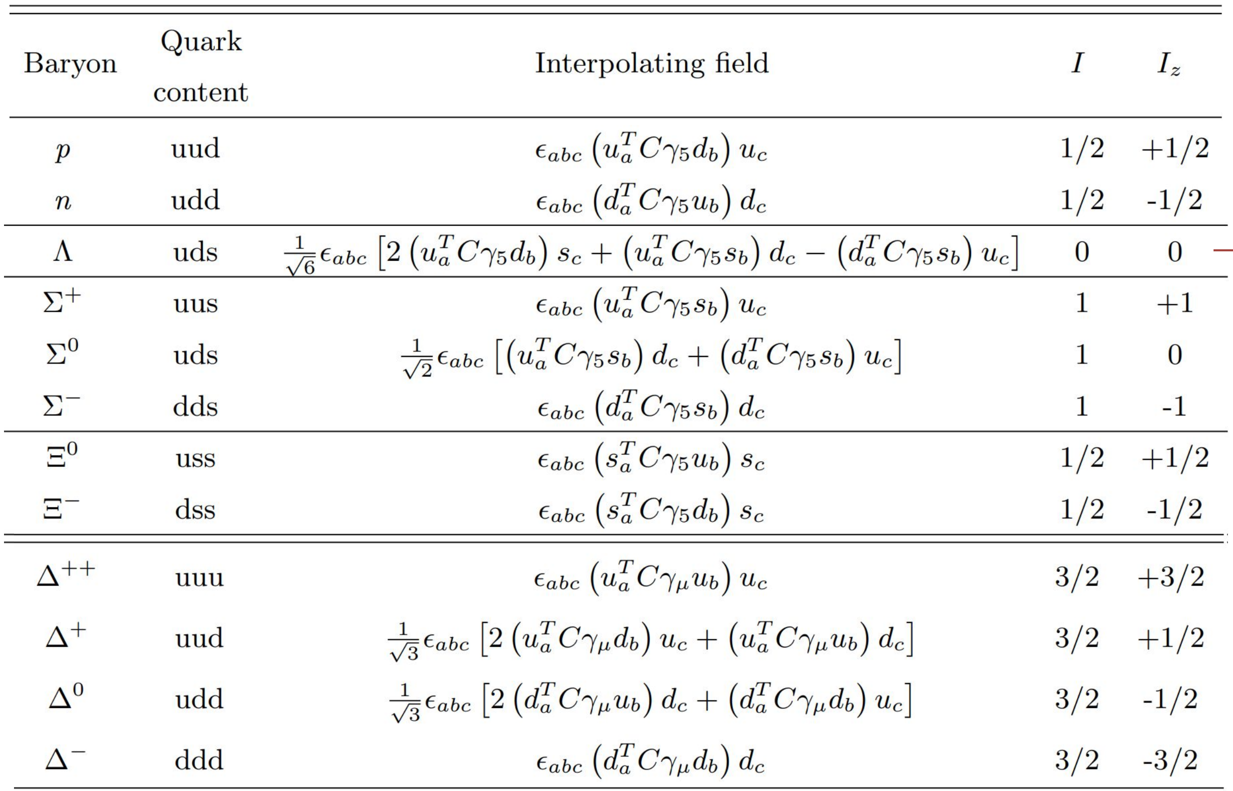
For **baryons**, there are various options to construct the interpolating field

$$\psi_{\text{baryon}}(x) = \epsilon_{abc}\, P_\pm\, \Gamma_A\, q_1(x)^a \left({q_2}^T(x)^b \Gamma_B\, q_3(x)^c\right)$$

This gives more precise results

- **Spin selection:**
  - $J = {}^1\!/_2 \rightarrow \;\; (\Gamma^A, \Gamma^B) = (\mathbb{1}, C\gamma_5),\ (\gamma_5, C),\ \text{or}\ (\mathbb{1}, \mathrm{i}\gamma_4 C\gamma_5)$
  - $J = {}^3\!/_2 \rightarrow \;\; (\Gamma^A, \Gamma^B) = (\mathbb{1}, C\gamma_j)$
- **Parity selection:** positive and negative parity selected using $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_4)$
- **Other symmetries:**
  - C is the charge conjugation matrix, ensuring the antisymmetry
  - ϵ_{abc} ensures antisymmetry in the color indices (so that the wavefunction is antisymmetric overall, respecting the Pauli exclusion principle).

Examples of Baryon Interpolating Fields



Baryon	Quark content	Interpolating field	I	I_z
p	uud	$\epsilon_{abc}\left(u_a^T C\gamma_5 d_b\right)u_c$	1/2	+1/2
n	udd	$\epsilon_{abc}\left(d_a^T C\gamma_5 u_b\right)d_c$	1/2	-1/2
Λ	uds	$\frac{1}{\sqrt{6}}\epsilon_{abc}\left[2\left(u_a^T C\gamma_5 d_b\right)s_c + \left(u_a^T C\gamma_5 s_b\right)d_c - \left(d_a^T C\gamma_5 s_b\right)u_c\right]$	0	0
Σ^+	uus	$\epsilon_{abc}\left(u_a^T C\gamma_5 s_b\right)u_c$	1	+1
Σ^0	uds	$\frac{1}{\sqrt{2}}\epsilon_{abc}\left[\left(u_a^T C\gamma_5 s_b\right)d_c + \left(d_a^T C\gamma_5 s_b\right)u_c\right]$	1	0
Σ^-	dds	$\epsilon_{abc}\left(d_a^T C\gamma_5 s_b\right)d_c$	1	-1
Ξ^0	uss	$\epsilon_{abc}\left(s_a^T C\gamma_5 u_b\right)s_c$	1/2	+1/2
Ξ^-	dss	$\epsilon_{abc}\left(s_a^T C\gamma_5 d_b\right)s_c$	1/2	-1/2
Δ^{++}	uuu	$\epsilon_{abc}\left(u_a^T C\gamma_\mu u_b\right)u_c$	3/2	+3/2
Δ^+	uud	$\frac{1}{\sqrt{3}}\epsilon_{abc}\left[2\left(u_a^T C\gamma_\mu d_b\right)u_c + \left(u_a^T C\gamma_\mu u_b\right)d_c\right]$	3/2	+1/2
Δ^0	udd	$\frac{1}{\sqrt{3}}\epsilon_{abc}\left[2\left(d_a^T C\gamma_\mu u_b\right)d_c + \left(d_a^T C\gamma_\mu d_b\right)u_c\right]$	3/2	-1/2
Δ^-	ddd	$\epsilon_{abc}\left(d_a^T C\gamma_\mu d_b\right)d_c$	3/2	-3/2

→ Odd under exchange of u↔d

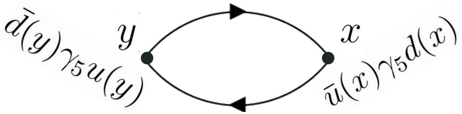
Even under exchange of u↔d

Example of spin 3/2

Wick Contractions

Now we need to construct correlation functions $C(t) \equiv \left\langle \psi(t)\psi^\dagger(0) \right\rangle$ of interpolating fields

We do this via Wick contractions. Example for the pion:



$$\langle \psi_{\pi^+}(y)\psi^\dagger_{\pi^+}(x)\rangle = \langle \bar{d}(y)\gamma_5 u(y)\bar{u}(x)\gamma_5 d(x)\rangle =$$

$$= \quad (\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'}\langle \bar{d}(y)^a_\alpha u(y)^a_\beta \bar{u}(x)^b_{\alpha'} d(x)^b_{\beta'}\rangle$$

$$= \quad -(\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'}\langle d(x)^b_{\beta'}\bar{d}(y)^a_\alpha\rangle\langle u(y)^a_\beta \bar{u}(x)^b_{\alpha'}\rangle$$

$$= \quad -(\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'} G_d(x;y)^{ba}_{\beta'\alpha} G_u(y;x)^{ab}_{\beta\alpha'}$$

$$= \quad -\mathrm{Tr}\left[\gamma_5 G_d(x;y)\gamma_5 G_u(y;x)\right]$$

Quark propagator

Steps in Wick contractions:

1. Explicitly write position, spin and color indices.
2. Construct all possible pairs of quark-antiquark and multiply by -1 for every commutation of quarks.
3. Write in terms of propagators and recombine indices, if possible.

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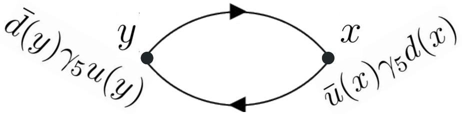

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$$= \quad -\mathrm{Tr}\Big[\gamma_5 G_d(x;y)\gamma_5 G_u(y;x)\Big] \quad \text{Quark propagator}$$

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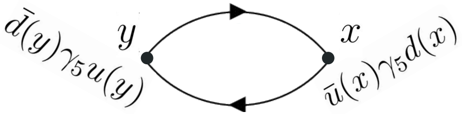

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$$= -(\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'}\langle d(x)^b_{\beta'}\bar{d}(y)^a_\alpha\rangle\langle u(y)^a_\beta \bar{u}(x)^b_{\alpha'}\rangle$$

$$= -(\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'} G_d(x;y)^{ba}_{\beta'\alpha} G_u(y;x)^{ab}_{\beta\alpha'}$$

$$= -\mathrm{Tr}\Big[\gamma_5 G_d(x;y)\gamma_5 G_u(y;x)\Big]$$

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Homework: Nucleon two-point functions

Now, follow the same example for the pion two-point function and compute the Wick contractions for the nucleon two-point functions.

The starting point is

$$\begin{aligned}\left\langle O_{N_\pm}(n)_\alpha \, \overline{O}_{N_\pm}(m)_\alpha \right\rangle \quad &= - \left\langle \overline{O}_{N_\pm}(m)_\alpha \, O_{N_\pm}(n)_\alpha \right\rangle_F \\ = - \Big\langle \epsilon_{abc} \, \epsilon_{a'b'c'} \; \left(\overline{u}(m)_a \, C\gamma_5 \, \overline{d}(m)_b^T \right) \, \overline{u}(m)_c \, P_\pm \, u(n)_{c'} \; & \left(u(n)_{a'}^T \, C\gamma_5 \, d(n)_{b'} \right) \Big\rangle\end{aligned}$$

The final result you should arrive to is

$$\boxed{D^{-1} = G}$$

$$\begin{aligned}= \epsilon_{abc} \, \epsilon_{a'b'c'} \, (C\gamma_5)_{\alpha'\beta'} \, (C\gamma_5)_{\alpha\beta} \, (P_\pm)_{\gamma\gamma'} \, D_d^{-1}(n|m)_{\substack{\beta'\beta \\ b'b}} \times \\ \left(D_u^{-1}(n|m)_{\substack{\alpha'\alpha \\ a'a}} \, D_u^{-1}(n|m)_{\substack{\gamma'\gamma \\ c'c}} - D_u^{-1}(n|m)_{\substack{\alpha'\gamma \\ a'c}} \, D_u^{-1}(n|m)_{\substack{\gamma'\alpha \\ c'a}} \right)\end{aligned}$$

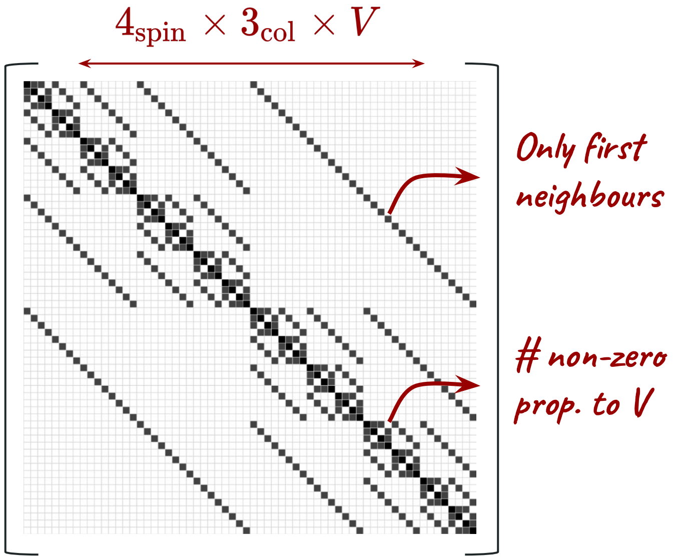
Part 3: Fermions on the Lattice

$$\underbrace{\langle \psi_j \bar{\psi}_i \rangle}_{G} = \frac{1}{Z}\int \mathcal{D}[U]\mathcal{D}[\psi\bar{\psi}]\, \psi_j\bar{\psi}_i\, e^{-S_{YM}(U)-\bar{\psi}D\psi}$$

$$= \frac{1}{Z}\int \mathcal{D}[U]\, D^{-1}_{ji}\, \det D\, e^{-S_{YM}(U)}$$

with $D \equiv D(U,\mu_f) \xrightarrow[a \to 0]{} (i\gamma^\mu D_\mu - m_f)$

- Many possible discretizations of the Dirac operator
    - *Wilson, **Twisted-Mass**, Domain-Wall, Overlap, etc.*
- The Dirac operator is a **sparse matrix** (first-neighbors only)
    - *Krylov methods are used to solve:* $D\vec{x} = \vec{b}$
    - e.g. $D\vec{x} = \vec{e}_i \implies x_j = D^{-1}_{ji}$
    - e.g. $D\vec{x} = \vec{\eta} \implies \vec{\eta}^{\dagger}\vec{x} \approx \mathrm{Tr}(D^{-1})$



Fermions on the Lattice

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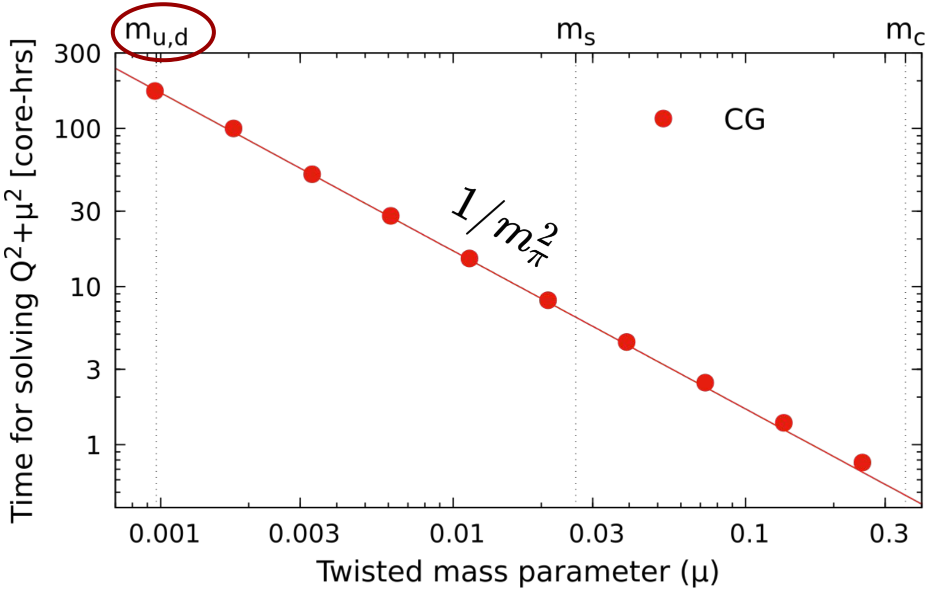
- Dirac operator is singular at *μ=0*
  - *Critical slowing down, e.g.*
  - *Light quark masses 100x more expensive than strange mass*
  - *Common to all discretizations*

~ **Simulations at larger pion mass**



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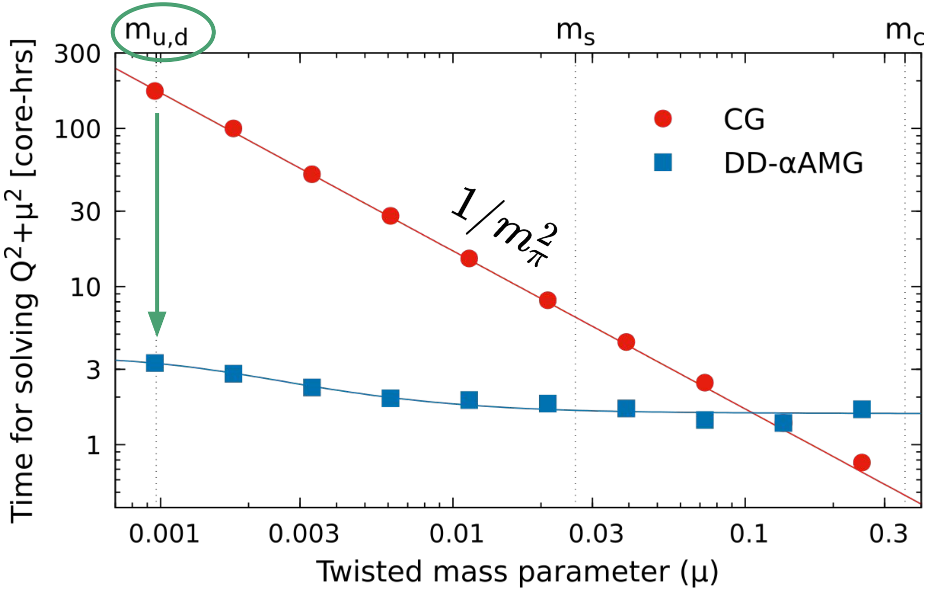
  - *Common to all discretizations*

✓ **Resolved by multigrid methods**



Stochastic propagators

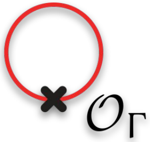
$$D\vec{x} = \vec{\eta} \implies \vec{\eta}^\dagger \vec{x} \approx \mathbf{Tr}(D^{-1})$$

- Hutchinson Trace Estimator:

Let $A \in \mathbb{C}^{D\times D}$ and $v \in \mathbb{C}^D$ be a random vector such that

$$\langle vv^\dagger\rangle = I \quad \Rightarrow \quad \mathrm{Tr}(A) = \langle v^\dagger A v\rangle$$

- Example 1: Quark Loop



$$\Gamma_{\mu\nu}\langle\psi_\nu(x)\bar\psi_\mu(x)\rangle = \langle\mathrm{Tr}(\Gamma D^{-1})\rangle_U = \langle\eta^\dagger\Gamma D^{-1}\eta\rangle_{U,\eta}$$
$$= \langle\eta^\dagger\Gamma x\rangle_{U,\eta} \quad \text{with} \quad Dx = \eta$$

  - Usually time-slice stochastic sources for allowing correlation in time

- Example 2: Pion two-point functions

$$\langle\psi_{\pi^+}(y)\psi^\dagger_{\pi^+}(x)\rangle = \langle\bar d(y)\gamma_5 u(y)\bar u(x)\gamma_5 d(x)\rangle =$$
$$= -\mathrm{Tr}\left[\gamma_5 G_d(x;y)\gamma_5 G_u(y;x)\right]$$

  - γ_5 - hermiticity: $\gamma_5 D \gamma_5 = D^\dagger$ (holds also for the inverse)

$$= -\mathrm{Tr}\left[G_u^\dagger(y;x)G_u(y;x)\right]$$

  - and using time-slice stochastic source in *t="0"*:

$$C(t) = \langle\psi_{\pi^+}(t)\psi_{\pi^+}(0)\rangle = \langle x^\dagger(t)x(t)\rangle_{U,\eta(0)}$$

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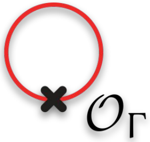

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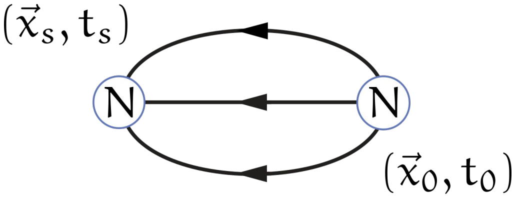
Point-to-all propagators

$$D\vec{x} = \vec{e}_i \implies x_j = D^{-1}_{ji}$$

Or we can just propagate from a single point to all.

- **Disadvantage:** compared to stochastic sources we do not exploit volume-average over the spatial volume, but use only one point.
- **Advantage:** any momentum can be inserted at the source!



$$\langle N(\vec{p}\,', t_s) N(\vec{p}, t_0)\rangle = \sum_{x_0, x_s} e^{i\vec{p}\,'\vec{x}_s} e^{i\vec{p}\vec{x}_0} \langle N(\vec{x}_s, t_s) N(\vec{x}_0, t_0)\rangle$$

if $N(\vec{x}_0, t_0)$ only defined in $\vec{x}_0$

$$= e^{i\vec{p}\vec{x}_0} \sum_{x_s} e^{i\vec{p}\,'\vec{x}_s} \langle N(\vec{x}_s, t_s) N(\vec{x}_0, t_0)\rangle$$

$$= e^{i\vec{p}\vec{x}_0} \langle N(\vec{p}\,', t_s) N(\vec{0}, t_0)\rangle$$

Note on stochastic sources:

while for mesons we use:

$$\langle \xi(\vec{x})\xi^\dagger(\vec{y})\rangle = \delta_{\vec{x},\vec{y}}$$

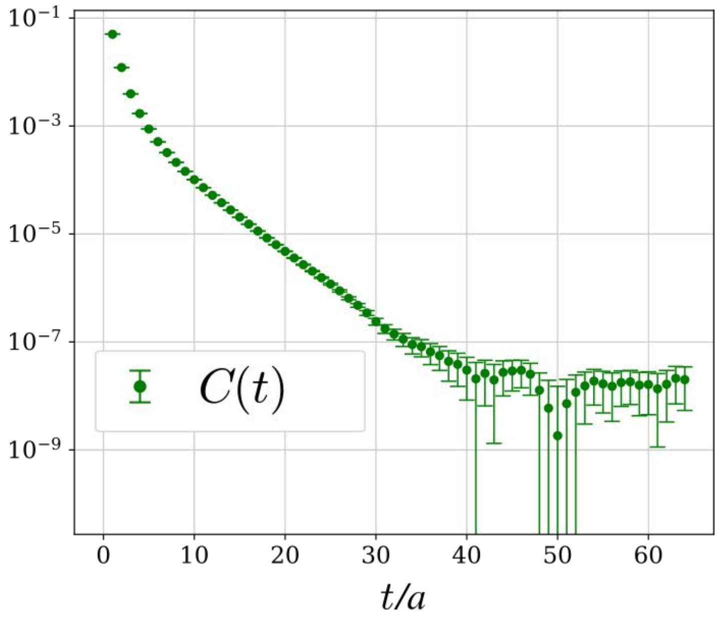
for baryons we should use:

$$\langle \xi(\vec{y})\xi(\vec{y}\,')\xi(\vec{y}\,'')\rangle = \delta_{\vec{y},\vec{y}\,'}\delta_{\vec{y}\,',\vec{y}\,''}$$

Two-point functions

Let's consider now the case of hadron two-point functions

$$C(t) \equiv \left\langle \psi(t)\psi^\dagger(0) \right\rangle = \sum_{n=0}^{\infty} \underbrace{|Z_n|^2}_{\langle n|\psi\rangle} e^{-E_n t}$$



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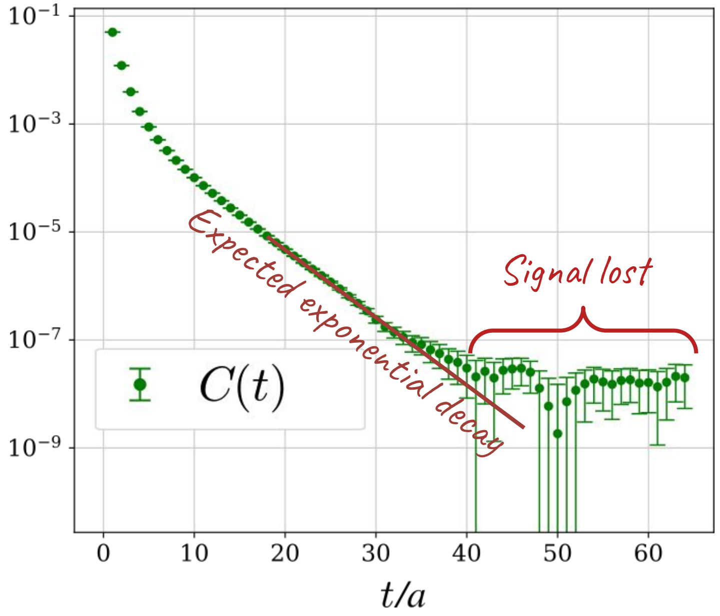
$Z_n \to \langle n|\psi \rangle$

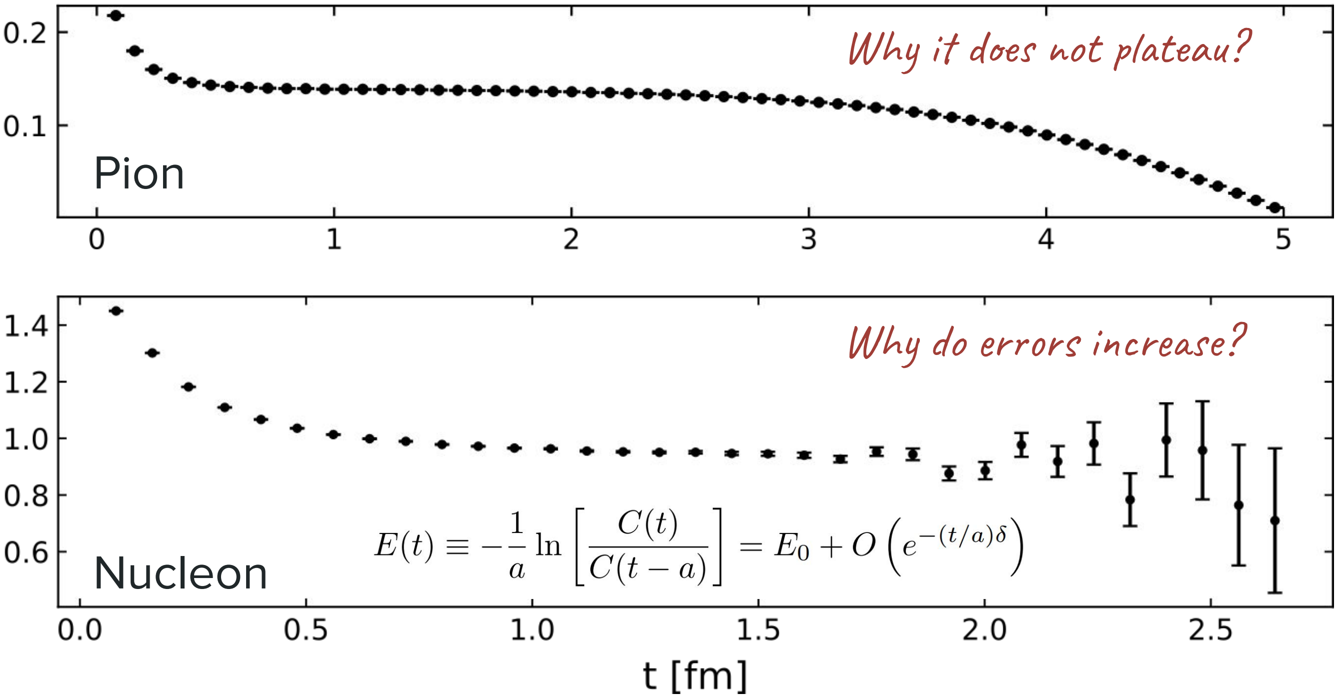
A main use of hadron two-point functions is to extract the energy spectrum of interpolating fields

$$E(t) \equiv -\frac{1}{a} \ln\left[\frac{C(t)}{C(t-a)}\right] = E_0 + O\left(e^{-(t/a)\delta}\right)$$

$E(t) \to$ Effective mass



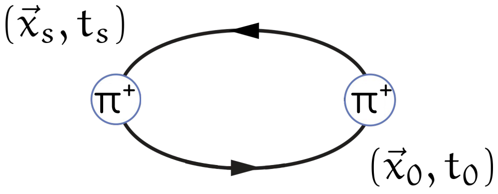
Pion and Nucleon effective masses

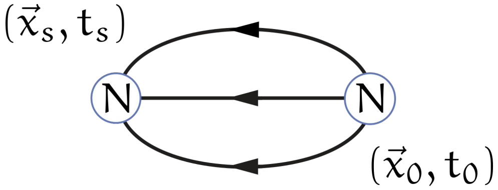


Why it does not plateau?

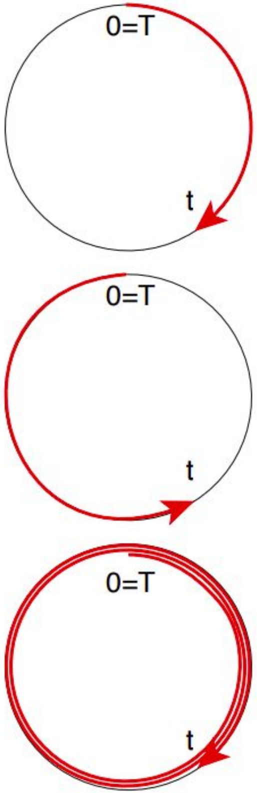
Why do errors increase?

$$E(t) \equiv -\frac{1}{a} \ln \left[\frac{C(t)}{C(t-a)} \right] = E_0 + O\left(e^{-(t/a)\delta} \right)$$





The effects of periodicity



The correlator propagating forward is $c_f(t) = c_f^0 e^{-mt}$

Enough for large enough masses

The correlator propagating backward is $c_b(t) = c_b^0 e^{-m(T-t)}$

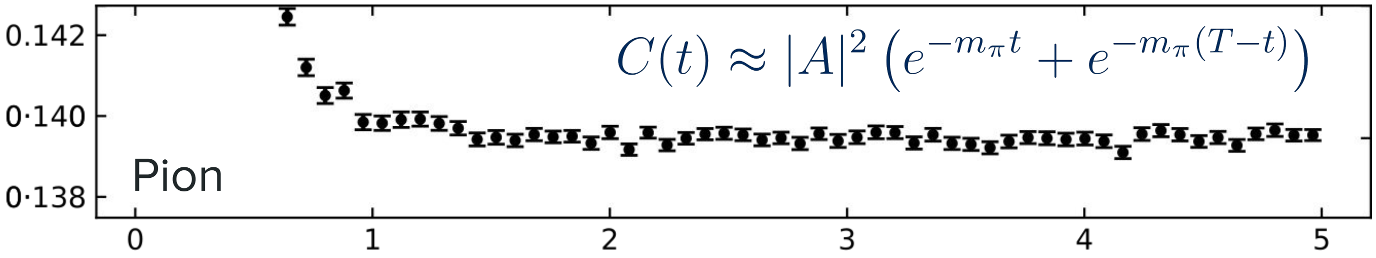
And accounting for all contributions we get

$$c_f(t) = c_f^0 \left(e^{-mt} + e^{-m(T+t)} + \dots \right)$$
$$= c_f^0 e^{-mt} \times \sum_{n=0}^{\infty} e^{-nmT}$$
$$= c_f^0 e^{-mt} \frac{1}{1 - e^{-mT}}$$

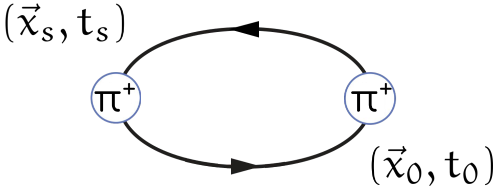
Enough for large enough time extent T

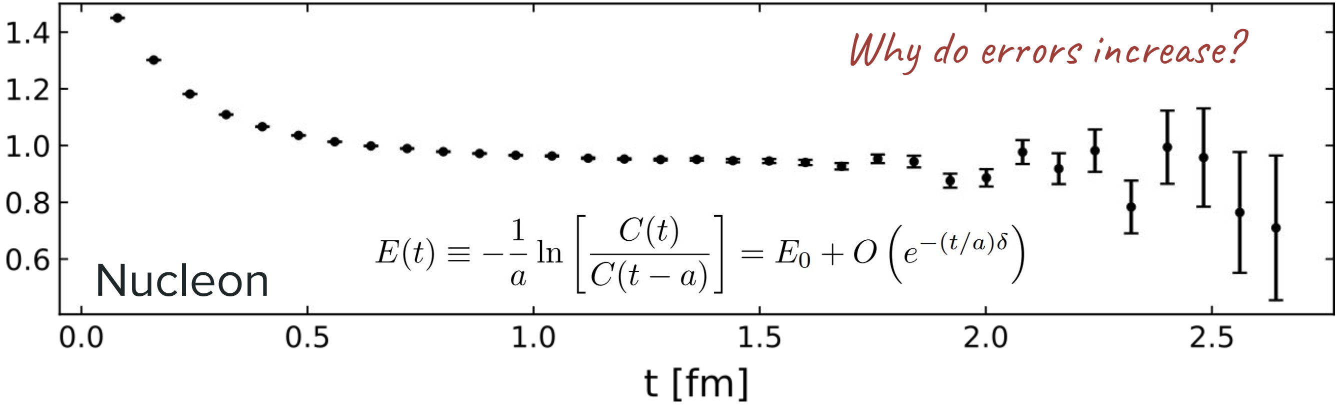
Required for finite temperature simulations

Pion and Nucleon effective masses



$$C(t) \approx |A|^2 \left(e^{-m_\pi t} + e^{-m_\pi (T-t)}\right)$$





Why do errors increase?

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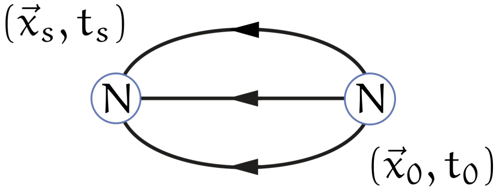

Exponential growth of the error

Why do errors grow exponentially for the nucleon and not for the pion?

- **Parisi-Lapage argument:**

$$\boxed{\mathrm{Var}(X) = E[X^2] - E[X]^2}$$

$$C(t) = \langle \psi^\dagger(t)\psi(0)\rangle$$

$$\sigma_C^2(t) = \langle \psi^\dagger(0)\psi(t)\psi^\dagger(t)\psi(0)\rangle - \langle \psi^\dagger(t)\psi(0)\rangle^2$$

$$\frac{\sigma_C(t)}{C(t)} = \sqrt{\frac{\langle \psi^\dagger(0)\psi(t)\psi^\dagger(t)\psi(0)\rangle}{\langle \psi^\dagger(t)\psi(0)\rangle^2} - 1}$$

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at large t

$$\approx \sqrt{\frac{|B|^2 e^{-tE_{0,C^2}}}{|A|^4 e^{-2tE_{0,C}}} - 1}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$\frac{\sigma_C(t)}{C(t)} \propto e^{t(E_{0,C} - \frac{1}{2}E_{0,C^2})}$$

Relative error [in most cases] increases exponentially in t since

$$E_{0,C^2} \leq 2E_{0,C}$$

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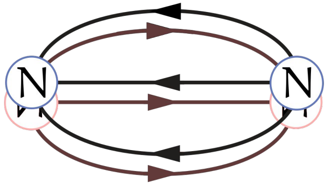
- **Pion:** $\frac{\sigma_C(t)}{C(t)} \propto e^{t\left(m_\pi - \frac{2}{2}m_\pi\right)} = \underline{\mathrm{const.}}$

- **Nucleon:** $\frac{\sigma_C(t)}{C(t)} \propto e^{t\left(m_N - \frac{3}{2}m_\pi\right)}$



Same quantum numbers of 3 pions!

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Relative error [in most cases] increases exponentially in t since

$$E_{0,C^2} \leq 2E_{0,C}$$

Noise-reduction techniques

How to tackle the exponential-growth of the noise?

by developing noise-reduction techniques!

There is a vast literature on this techniques for the signal-to-noise problem, including many experimental / exploratory / innovative approaches.

Here we will focus on two production-ready techniques:

- Smearing of interpolating fields
- Low-mode averaging (LMA)

See e.g.

"Path integral contour deformations"

[arXiv:2101.12668]

[arXiv:2304.03229]

1st technique: Smearing of interpolating fields

The interpolating fields we have considered are all local, but extended ones would still have the same quantum numbers, e.g.

Added a link to preserve gauge invariance

$$\psi_{\text{meson}}(x) = \sum_{a,b,\alpha,\beta} \bar{q}_1\,(x)^a_\alpha\,\Gamma_{\alpha,\beta}\,U^{ab}_\mu(x)\,q_2\,(x+\mu)^b_\beta$$

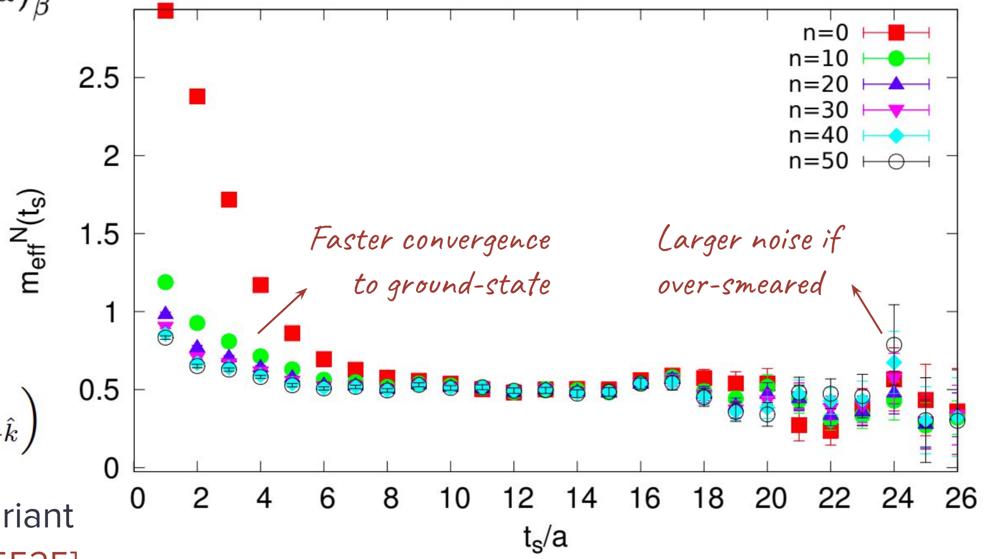
Wuppertal / Gaussian smearing

Parameters

$$q^{sm}(\vec{x},t) = \sum_{\vec{y}} \Big(\mathbb{1} + \alpha H(\vec{x},\vec{y};U(t))\Big)^n q(\vec{y},t)$$

$$H(\vec{x},\vec{y};U(t)) = \sum_{k=1}^{3}\Big(U_k(\vec{x},t)\delta_{\vec{x},\vec{y}-\hat{k}} + U_k^\dagger(\vec{x}-\hat{k},t)\delta_{\vec{x},\vec{y}+\hat{k}}\Big)$$

In case of boost, momentum smearing is a better variant [arXiv:1602.05525]



Faster convergence to ground-state

Larger noise if over-smeared

2^{nd} technique: Low-Mode Averaging (LMA)

Concept: Low-modes dominate at large distance in correlation functions!

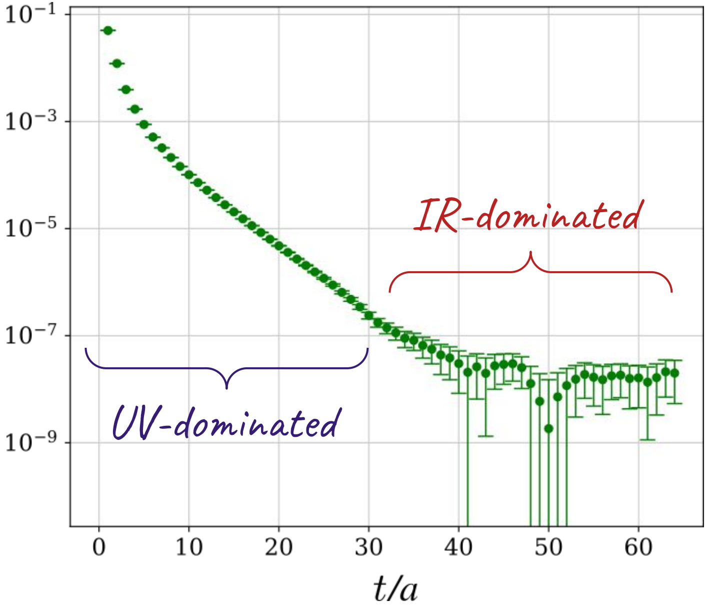
Quark Propagator

$$S_r(x,y) = \overbrace{|P_{\mathrm{IR}} Q_r^{-1} P_{\mathrm{IR}} \eta(x)\rangle \langle \eta(y)| \gamma_5}^{\text{IR contribution}} + \overbrace{|P_{\mathrm{UV}} Q_r^{-1} P_{\mathrm{UV}} \eta(x)\rangle \langle \eta(y)| \gamma_5}^{\text{UV contribution}}$$

$$= \underbrace{\sum_{j=1}^{K} \frac{|v_j(x)\rangle \langle v_j(y)| \gamma_5}{\lambda_j + ir\mu}}_{\text{Computed Exactly}} + \underbrace{\frac{1}{N} \sum_{\eta} |\widetilde{\phi}_r^{\eta}(x)\rangle \langle \eta(y)| \gamma_5 \Big|_{N \gg 1}}_{\text{Computed Stochastically}},$$

- In the correlation function computed stochastically, we replace the IR part with an **exact** knowledge of it

$$C_{\mathrm{LMA}}(t) = C_{\mathrm{stoch.}}(t) - C^{\mathrm{IR}}_{\mathrm{stoch.}}(t) + C^{\mathrm{IR}}_{\mathrm{exact.}}(t)$$



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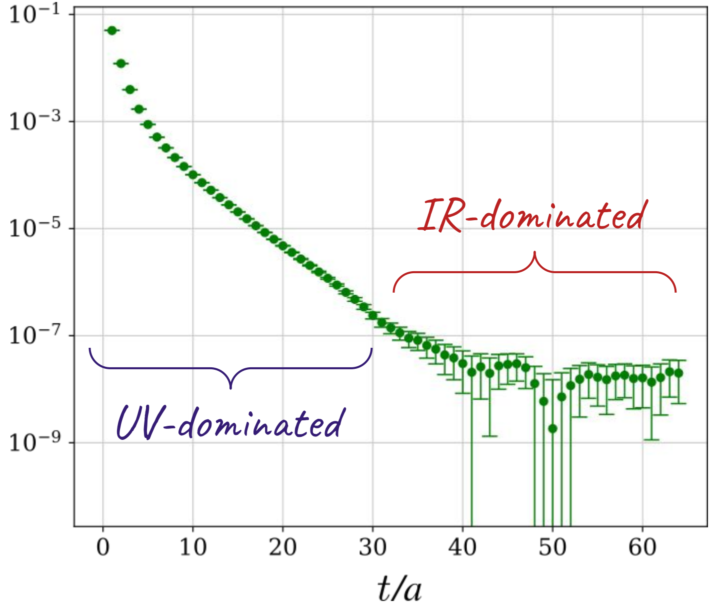

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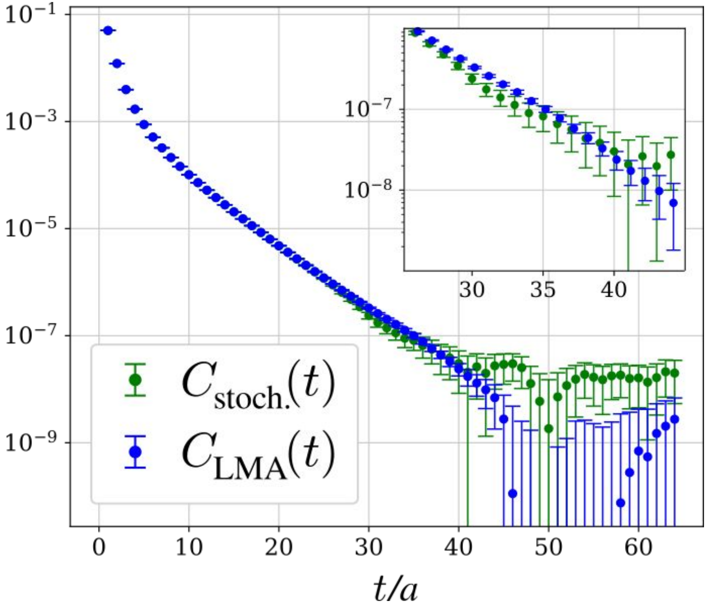
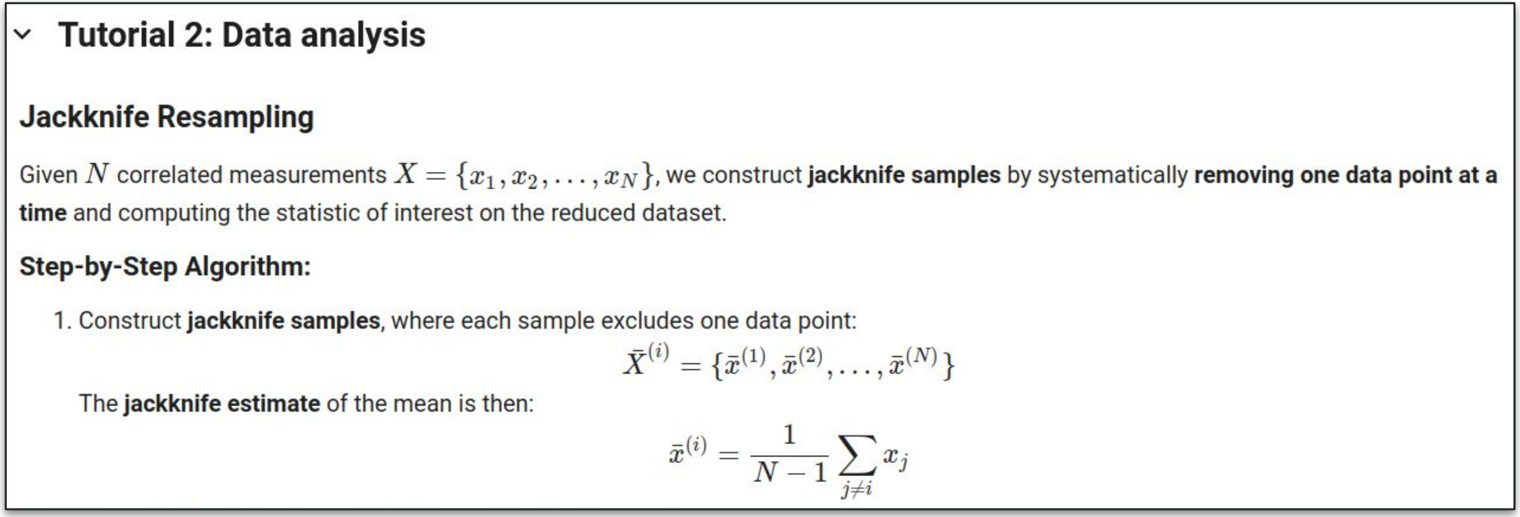


Tutorial 2: Pion and Nucleon effective masses

Now we will work on Part 2 of the Jupyter Notebook: `http://tiny.cc/ggi-lqcd-nb`

But first, let's see how we compute and propagate errors via **Jackknife resampling.**



Tutorial 2: Data analysis

Jackknife Resampling

Given N correlated measurements $X = \{x_1, x_2, \ldots, x_N\}$, we construct **jackknife samples** by systematically **removing one data point at a time** and computing the statistic of interest on the reduced dataset.

Step-by-Step Algorithm:

1. Construct **jackknife samples**, where each sample excludes one data point:

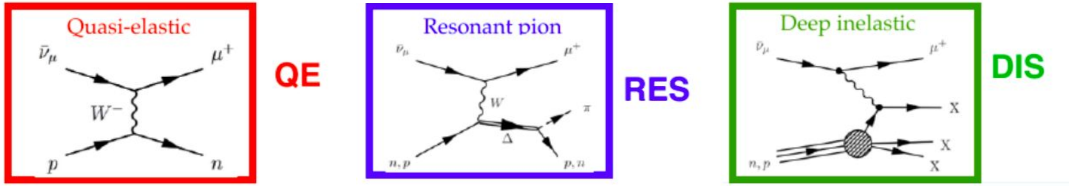
$$\bar{X}^{(i)} = \{\bar{x}^{(1)}, \bar{x}^{(2)}, \ldots, \bar{x}^{(N)}\}$$

The **jackknife estimate** of the mean is then:

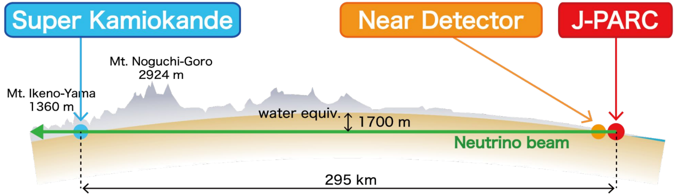
$$\bar{x}^{(i)} = \frac{1}{N-1} \sum_{j \neq i} x_j$$

Part 4: E.g. inputs to neutrino oscillation experiments

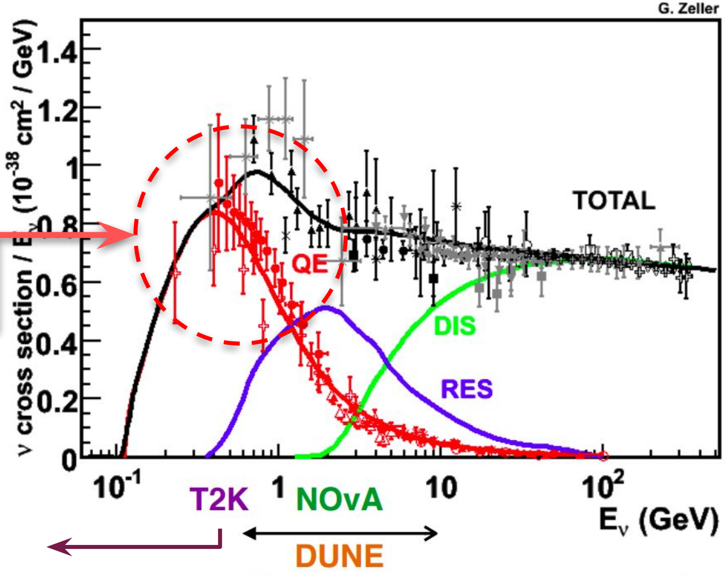
Monte-Carlo simulation needs input on the differential cross section to reconstruct the energy of the neutrino from the momentum of the detected charged lepton.



We focus on quasi-elastic scattering providing first-principle predictions on axial form factors



T2K: Tokai to Super-Kamiokande
$E = 0.6$ GeV, $L/E \simeq 500$ km/GeV.



J.A. Formaggio, G. Zeller, Reviews of Modern Physics, 84 (2012)

The weak axial-vector matrix element

The transition matrix element of the neutron β-decay is

$$\mathcal{M}(n \to p\, e^-\, \bar{\nu}_e) = \frac{G_F}{\sqrt{2}}\, V_{ud}\, \sum_\mu \underbrace{\langle p(p')|W_\mu|n(p)\rangle}\, L_\mu$$

with

$$W_\mu = V_\mu - A_\mu$$

$$V_\mu = \bar{u}\gamma_\mu d \longrightarrow$$

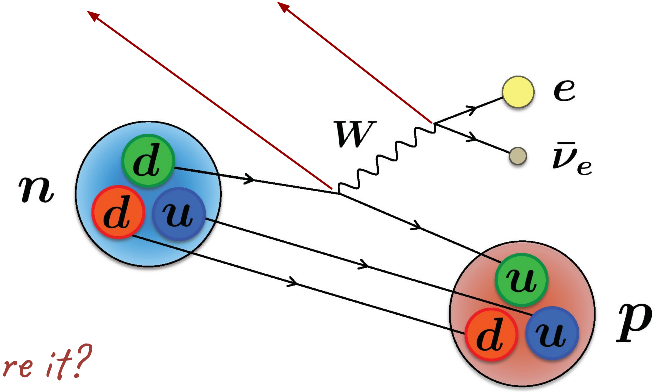
Vector contributions are well determined experimentally from lepton-nucleon scattering

$$\boxed{A_\mu = \bar{u}\gamma_\mu\gamma_5 d}$$

Axial-vector matrix element

$$\longrightarrow \langle p(p')|A_\mu|n(p)\rangle$$

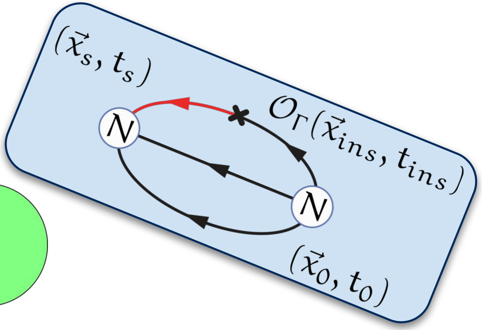
How to measure it?



Neutrino-nucleon scattering processes are related to matrix elements at finite momentum transfer.

Nucleon matrix elements

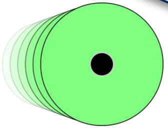
Nucleon matrix elements relate to moments of PDFs



Unpolarized
=
Vector struct.

$$\mathcal{O}_{V}^{\mu\mu_1\mu_2\ldots\mu_n} = \bar{\psi}\gamma^{\{\mu}iD^{\mu_1}iD^{\mu_2}\ldots iD^{\mu_n\}}\psi$$

$$\langle 1\rangle_{u-d} = g_V\,, \quad \langle x\rangle_{u-d}, \quad \ldots$$



Helicity
=
Axial struct.

$$\mathcal{O}_{A}^{\mu\mu_1\mu_2\ldots\mu_n} = \bar{\psi}\gamma_5\gamma^{\{\mu}iD^{\mu_1}iD^{\mu_2}\ldots iD^{\mu_n\}}\psi$$

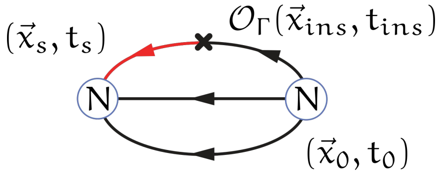
$$\langle 1\rangle_{\Delta u-\Delta d} = g_A, \quad \langle x\rangle_{\Delta u-\Delta d}, \quad \ldots$$

Transverse
=
Tensor struct.

$$\mathcal{O}_{T}^{\nu\mu\mu_1\mu_2\ldots\mu_n} = \bar{\psi}\sigma^{\nu\{\mu}iD^{\mu_1}iD^{\mu_2}\ldots iD^{\mu_n\}}\psi$$

$$\langle 1\rangle_{\delta u-\delta d} = g_T, \quad \langle x\rangle_{\delta u-\delta d}, \quad \ldots$$

From three-point functions to matrix elements



Three-point function *Matrix element*

How to go from $\langle \psi_N(\vec{p}', t_s) | \mathcal{O}(t_{ins}) | \psi_N(\vec{p}, 0)\rangle$ to $\langle N(\vec{p}') | \mathcal{O} | N(\vec{p})\rangle$?

We focus on the ground-state only:

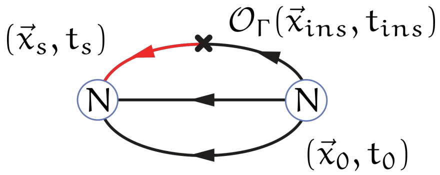
$$C_{3pt}(\vec{p}', \vec{p}; t_s, t_{ins}) \approx \langle \psi_N(\vec{p}', t_s) \overbrace{|N(\vec{p}', t_s)\rangle e^{-E_N(p')(t_s - t_{ins})} \langle N(\vec{p}', t_{ins})|} \mathcal{O}(t_{ins}) \overbrace{|N(\vec{p}, t_{ins})\rangle e^{-E_N(p)\, t_{ins}} \langle N(\vec{p}, 0)|} \psi_N(\vec{p}, 0)\rangle$$

$$= \langle \psi_N(\vec{p}') | N(\vec{p}')\rangle \langle N(\vec{p}') | \mathcal{O} | N(\vec{p})\rangle \langle N(\vec{p}) | \psi_N(\vec{p})\rangle \; e^{-E_N(p')(t_s - t_{ins}) - E_N(p)\, t_{ins}}$$

$$C_{2pt}(\vec{p}; t_s) \approx \langle \psi_N(\vec{p}) | N(\vec{p})\rangle \langle N(\vec{p}) | \psi_N(\vec{p})\rangle \; e^{-E_N(p)\, t_s}$$

$$\langle N(\vec{p}') | \mathcal{O} | N(\vec{p})\rangle = \lim_{\substack{t_s - t_{ins} \to \infty \\ t_{ins} \to \infty}} \underbrace{\frac{C_{3pt}(\vec{p}', \vec{p}; t_s, t_{ins})}{\sqrt{C_{2pt}(\vec{p}'; t_s) C_{2pt}(\vec{p}; t_s)}}}_{\text{Cancels the overlaps}} \underbrace{\sqrt{\frac{C_{2pt}(\vec{p}'; t_s - t_{ins}) C_{2pt}(\vec{p}; t_{ins})}{C_{2pt}(\vec{p}; t_s - t_{ins}) C_{2pt}(\vec{p}'; t_{ins})}}}_{\text{Cancels the residual exponential}}$$

From three-point functions to matrix elements



Three-point function *Matrix element*

How to go from $\langle \psi_N(\vec{p}\,', t_\mathrm{s}) | \mathcal{O}(t_\mathrm{ins}) | \psi_N(\vec{p}, 0) \rangle$ to $\langle N(\vec{p}\,') | \mathcal{O} | N(\vec{p}) \rangle$?

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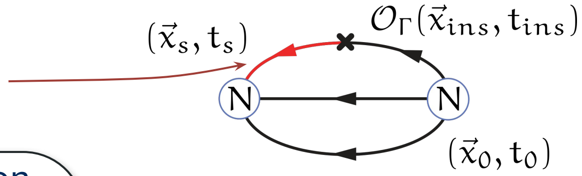
$$= \langle \psi_N(\vec{p}\,') | N(\vec{p}\,') \rangle \langle N(\vec{p}\,') | \mathcal{O} | N(\vec{p}) \rangle \langle N(\vec{p}) | \psi_N(\vec{p}) \rangle \; e^{-E_N(p')(t_\mathrm{s} - t_\mathrm{ins}) - E_N(p)\, t_\mathrm{ins}}$$

$$C_\mathrm{2pt}(\vec{p}; t_\mathrm{s}) \approx \langle \psi_N(\vec{p}) | N(\vec{p}) \rangle \langle N(\vec{p}) | \psi_N(\vec{p}) \rangle \; e^{-E_N(p)\, t_\mathrm{s}}$$

$$\boxed{\langle N(\vec{p}\,') | \mathcal{O} | N(\vec{p}) \rangle = \lim_{\substack{t_\mathrm{s} - t_\mathrm{ins} \to \infty \\ t_\mathrm{ins} \to \infty}} \underbrace{\frac{C_\mathrm{3pt}(\vec{p}\,', \vec{p}; t_\mathrm{s}, t_\mathrm{ins})}{\sqrt{C_\mathrm{2pt}(\vec{p}\,'; t_\mathrm{s}) C_\mathrm{2pt}(\vec{p}; t_\mathrm{s})}}}_{\text{Cancels the overlaps}} \underbrace{\sqrt{\frac{C_\mathrm{2pt}(\vec{p}\,'; t_\mathrm{s} - t_\mathrm{ins}) C_\mathrm{2pt}(\vec{p}; t_\mathrm{ins})}{C_\mathrm{2pt}(\vec{p}; t_\mathrm{s} - t_\mathrm{ins}) C_\mathrm{2pt}(\vec{p}\,'; t_\mathrm{ins})}}}_{\text{Cancels the residual exponential}}}$$

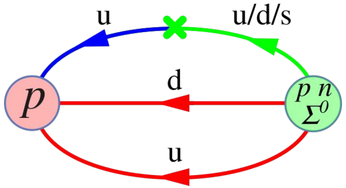
How to compute three-point functions?

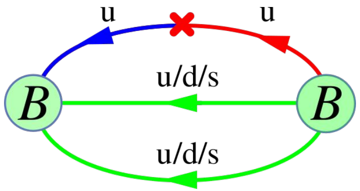
Three-point functions require computing a *sequential propagator*.



It can be done in two ways:

- Fixed-sink
- Fixed-insertion





● Fixed ● Free ● Sequential

- Fixed-sink:
  - Fix sink time t_s
  - Fix sink mom. p'
  - Fix sink interp. field
  - Get any ins. operator
  - Get any ins. mom.
  - Get any source mom. (pt. source)
  - Get multiple source interp. fields

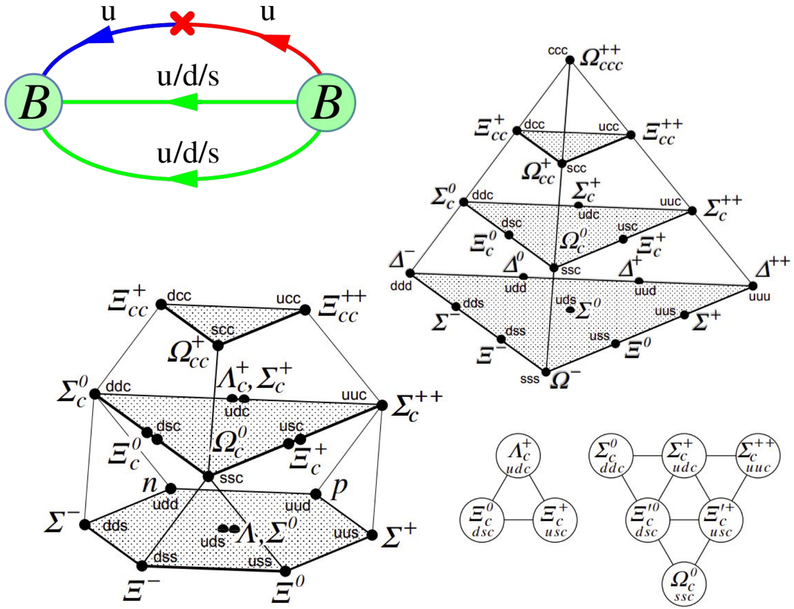
- Fixed-insertion:
  - Fix ins. time t_{ins}
  - Fix ins. mom. q
  - Fix ins. operator
  - Get any sink interp. field current
  - Get any source interp. fields
  - Get any sink time t_s
  - Get any sink mom.
  - Get any source mom. (pt. source)

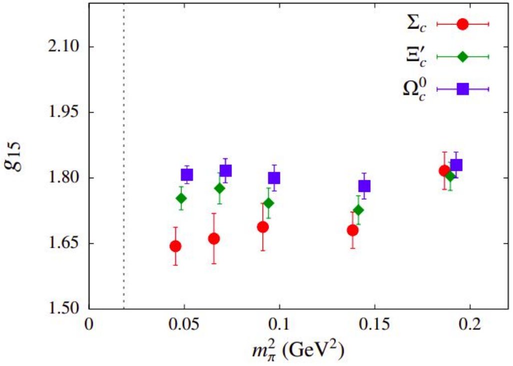
Example of fixed-insertion results

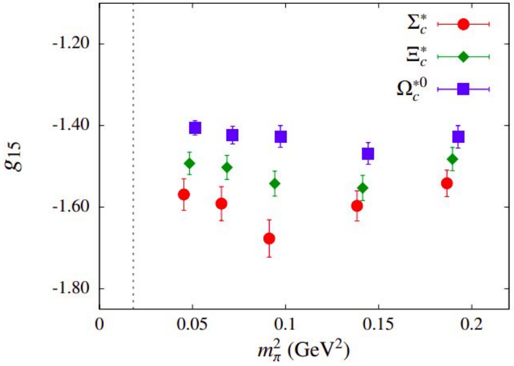
- “Axial charges of hyperons and charmed baryons using N_f=2+1+1 twisted mass fermions”

[arXiv:1606.01650]

  - Isovector *u-d*, *u+d-2s* and *u+d+s-3c* axial charges of 40 baryons
  - One in a kind study:
    - Computed using fixed-insertion
    - i.e. fixed current and momentum transfer
    - Got g_A for any possible baryon!



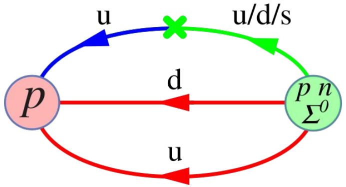


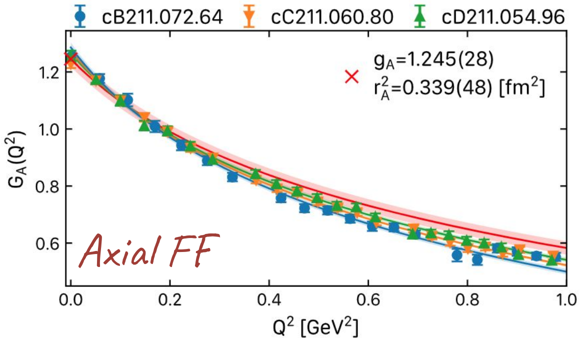


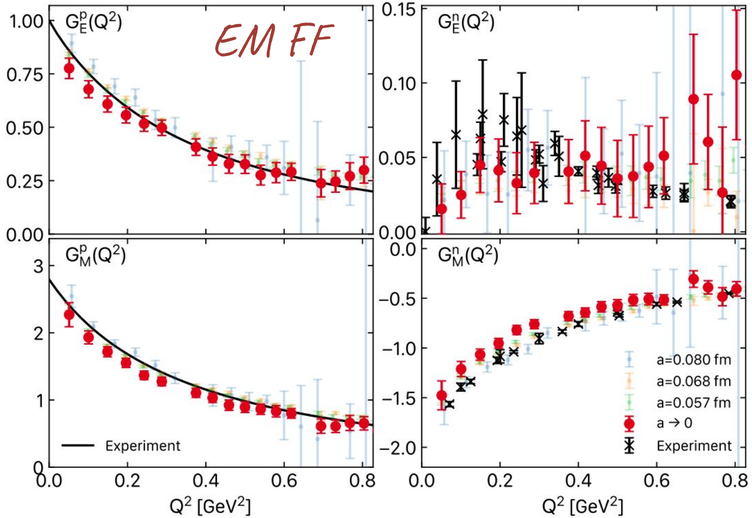
Example of fixed-sink results

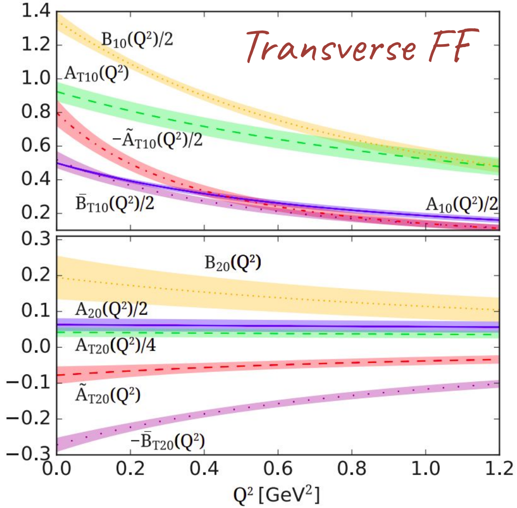
Most studies on nucleon structure uses fixed-sink since allows for a very reach programme.

- Any ins. current/operator: axial, vector, tensor, first and second Mellin moments, etc.
- Any momentum transfer: Form Factors (Q^2-dependence), extrapolation to Q^2=0 → charges/couplings







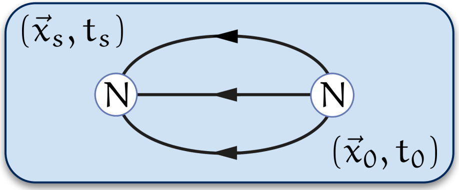


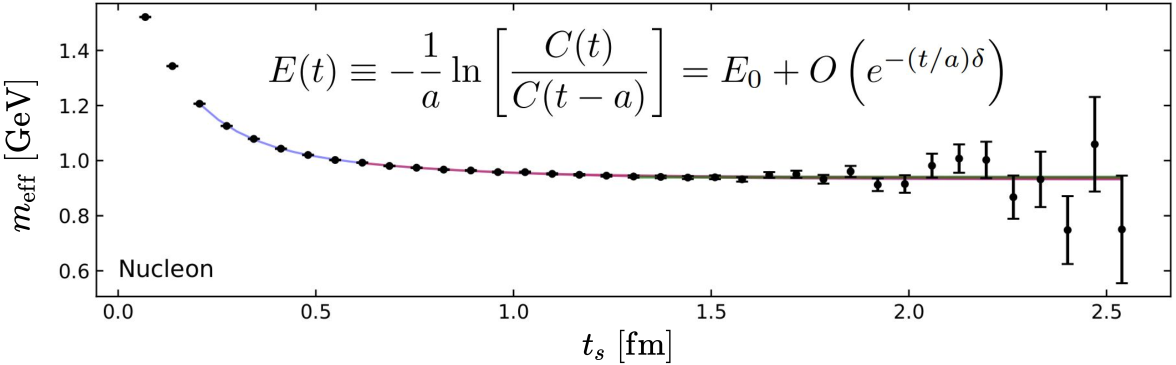
Nucleon two-point functions

$$G(t_s) = \sum_{\vec{x}} P_0^{\alpha\beta} \langle \bar{\chi}_N^{\beta}(\vec{x}_s, t_s) | \chi_N^{\alpha}(\vec{0}, 0) \rangle = \sum_k c_k e^{-t_s E_k}$$

with $\chi_N^{\alpha}(x) = \epsilon^{abc} u_{\alpha}^{a}(x)[u^{b}(x) C\gamma_5 d^{c}(x)]$

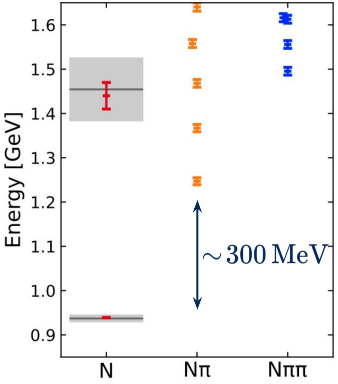
- Two-point functions
  - Ground state dominance at large-time limit $G(t_s) = c_0^{-t_s m_N}\Big|_{t_s \to \infty}$
  - Error increases exponentially with t
  - Density of excited states increases with volume





$$E(t) \equiv -\frac{1}{a} \ln \left[\frac{C(t)}{C(t-a)} \right] = E_0 + O\left(e^{-(t/a)\delta} \right)$$

$L = 64a,\ m_\pi L = 3.6$



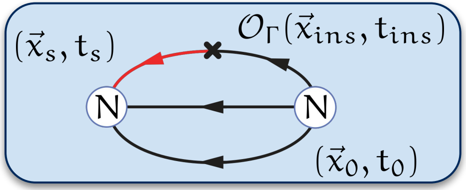
Nucleon three-point functions

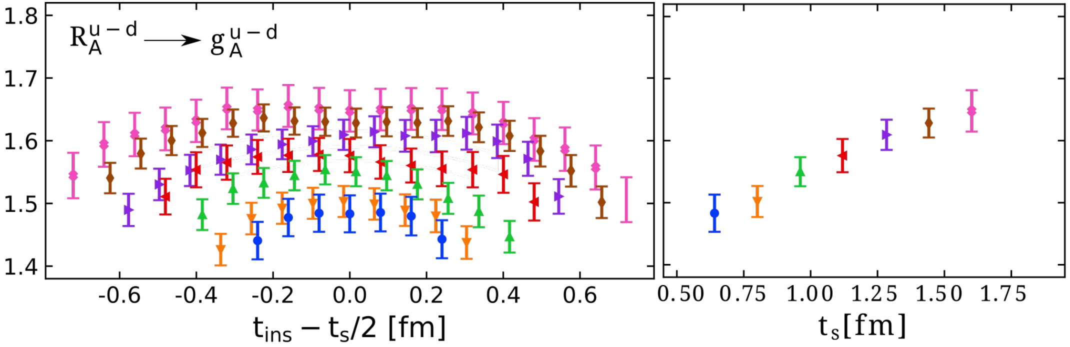
$$G_\Gamma(P;\vec{q};t_s,t_{ins}) = \sum_{\vec{x}_s,\vec{x}_{ins}} e^{-i\vec{q}\cdot\vec{x}_{ins}} P^{\alpha\beta} \langle \bar{\chi}^\beta_N(\vec{x}_s,t_s)|\mathcal{O}_\Gamma(\vec{x}_{ins},t_{ins})|\chi^\alpha_N(\vec{0},0)\rangle$$

suitable projector

e.g. $\mathcal{O}_A(x) = \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$

- Three-point functions
  - Ground state at $t_s \to \infty$, $(t_s - t_{ins}) \to \infty$
  - Error increases exponentially with t_s
  - Statistics increased to keep errors constant





×750 configurations

t_s/a	t_s[fm]	n_{src}
8	0.64	1
10	0.80	2
12	0.96	5
14	1.12	10
16	1.28	32
18	1.44	112
20	1.60	128
Nucleon 2pt		477

~30M inversions!

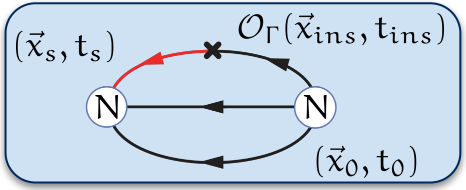
Nucleon three-point functions

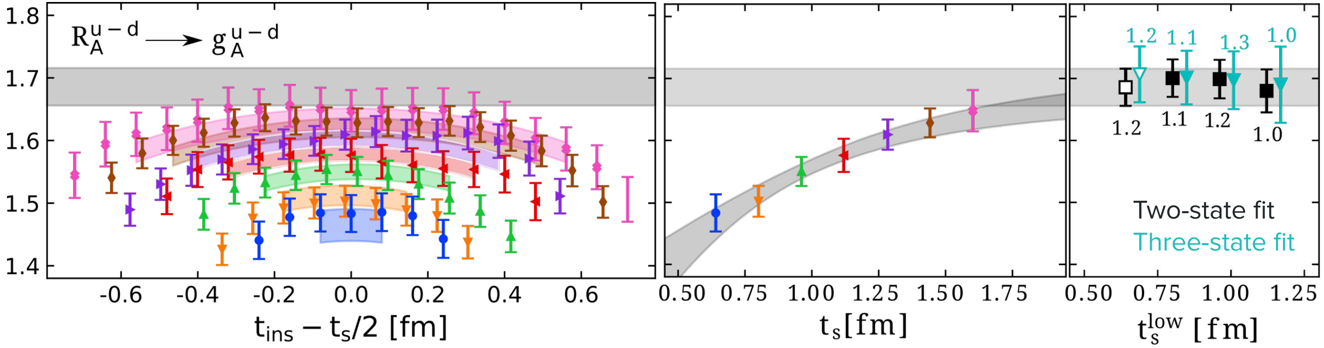
$$G_\Gamma(P;\vec{q};t_s,t_{ins}) = \sum_{\vec{x}_s,\vec{x}_{ins}} e^{-i\vec{q}\cdot\vec{x}_{ins}} P^{\alpha\beta} \langle \bar{\chi}^\beta_N(\vec{x}_s,t_s)|\mathcal{O}_\Gamma(\vec{x}_{ins},t_{ins})|\chi^\alpha_N(\vec{0},0)\rangle$$

$$G_\Gamma(t_s,t_{ins}) \simeq A_{00}e^{-m_N t_s} + A_{01}\left(e^{-E_1 t_{ins}} + e^{-E_1 t_s+(E_1-m_N)t_{ins}}\right) + A_{11}e^{-E_1 t_s}$$

$$G(t) \simeq c_0 e^{-m_N t_s} + c_1 e^{-E_1 t_s}$$

Desired matrix element: $\mathcal{M} = \frac{A_{00}}{c_0}$





[C. Alexandrou, S. B., et al. "Nucleon axial, tensor, and scalar charges and σ-terms in lattice QCD". Phys. Rev., D102(5):054517, 2020]

×750 configurations

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8	0.64	1
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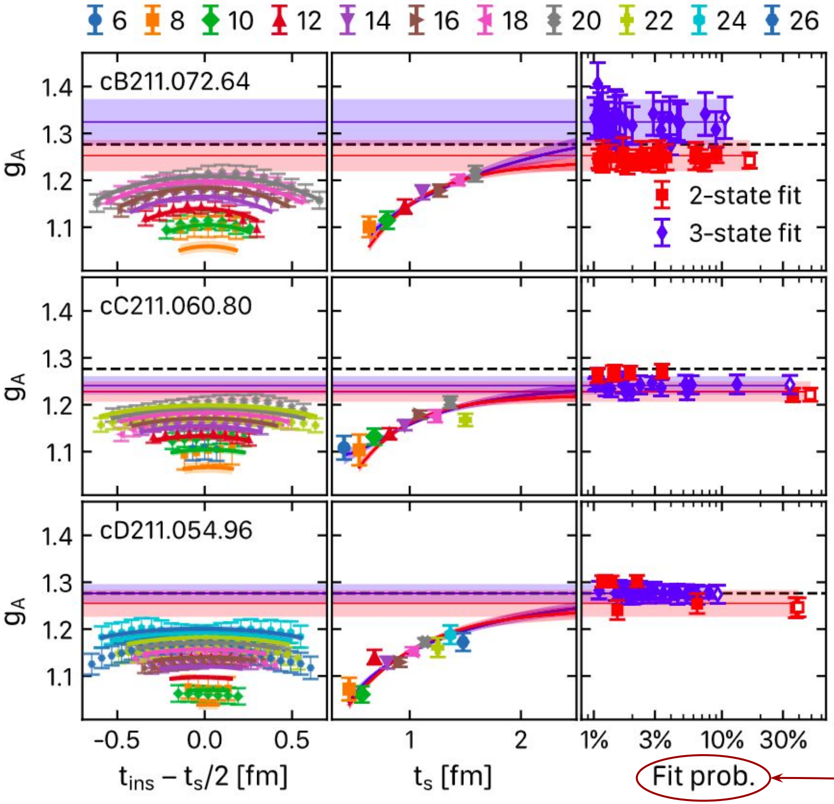
~30M inversions!

The three ensembles and model averaging



Ensemble	V/a^4	β	a [fm]	m_π [MeV]	$m_\pi L$
cB211.072.64	$64^3 \times 128$	1.778	0.07957(13)	140.2(2)	3.62
cC211.060.80	$80^3 \times 160$	1.836	0.06821(13)	136.7(2)	3.78
cD211.054.96	$96^3 \times 192$	1.900	0.05692(12)	140.8(2)	3.90

cB211.072.64		
750 configurations		
t_s/a	t_s[fm]	n_{src}
8	0.64	1
10	0.80	2
12	0.96	5
14	1.12	10
16	1.28	32
18	1.44	112
20	1.60	128
Nucleon 2pt		477

cC211.060.80		
400 configurations		
t_s/a	t_s[fm]	n_{src}
6	0.41	1
8	0.55	2
10	0.69	4
12	0.82	10
14	0.96	22
16	1.10	48
18	1.24	45
20	1.37	116
22	1.51	246
Nucleon 2pt		650

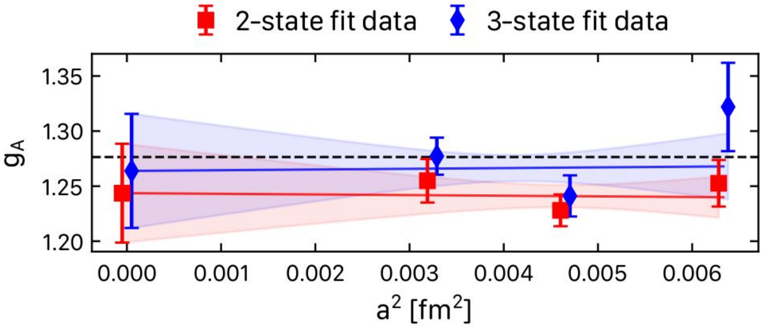
cD211.054.96		
500 configurations		
t_s/a	t_s[fm]	n_{src}
8	0.46	1
10	0.57	2
12	0.68	4
14	0.80	8
16	0.91	16
18	1.03	32
20	1.14	64
22	1.25	16
24	1.37	32
26	1.48	64
Nucleon 2pt		480

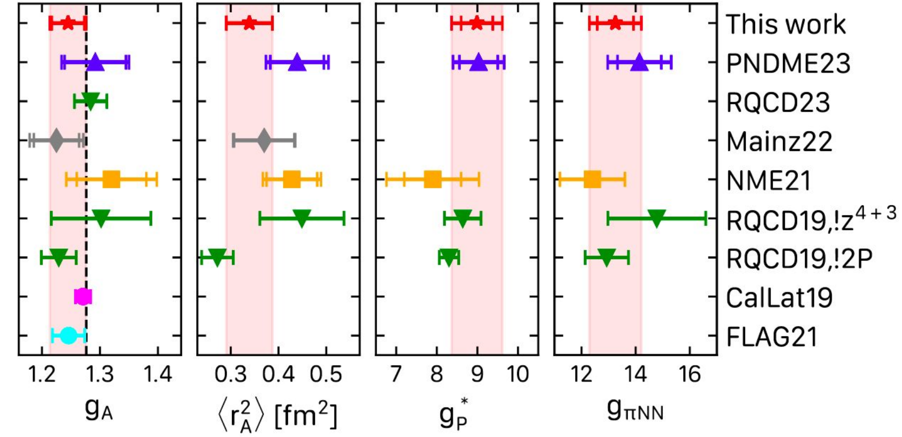
Up to 1.5fm for all ensembles

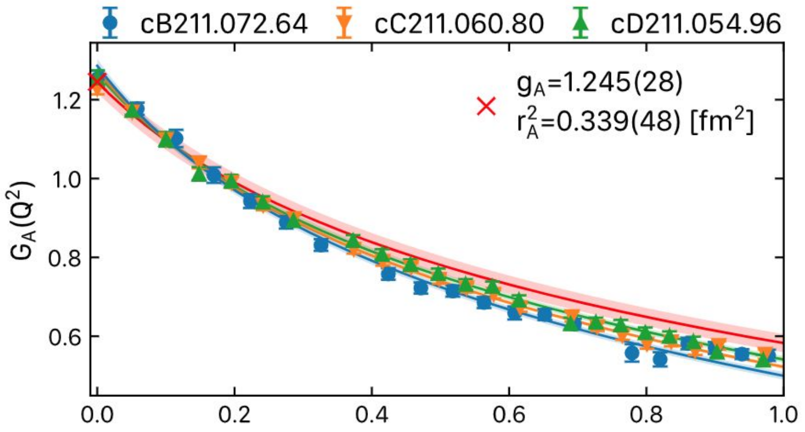
Model average over thousands of fits: $\log(w_i) = -\frac{\chi_i^2}{2} + N_{\mathrm{dof},i}$

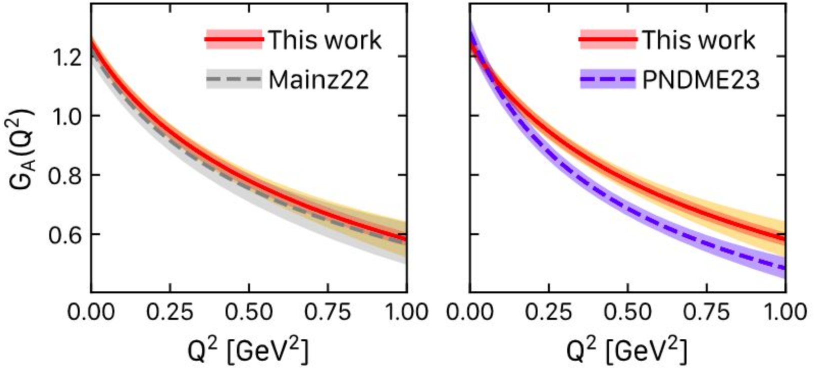
$p_i = \frac{w_i}{Z}$ with $Z = \sum_i w_i$. [E. T. Neil, J. W. Sitison, arXiv:2208.14983]

Continuum limit and comparison with other studies

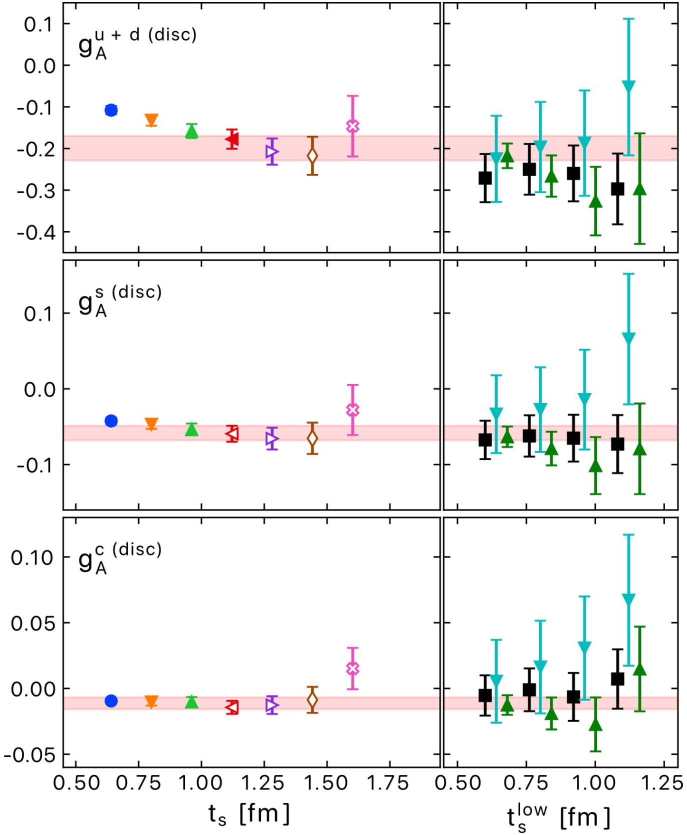




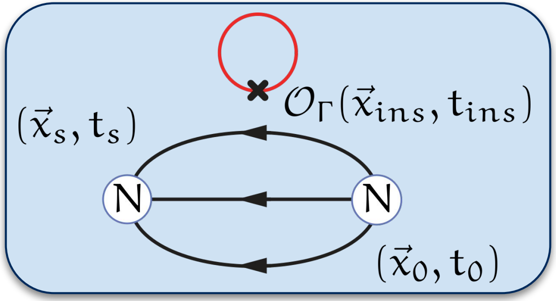




Disconnected contributions



- Loop calculations
  - Stochastic sources
  - Hierarchical probing
  - Low-mode deflation
- Disconnected contributions
  - Correlation between loops and two-point



The weak axial-vector matrix element

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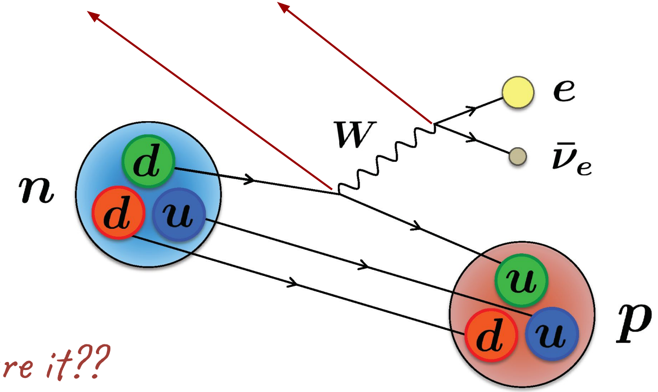
$$\mathcal{M}(n \to p\, e^- \bar{\nu}_e) = \frac{G_F}{\sqrt{2}}\, V_{ud} \sum_\mu \underbrace{\langle p(p')|W_\mu|n(p)\rangle}\, L_\mu$$



with
$$W_\mu = V_\mu - A_\mu$$
$$V_\mu = \bar{u}\gamma_\mu d$$
$$\boxed{A_\mu = \bar{u}\gamma_\mu\gamma_5 d}$$

Vector contributions are well determined experimentally from lepton-nucleon scattering

Axial-vector matrix element

$$\langle p(p')|A_\mu|n(p)\rangle$$

How to measure it??

Neutrino-nucleon scattering processes are related to matrix elements at finite momentum transfer.

The axial and induced pseudoscalar FF

Neglecting isospin-breaking effects, transition FFs are equivalent to isovector FFs

$$\langle p(p')|A_\mu|n(p)\rangle \qquad \xrightarrow[u=d]{} \qquad \langle N(p')|A_\mu^{\text{isov}}|N(p)\rangle$$

$$A_\mu = \bar{u}\gamma_\mu\gamma_5 d \qquad\qquad A_\mu^{\text{isov}} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$$

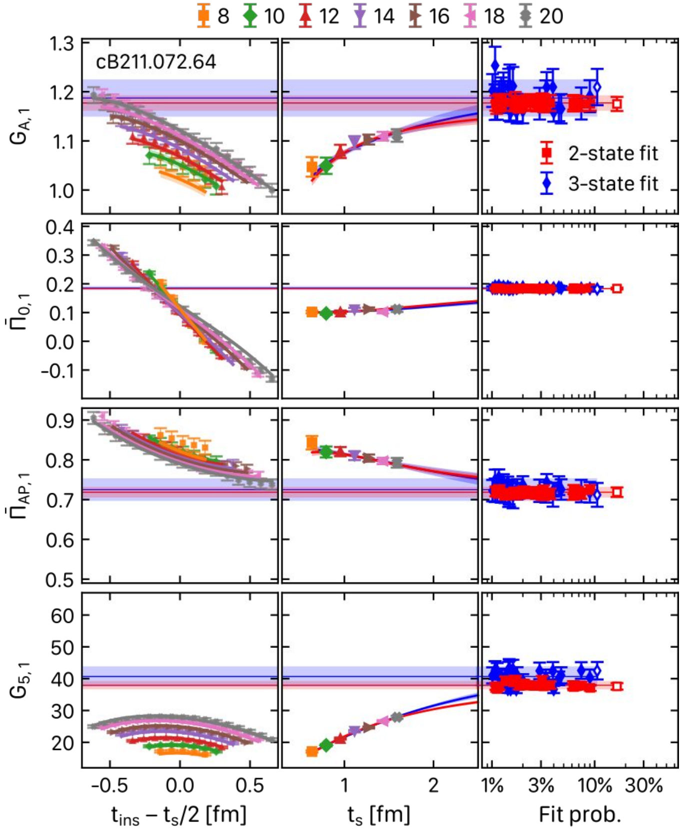
Matrix elements are decomposed into Lorentz-invariant form factors (FF)

$$\langle N(p',s')|A_\mu|N(p,s)\rangle = \bar{u}_N(p',s')\left[\gamma_\mu G_A(Q^2) - \frac{Q_\mu}{2m_N}G_P(Q^2)\right]\gamma_5 u_N(p,s),$$

Axial FF

Induced pseudoscalar FF

… and at finite momentum transfer



$$\Pi_\mu(\Gamma_k;\vec{q}) = \frac{\mathcal{A}^{0,0}_\mu(\Gamma_k,\vec{q})}{\sqrt{c_0(\vec{0})c_0(\vec{q})}}$$

→ *Three-point ground state*

→ *Two-point ground state*

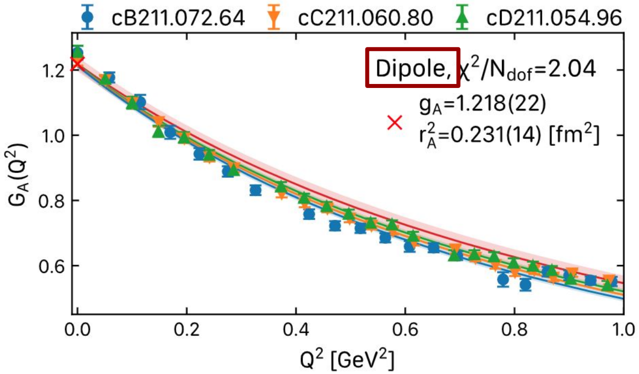
Combined fit of all three-point functions at the same Q^2

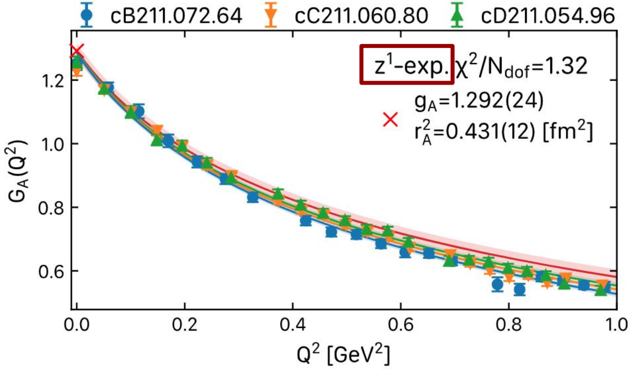
$$\Pi_i(\Gamma_k,\vec{q}) = \frac{i\mathcal{K}}{4m_N}\left[\frac{q_k q_i}{2m_N}G_P(Q^2) - \delta_{i,k}(m_N+E_N)G_A(Q^2)\right]$$

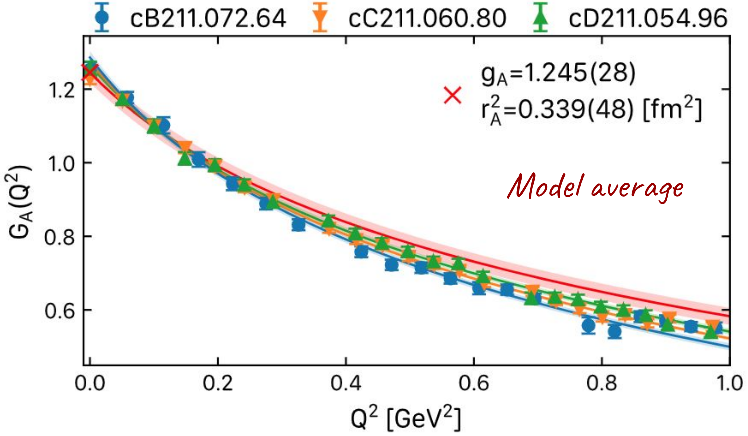
$$\Pi_0(\Gamma_k,\vec{q}) = -\frac{q_k\mathcal{K}}{2m_N}\left[G_A(Q^2) + \frac{(m_N-E_N)}{2m_N}G_P(Q^2)\right]$$

$$\Pi_5(\Gamma_k,\vec{q}) = -\frac{iq_k\mathcal{K}}{2m_N}G_5(Q^2) \longrightarrow$$ *Pseudoscalar FF*

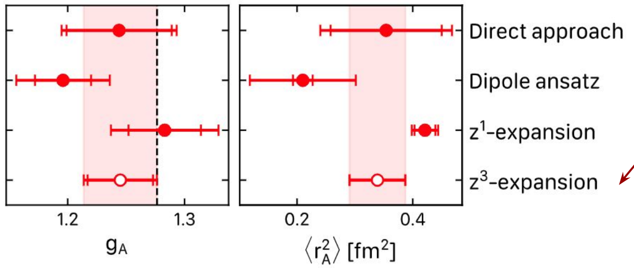
Dipole vs z-expansion





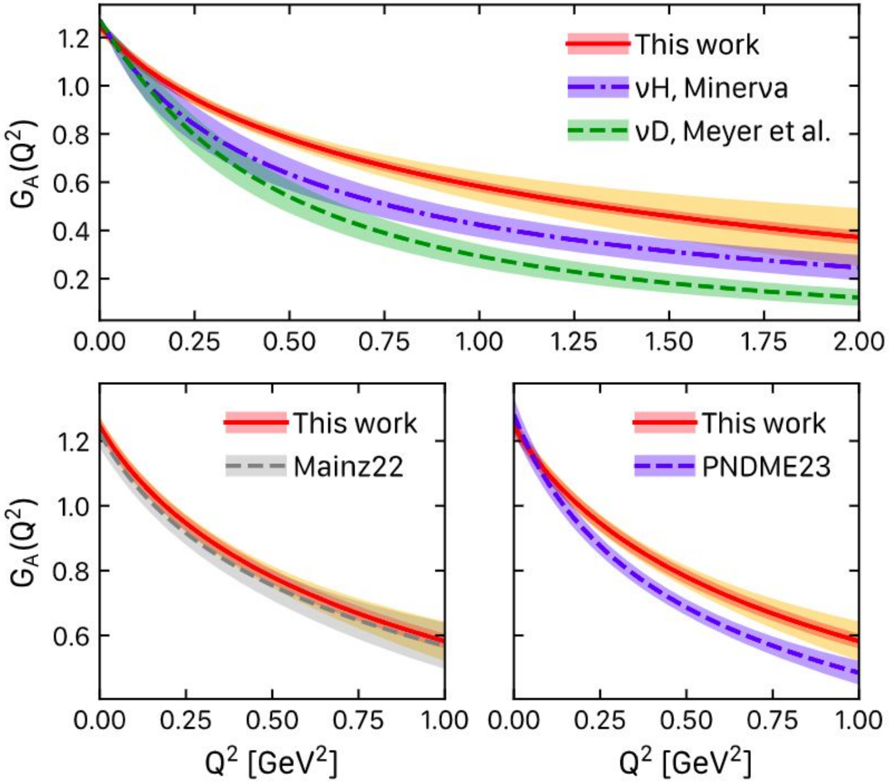


Model average



Compatible with the direct approach but smaller error because all information used

Comparison with other studies



- Overall good agreement between recent lattice results and better agreement with the very recent results from Minerva