Exotic hadrons

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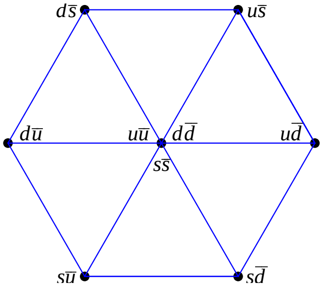
Light meson SU(3) [u, d, s] multiplets (octet + singlet):



- Vector mesons

meson	quark content	mass (MeV)
ρ^+/ρ^-	$u\bar{d}\,/d\bar{u}$	775
ρ^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	775
K^{*+}/K^{*-}	$u\bar{s}\,/s\bar{u}$	892
$K^{*0}/\bar{K}^{*0}$	$d\bar{s}\,/s\bar{d}$	896
ω	$(u\bar{u}+d\bar{d})/\sqrt{2}$	783
ϕ	$s\bar{s}$	1019

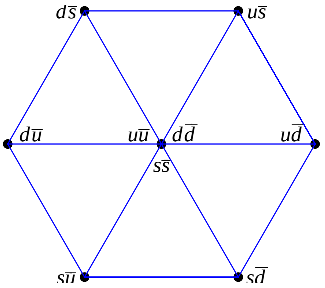
☞ approximate SU(3) symmetry

☞ very good isospin SU(2) symmetry

$$m_{\rho^0}-m_{\rho^\pm}=(-0.7\pm 0.8)\text{ MeV},\quad m_{K^{*0}}-m_{K^{*\pm}}=(6.7\pm 1.2)\text{ MeV}$$

Light vector and pseudoscalar mesons

Light meson SU(3) [u, d, s] multiplets (octet + singlet):



- Pseudoscalar mesons

meson	quark content	mass (MeV)
π^+/π^-	$u\bar{d}\,/\,d\bar{u}$	140
π^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	135
K^+/K^-	$u\bar{s}\,/\,s\bar{u}$	494
$K^0/\bar{K}^0$	$d\bar{s}\,/\,s\bar{d}$	498
η	$\sim (u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}$	548
η'	$\sim (u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$	958

☞ very good isospin SU(2) symmetry

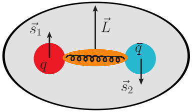
$$m_{\pi^\pm} - m_{\pi^0} = (4.5936 \pm 0.0005)\ \text{MeV}, \quad m_{K^0} - m_{K^\pm} = (3.937 \pm 0.028)\ \text{MeV}$$

☞ Why are the pions so light? Pseudo-Nambu-Goldstone bosons of the spontaneous breaking of chiral symmetry: $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \to \mathrm{SU}(2)_V$

Lectures by Juan Nieves

- J^{PC} of regular $q\bar{q}$ mesons

  L: orbital angular momentum
  $S = (0, 1)$: total spin of q and $\bar{q}$
  $P = (-1)^{L+1}$ $[Y_{Lm}(\theta - \pi, \phi + \pi) = (-1)^L Y_{Lm}(\theta, \phi)]$
  $C = (-1)^{L+S} = (-1)^{L+1+S+1}$ for flavor-neutral mesons



$$S = 0:\ \frac{1}{\sqrt{2}} |\uparrow_q\downarrow_{\bar{q}} - \downarrow_q\uparrow_{\bar{q}}\rangle; \quad S = 1:\ \left\{ |\uparrow_q\uparrow_{\bar{q}}\rangle, \frac{1}{\sqrt{2}} |\uparrow_q\downarrow_{\bar{q}} + \downarrow_q\uparrow_{\bar{q}}\rangle, |\downarrow_q\downarrow_{\bar{q}}\rangle \right\}$$

☞ For $S = 0$, the meson spin $J = L$, one has $P = (-1)^{J+1}$ and $C = (-1)^J$. Hence,

$$J^{PC} = \text{even}^{-+} \text{ and odd}^{+-}$$

☞ For $S = 1$, one has $P = C = (-1)^{L+1}$. Hence,

$$J^{PC} = 1^{--}, \{0, 1, 2\}^{++}, \{1, 2, 3\}^{--}, \ldots$$

- Exotic J^{PC} for mesons:

$$\boxed{J^{PC} = 0^{--}, \text{even}^{+-} \text{ and odd}^{-+}}$$

Digression: Mesons with exotic quantum numbers

- π_1 listed as established particles by the Particle Data Group (PDG)

$\pi_1(1400)$ $\quad I^G(J^{PC}) = 1^-(1^{-+})$

See also the mini-review under non- $q\bar{q}$ candidates in PDG 2006 , Journal of Physics G33 1 (2006).

$\pi_1(1400)$ MASS	1354 ± 25 MeV (S = 1.8)
$\pi_1(1400)$ WIDTH	330 ± 35 MeV

Decay Modes

Mode		Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P (MeV/c)
Γ_1	$\eta\pi^0$	seen		557
Γ_2	$\eta\pi^-$	seen		556
Γ_3	$\eta'\pi$			318
Γ_4	$\rho(770)\pi$	not seen		442

$\pi_1(1600)$ $\quad I^G(J^{PC}) = 1^-(1^{-+})$

$\pi_1(1600)$ MASS	1660^{+15}_{-11} MeV (S = 1.2)
$\pi_1(1600)$ WIDTH	257 ± 60 MeV (S = 1.9)

Decay Modes

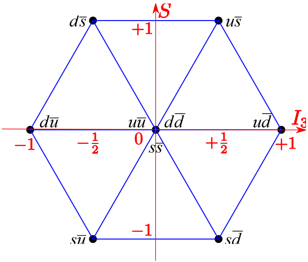
Mode		Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P (MeV/c)
Γ_1	$\pi\pi\pi$	seen		802
Γ_2	$\rho^0\pi^-$	seen		640
Γ_3	$f_2(1270)\pi^-$	not seen		316
Γ_4	$b_1(1235)\pi$	seen		355
Γ_5	$\eta'(958)\pi^-$	seen		542
Γ_6	$f_1(1285)\pi$	seen		312

- $J^{PC} = 1^{-+}$ $\eta_1(1855) \to \eta\eta'$ recently observed by BESIII BESIII, PRL129(2022)192002

It is unclear what they are: hybrids? hadronic molecules? or sth. else?

Some trivial facts about additive quantum numbers of regular mesons

- Light-flavor mesons (here S =strangeness)
  - Nonstrange mesons: $S=0$, $I=0,1$
  - Strange mesons: $S=\pm1$, $I=\frac{1}{2}$



- Open-flavor heavy mesons
  - $Q\bar{q}(q=u,d)$: $S=0$, $I=1/2$
  - $Q\bar{s}$: $S=1$, $I=0$
- Heavy quarkonia ($Q\bar{Q}$): $S=0, I=0$, neutral

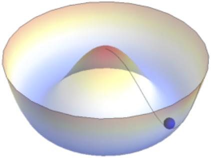
> Charge, isospin, strangeness etc. which cannot be achieved in the $q\bar{q}$ and qqq scheme would be a smoking gun for an exotic nature

$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + \frac{g_s^2\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$$

- Exact: Lorentz-invariance, SU$(3)_c$ gauge, C (for $\theta = 0$ w/ real m_f, P, T as well)
- Approximate: u ~2 MeV, d ~5 MeV, s ~100 MeV $\ll \Lambda_{\rm QCD} \ll$ c ~1.3 GeV, b ~4.2 GeV, t ~173 GeV

Light quarks: u, d, s — Heavy quarks: c, b, t

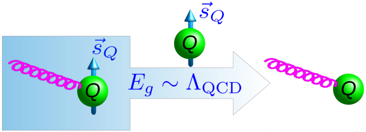
☞ Spontaneously broken chiral symmetry:
SU$(N_f)_L \times$ SU$(N_f)_R \xrightarrow{\text{SSB}}$ SU$(N_f)_V$



☞ Heavy quark spin symmetry (HQSS)

☞ Heavy quark flavor symmetry (HQFS)

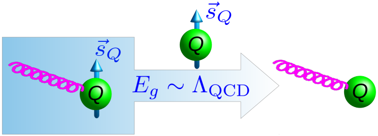
☞ Heavy aiquark-diquark symmetry (HADS)



Heavy quark symmetries

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer Λ_{QCD}

  ☞ heavy quark spin symmetry (HQSS):
  chromomag. interaction $\propto \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{m_Q}$
  spin of the heavy quark decouples



Let total angular momentum $\boldsymbol{J} = \boldsymbol{s}_Q + \boldsymbol{s}_\ell$,

$\boldsymbol{s}_Q$: heavy quark spin,

$\boldsymbol{s}_\ell$: spin of the light degrees of freedom (including orbital angular momentum)

✓ HQSS:

s_ℓ and s_Q are conserved separately in the heavy quark limit!

✓ spin multiplets:

for singly heavy mesons, e.g. $\{D, D^*\}$, $\{B, B^*\}$ with $s_\ell^P = \frac{1}{2}^-$;

for heavy quarkonia, e.g. S-wave: $\{\eta_c, J/\psi\}$, $\{\eta_b, \Upsilon\}$;

P-wave: $\{h_c, \chi_{c0,c1,c2}\}$, $\{h_b, \chi_{b0,b1,b2}\}$

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer Λ_{QCD}
  - ☞ heavy quark flavor symmetry (HQFS) for any hadron containing one heavy quark:

    velocity remains unchanged in the limit $m_Q \to \infty$: $\Delta v = \frac{\Delta p}{m_Q} = \frac{\Lambda_{\text{QCD}}}{m_Q}$

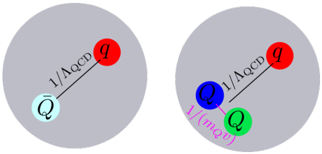
    $\Rightarrow$ heavy quark is like a static color triplet source, m_Q is irrelevant
  - ☞ heavy anti-quark–diquark symmetry

    Savage, Wise (1990)

    $m_Q v \gg \Lambda_{\text{QCD}}$,

    the diquark serves as a point-like color-$\bar{3}$ source, like a heavy anti-quark.

    It relates doubly-heavy baryons to anti-heavy mesons



Heavy quark fields

- In the heavy quark limit $m_Q \to \infty$, consider the quark propagator

$$i\frac{\not{p}+m_Q}{p^2-m_Q^2+i\epsilon} = i\frac{m_Q(\not{v}+1)+\not{k}}{2m_Q v\cdot k+k^2+i\epsilon} \xrightarrow{m_Q\to\infty} \frac{1+\not{v}}{2}\frac{i}{v\cdot k+i\epsilon}$$

here $p = m_Q v + k$, with a residual momentum $k \sim \Lambda_{\text{QCD}}$.

- Decompose heavy quark field into v-dep. fields $Q(x) = e^{-im_Q v\cdot x}\left[Q_v(x)+q_v(x)\right]$:

$$Q_v(x) = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q(x), \qquad q_v(x) = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q(x)$$

- At leading order (LO) of the $1/m_Q$ expansion:

$$\mathcal{L}_Q = \bar{Q}(i\not{D}-m_Q)Q = \bar{Q}_v(iv\cdot D)Q_v + \mathcal{O}\left(m_Q^{-1}\right)$$

  - ☞ No Dirac gamma matrices in the LO Lagrangian, so invariant under spin rotation ⇒ HQSS
  - ☞ No heavy quark mass term ⇒ HQFS

Applications to states of current interest

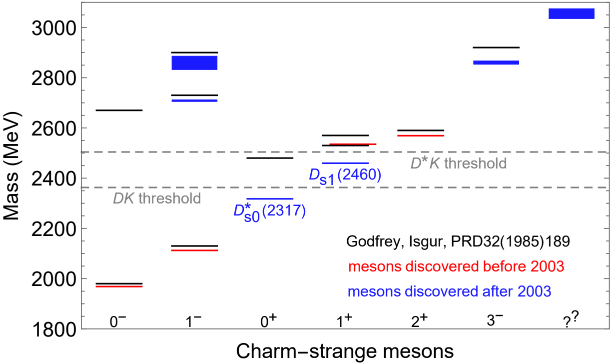
Examples of HQSS phenomenology:

- In the Review of Particle Physics (RPP) by the Particle Data Group (PDG), there are two D_1 ($J^P = 1^+$) mesons with very different widths
  - ☞ $\Gamma[D_1(2420)] = (27.4 \pm 2.5)$ MeV $\ll$ $\Gamma[D_1(2430)] = (384^{+130}_{-110})$ MeV
  - ☞ $\boldsymbol{s}_\ell = \boldsymbol{s}_q + \boldsymbol{L} \;\;\Rightarrow$ for P-wave charmed mesons: $s_\ell^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$
  - ☞ for decays $D_1 \to D^*\pi$:

    $\frac{1}{2}^+ \to \frac{1}{2}^- + 0^-$ in S-wave $\Rightarrow$ large width

    $\frac{3}{2}^+ \to \frac{1}{2}^- + 0^-$ in D-wave $\Rightarrow$ small width
  - ☞ thus, dominant components: $D_1(2420)$: $s_\ell = \frac{3}{2}$, $\quad D_1(2430)$: $s_\ell = \frac{1}{2}$
- Suppression of the S-wave production of $\frac{3}{2}^+ + \frac{1}{2}^-$ heavy meson pairs in e^+e^- annihilation Table VI in E.Eichten et al., PRD**17**(1978)3090; X. Li, M. Voloshin, PRD**88**(2013)034012

Exercise: Try to understand this statement as a consequence of HQSS.
Hint: in e^+e^- collisions, the leading production mechanism of heavy meson pairs is from the vector current $\bar{Q}\gamma^\mu Q$ which couples to the virtual photon, *i.e.*, $e^+e^- \to \gamma^* \to \bar{Q}Q$ with the $Q\bar{Q}$ pair in an S-wave.



- $D^*_{s0}(2317)$: BaBar (2003)
  $J^P = 0^+$, $\Gamma < 3.8$ MeV
- $D_{s1}(2460)$: CLEO (2003)
  $J^P = 1^+$, $\Gamma < 3.5$ MeV
- no isospin partner observed, tiny widths
  $\Rightarrow I = 0$

- Mystery 1: Mass problem: Why are $D^*_{s0}(2317)$ and $D_{s1}(2460)$ so light?
- Mystery 2: Naturalness problem:

$$\text{Why } \underbrace{M_{D_{s1}(2460)} - M_{D^*_{s0}(2317)}}_{(141.8\pm0.8)\text{ MeV}} \simeq \underbrace{M_{D^{*\pm}} - M_{D^\pm}}_{(140.67\pm0.08)\text{ MeV}} ?$$

- HQFS: for a singly-heavy hadron, $M_{H_Q} = m_Q + A + \mathcal{O}\left(m_Q^{-1}\right)$
- rough estimates of bottom analogues whatever the D_{sJ} states: Defining the spin-averaged mass for charmed mesons:
  $\bar{M}_c = \frac{1}{4}\left[M_{D_{s0}^*(2317)} + 3M_{D_{s1}(2460)}\right] \simeq 2.424$ GeV, then

$$M_{B_{s0}^*} = \bar{M}_c + \Delta_{b-c} + \left(M_{D_{s0}^*} - \bar{M}_c\right)\frac{m_c}{m_b} \simeq 5.71 \text{ GeV}$$

$$M_{B_{s1}} = \bar{M}_c + \Delta_{b-c} + \left(M_{D_{s1}} - \bar{M}_c\right)\frac{m_c}{m_b} \simeq 5.76 \text{ GeV}$$

  here $\Delta_{b-c} \equiv m_b - m_c \simeq \overline{M}_{B_s} - \overline{M}_{D_s} \simeq 3.33$ GeV, where $\overline{M}_{B_s} = 5.403$ GeV, $\overline{M}_{D_s} = 2.076$ GeV: spin-averaged g.s. $Q\bar{s}$ meson masses
  - ☞ comparing with the lattice QCD results: Lang et al., PLB750(2015)17

$$M_{B_{s0}^*}^{\text{lat.}} = (5.711 \pm 0.013 \pm 0.019) \text{ GeV}$$

$$M_{B_{s1}}^{\text{lat.}} = (5.750 \pm 0.017 \pm 0.019) \text{ GeV}$$

  - ☞ both to be discovered [1]
- more precise predictions can be made in a given model, e.g. hadronic molecules

[1] The established meson $B_{s1}(5830)$ is probably the bottom partner of $D_{s1}(2536)$.

Charm-strange mesons (3)

- In the hadronic molecular model, the main component: $D^*_{s0}(2317) : DK(I=0)$, $D_{s1}(2460) : D^*K(I=0)$
  as a consequence of HQSS:

  similar binding energies $M_D + M_K - M_{D^*_{s0}} \simeq 45$ MeV

  $$M_{D_{s1}(2460)} - M_{D^*_{s0}(2317)} \simeq M_{D^*} - M_D \text{ is natural}$$

- predicting the bottom-partner masses in one minute:

$$M_{B^*_{s0}} \simeq M_B + M_K - 45\,\text{MeV} \simeq 5.73 \text{ GeV}$$
$$M_{B_{s1}} \simeq M_{B^*} + M_K - 45\,\text{MeV} \simeq 5.78 \text{ GeV}$$

to be compared with lattice results for the lowest positive-parity bottom-strange mesons: Lang, Mohler, Prelovsek, Woloshyn, PLB750(2015)17

$$M^{\text{lat.}}_{B^*_{s0}} = (5.711 \pm 0.013 \pm 0.019) \text{ GeV}$$
$$M^{\text{lat.}}_{B_{s1}} = (5.750 \pm 0.017 \pm 0.019) \text{ GeV}$$

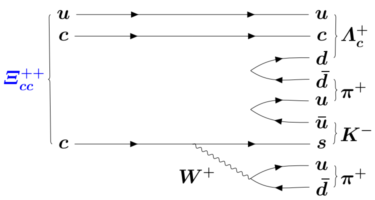
Development inspired by the LHCb discovery of the $\Xi_{cc}(3620)^{++}$

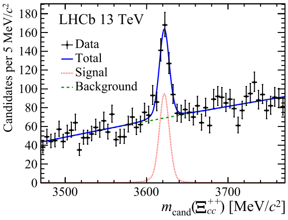
- Heavy antiquark-diquark symmetry (HADS):
  repalcing $\bar{Q}$ in $\bar{Q}q$ by QQ $[\bar{3}_{\text{color}}]$ $\Rightarrow$ QQq;
  repalcing $\bar{Q}$ in $\bar{Q}\bar{q}\bar{q}$ by QQ $[\bar{3}_{\text{color}}]$ $\Rightarrow$ $QQ\bar{q}\bar{q}$;

$$\begin{array}{lcccccccc} & \bar{Q}q & \Rightarrow & QQq, & \bar{Q}\bar{q}\bar{q} & \Rightarrow & QQ\bar{q}\bar{q} \\ \text{mass} \approx & m_Q + A & \Rightarrow & m_{QQ} + A, & m_Q + B & \Rightarrow & m_{QQ} + B \end{array}$$

  Prediction: $M_{QQ\bar{q}\bar{q}} - M_{\bar{Q}\bar{q}\bar{q}} \simeq M_{QQq} - M_{\bar{Q}q}$

- Doubly-charmed baryon discovered by LHCb PRL119(2017)112001 [arXiv:1707.01621]





$M_{\Xi_{cc}^{++}} = (3621.40 \pm 0.78)$ MeV can be used as input

Applications from heavy baryons to doubly-heavy tetraquarks (2)

TABLE II. Expectations for the ground-state tetraquark masses, in MeV.[a] The column labeled "HQS relation" is the result of our heavy-quark symmetry relations and is explicitly given by the sum of the right-hand side of Eq. (1) and the kinetic-energy mass shifts of Eq. (7). Here q denotes an up or down quark. For stable tetraquark states the $\mathcal{Q}$ value is highlighted in a box.

State	J^P	j_ℓ	$m(Q_iQ_jq_m)$ (c.g.)	HQS relation	$m(Q_iQ_j\bar{q}_k\bar{q}_l)$	Decay channel	$\mathcal{Q}$ (MeV)
$\{cc\}[\bar{u}\,\bar{d}]$	1^+	0	3663[b]	$m(\{cc\}u)+315$	3978	D^+D^{*0} 3876	102
$\{cc\}[\bar{q}_k\bar{s}]$	1^+	0	3764[c]	$m(\{cc\}s)+392$	4156	$D^+D_s^{*-}$ 3977	179
$\{cc\}\{\bar{q}_k\bar{q}_l\}$	$0^+,1^+,2^+$	1	3663	$m(\{cc\}u)+526$	4146,4167,4210	D^+D^0, D^+D^{*0} 3734,3876	412,292,476
$[bc][\bar{u}\,\bar{d}]$	0^+	0	6914	$m([bc]u)+315$	7229	B^-D^+/B^0D^0 7146	83
$[bc][\bar{q}_k\bar{s}]$	0^+	0	7010[d]	$m([bc]s)+392$	7406	B_sD 7236	170
$[bc]\{\bar{q}_k\bar{q}_l\}$	1^+	1	6914	$m([bc]u)+526$	7439	B^*D/BD^* 7190/7290	249
$\{bc\}[\bar{u}\,\bar{d}]$	1^+	0	6957	$m(\{bc\}u)+315$	7272	B^*D/BD^* 7190/7290	82
$\{bc\}[\bar{q}_k\bar{s}]$	1^+	0	7053[d]	$m(\{bc\}s)+392$	7445	DB_s^* 7282	163
$\{bc\}\{\bar{q}_k\bar{q}_l\}$	$0^+,1^+,2^+$	1	6957	$m(\{bc\}u)+526$	7461,7472,7493	BD/B^*D 7146/7190	317,282,349
$\{bb\}[\bar{u}\,\bar{d}]$	1^+	0	10 176	$m(\{bb\}u)+306$	10 482	$B^-\bar{B}^{*0}$ 10 603	$\boxed{-121}$
$\{bb\}[\bar{q}_k\bar{s}]$	1^+	0	10 252[c]	$m(\{bb\}s)+391$	10 643	$\bar{B}\bar{B}_s^*/\bar{B}_s\bar{B}^*$ 10 695/10 691	$\boxed{-48}$
$\{bb\}\{\bar{q}_k\bar{q}_l\}$	$0^+,1^+,2^+$	1	10 176	$m(\{bb\}u)+512$	10 674,10 681,10 695	B^-B^0, B^-B^{*0} 10 559,10 603	115,78,136

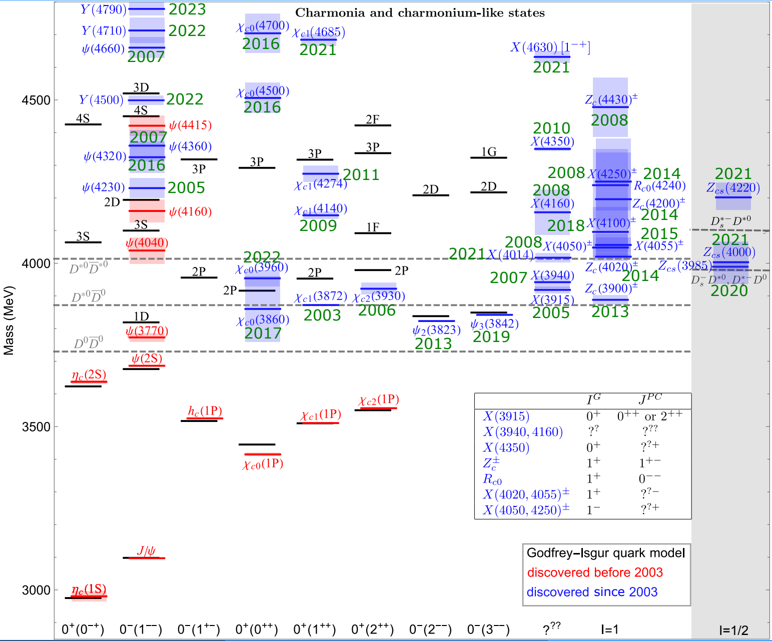
[a]Masses of the unobserved doubly heavy baryons are taken from Ref. [14]; for lattice evaluations of b-baryon masses, see Ref. [15].
[b]Based on the mass of the LHCb Ξ_{cc}^{++} candidate, 3621.40 MeV, Ref. [10].
[c]Using the s/d mass differences of the corresponding heavy-light mesons.
[d]Evaluated as $\frac{1}{2}[m(c\bar{s})-m(c\bar{d})+m(b\bar{s})-m(b\bar{d})]+m(bcd)$.

Eichten, Quigg, PRL119(2017)202002

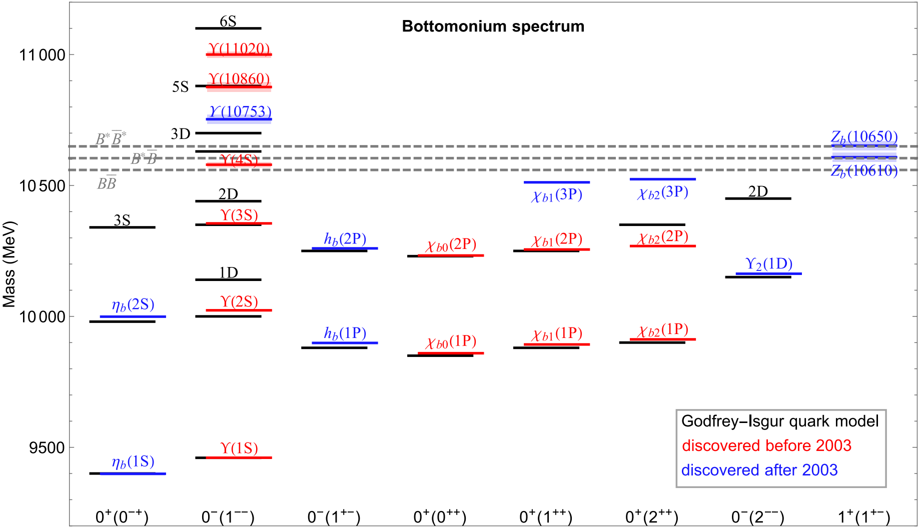
- HADS ⇒ stable doubly-bottom tetraquarks (only decay weakly) are likely to exist

  see also Carlson, Heller, Tjon, PRD37(1988)744; Manohar, Wise, NPB399(1993)17; Karliner, Rosner, PRL119(2017)202001; Czarnecki, Leng, Voloshin, PLB778(2018)233; ...
- support from lattice QCD — Francis, Hudspith, Lewis, Maltman, PRL118(2017)142001; . . .
- Possible detecting method: T. Gershon, A. Poluektov, *Displaced B_c mesons as an inclusive signature of weakly decaying double beauty hadrons*, JHEP 01 (2019) 019

Charmonium spectrum: current status



Bottomonium spectrum: current status



Naming convention

For states with properties in conflict with naive quark model (normally):

- X: $I = 0$, J^{PC} other than 1^{--} or unknown
- Y: $I = 0$, $J^{PC} = 1^{--}$
- Z: $I = 1$

PDG naming scheme:

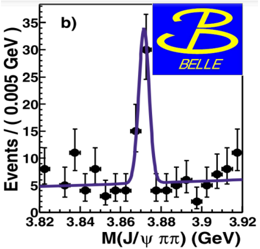
Minimal quark content	$J^{PC} =$ 0^{-+}, 2^{-+}, $\vdots$	1^{+-}, 3^{+-}, $\vdots$	1^{--}, 2^{--}, $\vdots$	0^{++}, 1^{++}, $\vdots$
$u\bar{d}, u\bar{u} - d\bar{d}, d\bar{u}$ $(I = 1)$	π	b	ρ	a
$d\bar{d} + u\bar{u}$ and/or $s\bar{s}$ $(I = 0)$	η, η'	h, h'	ω, ϕ	f, f'
$c\bar{c}$	η_c	h_c	$\psi^\dagger$	χ_c
$b\bar{b}$	η_b	h_b	Υ	χ_b
$I = 1$ with $c\bar{c}$	(Π_c)	Z_c	R_c	(W_c)
$I = 1$ with $b\bar{b}$	(Π_b)	Z_b	(R_b)	(W_b)

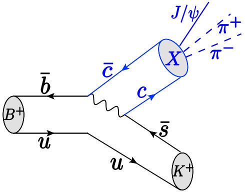
†The J/ψ remains the J/ψ.

"Young man, if I could remember the names of these particles, I would have been a botanist." *— Enrico Fermi*

$X(3872)$ (1)

Belle, PRL91(2003)262001 [hep-ex/0309032]





- The beginning of the XYZ story, discovered in $B^\pm \to K^\pm J/\psi\pi\pi$

  $M_X = (3871.69 \pm 0.17)$ MeV
- $\Gamma < 1.2$ MeV — Belle, PRD84(2011)052004
- Confirmed in many experiments: Belle, BaBar, BESIII, CDF, CMS, D0, LHCb, ...
- 10 years later, $J^{PC} = 1^{++}$

  LHCb, PRL110(2013)222001

  $\Rightarrow$ S-wave coupling to $D\bar{D}^*$

Mysterious properties:

- Mass coincides with the $D^0\bar{D}^{*0}$ threshold:

  $M_{D^0} + M_{D^{*0}} - M_X = (0.01 \pm 0.14)$ MeV

  LHCb, PRD102(2020)092005

$X(3872)$ (2)

Mysterious properties (cont.):

- Large coupling to $D^0\bar{D}^{*0}$:

  $\mathcal{B}(X \to D^0\bar{D}^{*0}) > 30\%$ Belle, PRD81(2010)031103

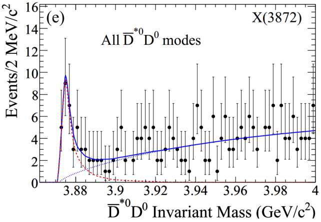
  $\mathcal{B}(X \to D^0\bar{D}^0\pi^0) > 40\%$ Belle, PRL97(2006)162002

- No isospin partner observed $\Rightarrow I = 0$

  but, large isospin breaking:

$$\frac{\mathcal{B}(X \to \omega J/\psi)}{\mathcal{B}(X \to \pi^+\pi^- J/\psi)} = 0.8 \pm 0.3$$

$$C(X) = +, C(J/\psi) = - \quad \Rightarrow C(\pi^+\pi^-) = - \quad \Rightarrow I(\pi^+\pi^-) = 1$$



BaBar, PRD77(2008)011102

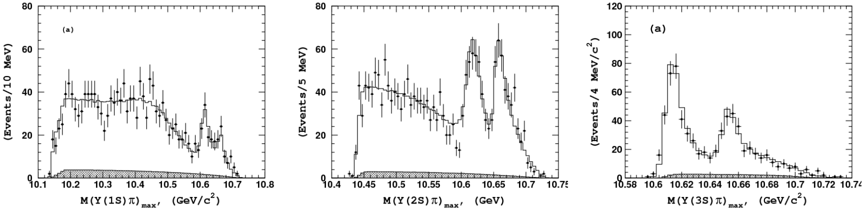
Exercise:

Why is the isospin of the negative C-parity $\pi^+\pi^-$ system equal to 1?

- $Z_c^\pm$, $Z_b^\pm$: charged structures in heavy quarkonium mass region, $Q\bar{Q}\bar{d}u, Q\bar{Q}\bar{u}d$
  $Z_c(3900)$, $Z_c(4020)$, $Z_c(4200)$, $Z_c(4430)$, ...
- $Z_b(10610)$ and $Z_b(10650)$ (the latter also called Z_b' sometimes):

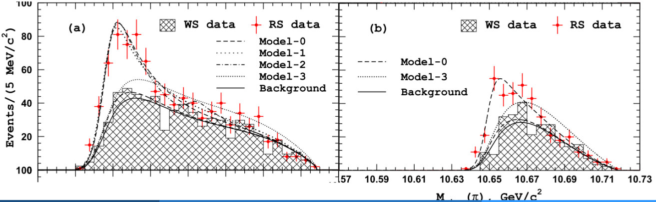
Belle, arXiv:1105.4583; PRL108(2012)122001

observed in $\Upsilon(10860) \to \pi^\mp[\pi^\pm\Upsilon(1S,2S,3S)/h_b(1P,2P)]$

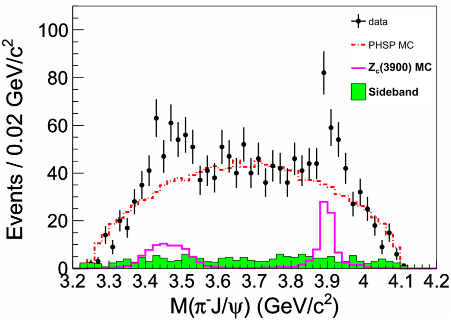


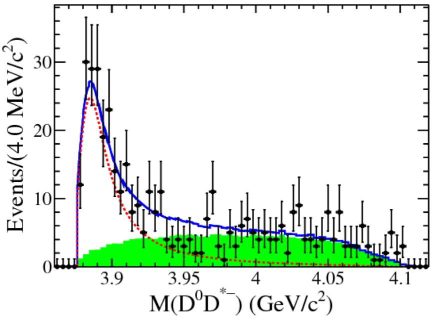
also in $\Upsilon(10860) \to \pi^\mp[B^{(*)}\bar{B}^*]^\pm$

Belle, arXiv:1209.6450; PRL116(2016)212001



- $Z_c(3900/3885)^\pm$: structure around 3.9 GeV seen in $J/\psi\pi$ by BESIII and Belle in $Y(4260) \to J/\psi\pi^+\pi^-$, BESIII, PRL110(2013)252001; Belle, PRL110(2013)252002
  and in $D\bar{D}^*$ by BESIII in $Y(4260) \to \pi^\pm(D\bar{D}^*)^\mp$ BESIII, PRD92(2015)092006





- $Z_c(4020)^\pm$ observed in $h_c\pi^\pm$ and $(\bar{D}^*D^*)^\pm$ distributions

BESIII, PRL111(2013)242001; PRL112(2014)132001

HQSS for XYZ (1)

- Hadronic molecular model: $X(3872)$: $D\bar{D}^*$; $Z_b(10610, 10650)$: $B\bar{B}^*$ and $B^*\bar{B}^*$
- Consider S-wave interaction between a pair of $s_\ell^P = \frac{1}{2}^-$ (anti-)heavy mesons:

$$0^{++}: \quad D\bar{D}, \quad D^*\bar{D}^*$$

$$1^{+-}: \quad \frac{1}{\sqrt{2}}\left(D\bar{D}^* + D^*\bar{D}\right), \quad D^*\bar{D}^*$$

$$1^{++}: \quad \frac{1}{\sqrt{2}}\left(D\bar{D}^* - D^*\bar{D}\right); \qquad 2^{++}: \quad D^*\bar{D}^*$$

here, phase convention: $D \xrightarrow{C} +\bar{D}$, $D^* \xrightarrow{C} -\bar{D}^*$

- Heavy quark spin irrelevant ⇒ interaction matrix elements:

$$\left\langle s_{1c}, s_{2c}, s_{c\bar{c}}; s_{1\,\ell}, s_{2\,\ell}, s_L; J \left| \hat{\mathcal{H}} \right| s'_{1c}, s'_{2c}, s'_{c\bar{c}}; s'_{1\,\ell}, s'_{2\,\ell}, s'_L; J' \right\rangle$$

$$= \left\langle s_{1\,\ell}, s_{2\,\ell}, s_L \left| \hat{\mathcal{H}} \right| s'_{1\,\ell}, s'_{2\,\ell}, s_L \right\rangle \delta_{s_{c\bar{c}}, s'_{c\bar{c}}} \delta_{s_L, s'_L} \delta_{JJ'}$$

For each isospin, 2 independent terms

$$\left\langle \frac{1}{2}, \frac{1}{2}, 0 \left| \hat{\mathcal{H}} \right| \frac{1}{2}, \frac{1}{2}, 0 \right\rangle, \qquad \left\langle \frac{1}{2}, \frac{1}{2}, 1 \left| \hat{\mathcal{H}} \right| \frac{1}{2}, \frac{1}{2}, 1 \right\rangle$$

⇒ 6 pairs grouped in 2 multiplets with $s_L = 0$ and 1, respectively

HQSS for XYZ (2)

- For the HQSS consequences, convenient to use the basis of states: $s_L^{PC} \otimes s_{c\bar{c}}^{PC}$
  - ☞ S-wave: $s_L^{PC}, s_{c\bar{c}}^{PC} = 0^{-+}$ or 1^{--}
  - ☞ multiplet with $s_L = 0$:

$$0_L^{-+} \otimes 0_{c\bar{c}}^{-+} = 0^{++}, \qquad 0_L^{-+} \otimes 1_{c\bar{c}}^{--} = 1^{+-}$$

  - ☞ multiplet with $s_L = 1$:

$$1_L^{--} \otimes 0_{c\bar{c}}^{-+} = 1^{+-}, \qquad 1_L^{--} \otimes 1_{c\bar{c}}^{--} = 0^{++} \oplus \boxed{\mathbf{1^{++}}} \oplus 2^{++}$$

- Multiplets in strict heavy quark limit:
  - ☞ $X(3872)$ has three partners with 0^{++}, 2^{++} and 1^{+-}

    Hidalgo-Duque et al., PLB727(2013)432; Baru et al., PLB763(2016)20
  - ☞ Z_b, Z_b' as $B^{(*)}\bar{B}^*$ molecules would imply 6 $I = 1$ hadronic molecules:
    $Z_b[1^{+-}], Z_b'[1^{+-}]$ and $W_{b0}[0^{++}]$, $W_{b0}'[0^{++}]$, $W_{b1}[1^{++}]$ and $W_{b2}[2^{++}]$

    Bondar et al., PRD84(2011)054010; Voloshin, PRD84(2011)031502;
    Mehen, Powell, PRD84(2011)114013

- Recall the exercise on page 50:

  Is $\Upsilon\pi^+\pi^-$ a good choice of final states for the search of X_b, the $J^{PC}=1^{++}$ bottom analogue of the $X(3872)$?

  Answer: No. $X_b \to \Upsilon\pi\pi$ breaks isospin symmetry

  FKG, Hidalgo-Duque, Nieves, Valderrama, PRD88(2013)054007; Karliner, Rosner, PRD91(2015)014014

  $M_{B^0} - M_{B^\pm} = (0.31 \pm 0.06)$ MeV [$M_{D^\pm} - M_{D^0} = (4.822 \pm 0.015)$ MeV]

- Negative results:

  CMS, *Search for a new bottomonium state decaying to $\Upsilon(1S)\pi^+\pi^-$ in pp collisions at $\sqrt{s}=8$ TeV*, PLB727(2013)57;

  ATLAS, *Search for the X_b and other hidden-beauty states in the $\pi^+\pi^-\Upsilon(1S)$ channel at ATLAS*, PLB740(2015)199

- The results can be reinterpreted as for the search of W_{bJ} ($I=1, J^{++}$)

$$1_L^{--} \otimes 1_{c\bar{c}}^{--} = 0^{++} \oplus \boxed{\mathbf{1}^{++}} \oplus 2^{++}$$

- Heavy quark spin selection rule for $X(3872)$: Voloshin, PLB604(2004)69
  for $X(3872)$ being a $\mathbf{1}^{++}$ $D\bar{D}^*$ molecule, $s_L = 1$, $s_{c\bar{c}} = 1$
- spin structure of $Q\bar{Q}$:

	s_L	$s_{c\bar{c}}$	J^{PC}	$c\bar{c}$
S-wave	0	0	0^{-+}	η_c
	0	1	1^{--}	J/ψ
P-wave	1	0	1^{+-}	h_c
	1	1	$(0,1,2)^{++}$	$\chi_{c0}, \chi_{c1}, \chi_{c2}$

- allowed: $X(3872) \to J/\psi\pi\pi$, $X(3872) \to \chi_{cJ}\pi$, $X(3872) \to \chi_{cJ}\pi\pi$
  suppressed: $X(3872) \to \eta_c\pi\pi$, $X(3872) \to h_c\pi\pi$
- Interesting feature of $Z_b^{(\prime)}$: observed with similar rates in both $\Upsilon\pi\pi[s_{b\bar{b}} = 1]$ and $h_b\pi\pi[s_{b\bar{b}} = 0]$ Bondar, Garmash, Milstein, Mizuk, Voloshin, PRD84(2011)054010

  $Z_b \sim B\bar{B}^* \sim 0_{b\bar{b}}^- \otimes 1_{q\bar{q}}^- - 1_{b\bar{b}}^- \otimes 0_{q\bar{q}}^-$, $\quad Z_b' \sim B^*\bar{B}^* \sim 0_{b\bar{b}}^- \otimes 1_{q\bar{q}}^- + 1_{b\bar{b}}^- \otimes 0_{q\bar{q}}^-$

HQSS for XYZ (5)

unitary transformation from two-meson basis to $|s_{1\,c}, s_{2\,c}, s_{c\bar c}; s_{1\,\ell}, s_{2\,\ell}, s_L; J\rangle$:

$$|s_{1\,c}, s_{1\,\ell}, j_1; s_{2\,c}, s_{2\,\ell}, j_2; J\rangle = \sum_{s_{c\bar c}, s_L} \sqrt{(2j_1+1)(2j_2+1)(2s_{c\bar c}+1)(2s_L+1)} \times \begin{Bmatrix} s_{1\,c} & s_{2\,c} & s_{c\bar c} \\ s_{1\,\ell} & s_{2\,\ell} & s_L \\ j_1 & j_2 & J \end{Bmatrix} |s_{1\,c}, s_{2\,c}, s_{c\bar c}; s_{1\,\ell}, s_{2\,\ell}, s_L; J\rangle$$

$j_{1,2}$: meson spins;

J: the total angular momentum of the whole system

$s_{1\,c(2\,c)} = \frac{1}{2}$: spin of the heavy quark in meson 1 (2)

$s_{1\,\ell(2\,\ell)} = \frac{1}{2}$: angular momentum of the light quarks in meson 1 (2)

- $s_{c\bar c} = 0, 1$: total spin of $c\bar c$, conserved but decoupled
- $s_L = 0, 1$: total angular momentum of the light-quark system, conserved
- only two independent $\left\langle s_{\ell 1}, s_{\ell 2}, s_L \middle| \hat{\mathcal{H}} \middle| s'_{\ell 1}, s'_{\ell 2}, s_L \right\rangle_I$ terms for each isospin I:

$$F_{I0} = \left\langle \tfrac{1}{2}, \tfrac{1}{2}, 0 \middle| \hat{\mathcal{H}} \middle| \tfrac{1}{2}, \tfrac{1}{2}, 0 \right\rangle_I, \quad F_{I1} = \left\langle \tfrac{1}{2}, \tfrac{1}{2}, 1 \middle| \hat{\mathcal{H}} \middle| \tfrac{1}{2}, \tfrac{1}{2}, 1 \right\rangle_I$$

Exercise:

Show that the combinations of the LO contact terms in the S-wave interaction matrix elements for $D^{(*)}\bar{D}^{(*)}$ are as follows:

$$\begin{pmatrix} D\bar{D} \\ D^*\bar{D}^* \end{pmatrix}: \quad V^{(0^{++})} = \begin{pmatrix} C_{IA} & \sqrt{3}C_{IB} \\ \sqrt{3}C_{IB} & C_{IA} - 2C_{IB} \end{pmatrix},$$

$$\begin{pmatrix} D\bar{D}^* \\ D^*\bar{D}^* \end{pmatrix}: \quad V^{(1^{+-})} = \begin{pmatrix} C_{IA} - C_{IB} & 2C_{IB} \\ 2C_{IB} & C_{IA} - C_{IB} \end{pmatrix},$$

$$D\bar{D}^*: \quad V^{(1^{++})} = C_{IA} + C_{IB},$$

$$D^*\bar{D}^*: \quad V^{(2^{++})} = C_{IA} + C_{IB},$$

here, $C_{IA} = \frac{1}{4}(3F_{I1} + F_{I0})$, $C_{IB} = \frac{1}{4}(F_{I1} - F_{I0})$

- This would suggest spin multiplets. Good candidates:
  - ☞ $X(3872)$ and $X_2(4013)$ (not observed yet!); $Z_c(3900)$ and $Z_c(4020)$

    Nieves, Valderrama, PRD86(2012)056004; ...

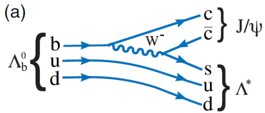
$$M_{X_2(4013)} - M_{X(3872)} \approx M_{Z_c(4020)} - M_{Z_c(3900)} \approx M_{D^*} - M_D$$

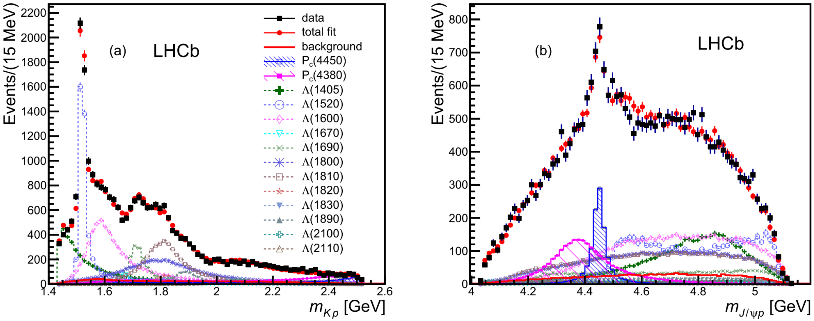
  - ☞ $Z_b(10610)$ and $Z_b(10650)$: Bondar et al., PRD84(2011)054010; ...

$$M_{Z_b(10650)} - M_{Z_b(10610)} \approx M_{B^*} - M_B$$

  - ☞ Z_c and Z_b states need a suppression of coupled-channel effect (reason?)

Discovered in $\Lambda_b^0 \to {\color{red}J/\psi p}K^-$ LHCb, PRL115(2015)072001 [arXiv:1507.03414]





Two Breit–Wigner resonances needed:

$$M_1 = (4380 \pm 8 \pm 29)\ \text{MeV}, \qquad \Gamma_1 = (205 \pm 18 \pm 86)\ \text{MeV},$$

$$M_2 = (4449.8 \pm 1.7 \pm 2.5)\ \text{MeV}, \qquad \Gamma_2 = (39 \pm 5 \pm 19)\ \text{MeV}.$$

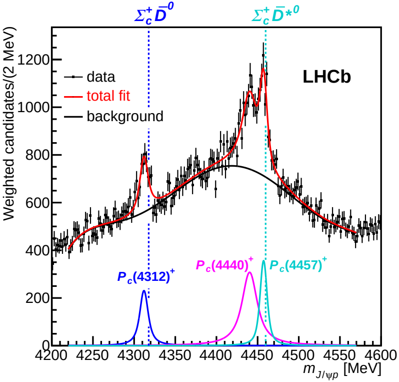
LHCb's P_c (2)

- In $J/\psi\, p$ invariant mass distribution, with hidden charm
  $\Rightarrow$ pentaquarks if they are hadron resonances
- Quantum numbers not fully determined, for ($P_c(4380), P_c(4450)$):
  $(3/2^-, 5/2^+)$, $(3/2^+, 5/2^-)$, $(5/2^+, 3/2^-)$, $\ldots$ LHCb, PRL115(2015)072001

  From a reanalysis using an extended Λ^* model: N. Jurik, CERN-THESIS-2016-086

		$P_c(4380)$		$P_c(4450)$	
$J^p(4380, 4450)$	$(\sqrt{\Delta(-2\ln\mathcal{L})})^2$	M_0	Γ_0	M_0	Γ_0
	$(3/2-, 5/2^+)$ solution				
$3/2^-, 5/2^+$	$--$	4359	151	4450.1	49
	Δ from $(3/2-, 5/2^+)$ solution				
$5/2^+, 3/2^-$	-3.6^2	10	-7	-1.6	-6
$5/2^-, 3/2^+$	-2.7^2	-4	-9	-3.6	-2
$3/2^-, 5/2^+$	$-$	$-$	$-$	$-$	$-$

- Early prediction:
  Prediction of narrow N^* and Λ^* resonances with hidden charm above 4 GeV,
  J.-J. Wu, R. Molina, E. Oset, B.-S. Zou, PRL105(2010)232001



State	M [MeV]	Γ [MeV]	(95% CL)	$\mathcal{R}$ [%]
$P_c(4312)^+$	$4311.9 \pm 0.7^{+6.8}_{-0.6}$	$9.8 \pm 2.7^{+\ 3.7}_{-\ 4.5}$	(< 27)	$0.30 \pm 0.07^{+0.34}_{-0.09}$
$P_c(4440)^+$	$4440.3 \pm 1.3^{+4.1}_{-4.7}$	$20.6 \pm 4.9^{+\ 8.7}_{-10.1}$	(< 49)	$1.11 \pm 0.33^{+0.22}_{-0.10}$
$P_c(4457)^+$	$4457.3 \pm 0.6^{+4.1}_{-1.7}$	$6.4 \pm 2.0^{+\ 5.7}_{-\ 1.9}$	(< 20)	$0.53 \pm 0.16^{+0.15}_{-0.13}$

$$\mathcal{R} \equiv \mathcal{B}(\Lambda_b^0 \to P_c^+ K^-)\mathcal{B}(P_c^+ \to J/\psi p)/\mathcal{B}(\Lambda_b^0 \to J/\psi p K^-)$$

HQSS for P_c (1)

The LHCb P_c states might be $\Sigma_c^{(*)}\bar{D}^{(*)}$ molecules predicted in Wu, Molina, Oset, Zou (2010)

$P_c(4312) \sim \Sigma_c\bar{D}$, $P_c(4440, 4457) \sim \Sigma_c\bar{D}^*$

Consider S-wave pairs of $\Sigma_c^{(*)}\bar{D}^{(*)}$ [$J_{\Sigma_c} = \frac{1}{2}, J_{\Sigma_c^*} = \frac{3}{2}$]:

$$\begin{aligned} J^P &= \frac{1}{2}^- : \quad \Sigma_c\bar{D},\ \Sigma_c\bar{D}^*,\ \Sigma_c^*\bar{D}^* \\ J^P &= \frac{3}{2}^- : \quad \Sigma_c^*\bar{D},\ \Sigma_c\bar{D}^*,\ \Sigma_c^*\bar{D}^* \\ J^P &= \frac{5}{2}^- : \quad \Sigma_c^*\bar{D}^* \end{aligned}$$

Spin of the light degrees of freedom s_ℓ: $s_\ell(D^{(*)}) = \frac{1}{2}$, $s_\ell(\Sigma_c^{(*)}) = 1$. Thus, $s_L = \frac{1}{2}, \frac{3}{2}$

For each isospin, 2 independent terms

$$\left\langle 1, \frac{1}{2}, \frac{1}{2} \middle| \hat{\mathcal{H}} \middle| 1, \frac{1}{2}, \frac{1}{2} \right\rangle, \qquad \left\langle 1, \frac{1}{2}, \frac{3}{2} \middle| \hat{\mathcal{H}} \middle| 1, \frac{1}{2}, \frac{3}{2} \right\rangle$$

Thus, the 7 pairs are in two spin multiplets: 3 with $s_L = \frac{1}{2}$ and 4 with $s_L = \frac{3}{2}$

HQSS for P_c (2)

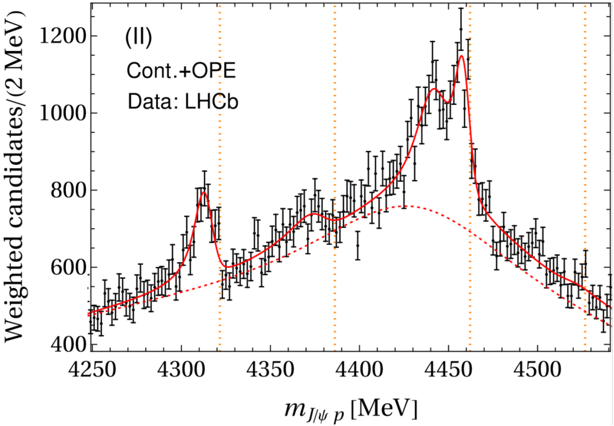
Seven P_c generally expected in this hadronic molecular model Xiao, Nieves, Oset (2013); Liu et al. (2018, 2019); Sakai et al. (2019); ...

Predictions using the masses of $P_c(4440, 4457)$ as inputs Liu et al., PRL122(2019)242001

Scenario	Molecule	J^P	B (MeV)	M (MeV)
A	$\bar{D}\Sigma_c$	$\frac{1}{2}^-$	7.8 – 9.0	4311.8 – 4313.0
A	$\bar{D}\Sigma_c^*$	$\frac{3}{2}^-$	8.3 – 9.2	4376.1 – 4377.0
A	$\bar{D}^*\Sigma_c$	$\frac{1}{2}^-$	Input	4440.3
A	$\bar{D}^*\Sigma_c$	$\frac{3}{2}^-$	Input	4457.3
A	$\bar{D}^*\Sigma_c^*$	$\frac{1}{2}^-$	25.7 – 26.5	4500.2 – 4501.0
A	$\bar{D}^*\Sigma_c^*$	$\frac{3}{2}^-$	15.9 – 16.1	4510.6 – 4510.8
A	$\bar{D}^*\Sigma_c^*$	$\frac{5}{2}^-$	3.2 – 3.5	4523.3 – 4523.6
B	$\bar{D}\Sigma_c$	$\frac{1}{2}^-$	13.1 – 14.5	4306.3 – 4307.7
B	$\bar{D}\Sigma_c^*$	$\frac{3}{2}^-$	13.6 – 14.8	4370.5 – 4371.7
B	$\bar{D}^*\Sigma_c$	$\frac{1}{2}^-$	Input	4457.3
B	$\bar{D}^*\Sigma_c$	$\frac{3}{2}^-$	Input	4440.3
B	$\bar{D}^*\Sigma_c^*$	$\frac{1}{2}^-$	3.1 – 3.5	4523.2 – 4523.6
B	$\bar{D}^*\Sigma_c^*$	$\frac{3}{2}^-$	10.1 – 10.2	4516.5 – 4516.6
B	$\bar{D}^*\Sigma_c^*$	$\frac{5}{2}^-$	25.7 – 26.5	4500.2 – 4501.0

Fit to the LHCb measured $J/\psi p$ invariant mass distribution using hadronic molecular model with HQSS

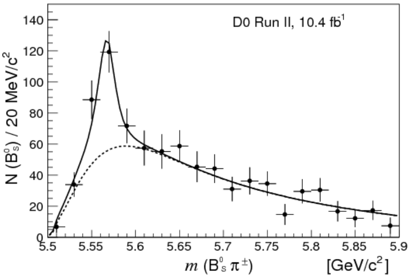
M.-L. Du, V. Baru, F.-K. Guo, C. Hanhart, U.-G. Meißner, J. A. Oller, Q. Wang, PRL124(2020)072001



Scenario B in the previous page in favored in the fit after considering the one-pion exchange.

Explicitly exotic: $X(5568)$ (1)

- $X(5568)$ by D0 Collaboration ($p\bar{p}$ collisions) PRL117(2016)022003



$$M = \left(5567.8 \pm 2.9^{+0.9}_{-1.9}\right) \text{ MeV}$$

$$\Gamma = \left(21.9 \pm 6.4^{+5.0}_{-2.5}\right) \text{ MeV}$$

- Observed in $B_s^{(*)0}\pi^+$, sizeable width $\Rightarrow I = 1$:

  minimal quark contents is $\bar{b}s\bar{d}u$!

- a favorite mulqituark candidate:

  explicitly flavor exotic, minimal number of quarks ≥ 4

Estimate of isospin breaking decay width:

$$\Gamma_{\not{I}} \sim \left(\frac{\left(\frac{m_d - m_u}{\Lambda_{\rm QCD}}\right)^2}{\alpha^2} \right) \times \mathcal{O}(100\text{ MeV})$$

$$= \mathcal{O}(10\text{ keV})$$

$X(5568)$ (2)

FKG, Meißner, Zou, *How the X(5568) challenges our understanding of QCD*, Commun.Theor.Phys. 65 (2016) 593

- mass too low for $X(5568)$ to be a $\bar{b}s\bar{u}d$: $M \simeq M_{B_s} + 200$ MeV
  - ☞ $M_\pi \simeq 140$ MeV because pions are pseudo-Goldstone bosons
  - ☞ For any matter field: $M_R \gg M_\pi$; we expect $M_{\bar{q}q} \sim M_R \gtrsim M_\sigma$

$$M_{\bar{b}s\bar{u}d} \gtrsim M_{B_s} + 500 \text{ MeV} \sim 5.9 \text{ GeV}$$

- HQFS predicts an isovector X_c:

$$M_{X_c} = M_{X(5568)} - \Delta_{b-c} + \mathcal{O}\left(\Lambda^2_{\text{QCD}}\left(\frac{1}{m_c} - \frac{1}{m_b}\right)\right) \simeq (2.24 \pm 0.15) \text{ GeV}$$

but in $D_s\pi$, only the isoscalar $D^*_{s0}(2317)$ was observed! BaBar (2003)

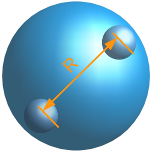
☞ negative results reported by LHCb, LHCb, PRL117(2016)152003
by CMS, CMS, PRL120(2018)202005
by CDF, CDF, PRL120(2018)202006
by ATLAS ATLAS, PRL120(2018)202007

Hadronic molecules

FKG, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004

- Hadronic molecule:
  dominant component is a composite state of 2 or more hadrons
- Concept at large distances, so that can be approximated by system of multi-hadrons at low energies
  Consider a 2-body bound state with a mass $M = m_1 + m_2 - E_B$

size: $\sim \frac{1}{\sqrt{2\mu E_B}} \gg r_{\text{hadron}}$



- scale separation $\Rightarrow$ (nonrelativistic) EFT applicable!
- Only narrow hadrons can be considered as components of hadronic molecules, $\Gamma_h \ll 1/r$, r: range of forces

Filin *et al.*, PRL105(2010)019101; FKG, Meißner, PRD84(2011)014013

Compositeness (1)

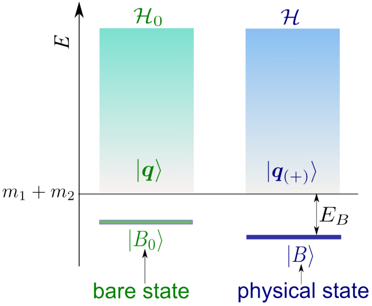
S. Weinberg, PR137(1965)B672; V. Baru *et al.*, PLB586(2004)53; T. Hyodo, IJMPA28(2013)1330045, ...

Model-independent result for S-wave loosely bound composite states:

Consider a system with Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + V$$

$\mathcal{H}_0$: free Hamiltonian, V: interaction potential



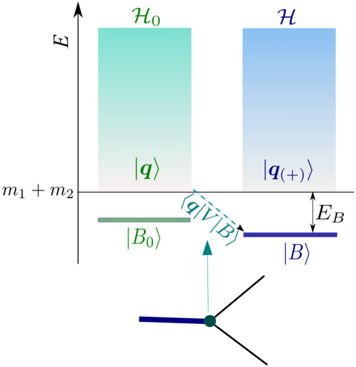
- Compositeness:
  the probability of finding the physical state $|B\rangle$ in the 2-body continuum $|\boldsymbol{q}\rangle$

$$1 - Z = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \left|\langle \boldsymbol{q}|B\rangle\right|^2$$

- $Z = \left|\langle B_0|B\rangle\right|^2, \qquad 0 \le (1 - Z) \le 1$
  - ☞ $Z = 0$: pure bound (composite) state
  - ☞ $Z = 1$: pure elementary state

Compositeness (2)

Compositeness : $1 - Z = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} |\langle \boldsymbol{q}|B\rangle|^2$



- Schrödinger equation

$$(\mathcal{H}_0 + V)|B\rangle = -E_B|B\rangle$$

multiplying by $\langle \boldsymbol{q}|$ and using $\mathcal{H}_0|\boldsymbol{q}\rangle = \frac{\boldsymbol{q}^2}{2\mu}|\boldsymbol{q}\rangle$:
$\Rightarrow$ momentum-space wave function:

$$\langle \boldsymbol{q}|B\rangle = -\frac{\langle \boldsymbol{q}|V|B\rangle}{E_B + \boldsymbol{q}^2/(2\mu)}$$

- S-wave, small binding energy so that $R = 1/\sqrt{2\mu E_B} \gg r$, r: range of forces

$$\langle \boldsymbol{q}|V|B\rangle = g_{\rm NR}\left[1 + \mathcal{O}(r/R)\right]$$

- Compositeness:

$$1 - Z = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{g_{\rm NR}^2}{\left[E_B + \boldsymbol{q}^2/(2\mu)\right]^2}\left[1 + \mathcal{O}\left(\frac{r}{R}\right)\right] = \frac{\mu^2 g_{\rm NR}^2}{2\pi\sqrt{2\mu E_B}}\left[1 + \mathcal{O}\left(\frac{r}{R}\right)\right]$$

Compositeness (3)

- Coupling constant measures the compositeness for an S-wave shallow bound state
$$g_{\rm NR}^2 \approx (1-Z)\frac{2\pi}{\mu^2}\sqrt{2\mu E_B} \leq \frac{2\pi}{\mu^2}\sqrt{2\mu E_B}$$
bounded from the above

It can be shown that $g_{\rm NR}^2$ is the residue of the T-matrix element at the pole $E = -E_B$ ($E \equiv \sqrt{s} - m_1 - m_2$) :

$$g_{\rm NR}^2 = \lim_{E\to -E_B} (E+E_B)\langle \boldsymbol{k}|T_{\rm NR}|\boldsymbol{k}\rangle$$

here nonrelativistic normalization is used, comparing with the T-matrix using relativistic normalization in the first part of this lecture (also a sign difference in the definition), $T_{\rm NR} = -\frac{T}{4\mu\sqrt{s}} \simeq -\frac{T}{4m_1m_2}$.

Hint: use the Lippmann–Schwinger equation $T_{\rm NR} = V + V\frac{1}{E-\mathcal{H}_0+i\epsilon}T_{\rm NR}$ and the completeness relation $|B\rangle\langle B| + \int\frac{d^3q}{(2\pi)^3}|\boldsymbol{q}_{(+)}\rangle\langle \boldsymbol{q}_{(+)}| = 1$ to derive the Low equation (noticing $T_{\rm NR}|\boldsymbol{q}\rangle = V|\boldsymbol{q}_{(+)}\rangle$):

$$\langle \boldsymbol{k}'|T_{\rm NR}|\boldsymbol{k}\rangle = \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle + \frac{\langle \boldsymbol{k}'|V|B\rangle\langle B|V|\boldsymbol{k}\rangle}{E+E_B+i\epsilon} + \int\frac{d^3q}{(2\pi)^3}\frac{\langle \boldsymbol{k}'|T_{\rm NR}|\boldsymbol{q}\rangle\langle \boldsymbol{q}|T_{\rm NR}^\dagger|\boldsymbol{k}\rangle}{E-\boldsymbol{q}^2/(2\mu)+i\epsilon}$$

- Z can be related to scattering length a and effective range r_e Weinberg (1965)

$$a_0 = \frac{2R(1-Z)}{2-Z}\left[1+\mathcal{O}\left(\frac{r}{R}\right)\right], \quad r_{e0} = -\frac{RZ}{1-Z}\left[1+\mathcal{O}\left(\frac{r}{R}\right)\right]$$

Effective range expansion (S-wave): $f_0^{-1}(k) = -1/a_0 + r_{e0}k^2/2 - ik + \mathcal{O}\left(k^4\right)$

Derivation:

$$T_{\rm NR}(E) \equiv \langle k|T_{\rm NR}|k\rangle = -\frac{2\pi}{\mu}f_0(k) \ \Rightarrow \ \mathrm{Im}\,T_{\rm NR}^{-1}(E) = \frac{\mu}{2\pi}\sqrt{2\mu E}\,\theta(E)$$

Twice-subtracted dispersion relation for $t^{-1}(E)$

$$T_{\rm NR}^{-1}(E) = \frac{E+E_B}{g_{\rm NR}^2} + \frac{(E+E_B)^2}{\pi}\int_0^{+\infty} dw \frac{\mathrm{Im}\,T_{\rm NR}^{-1}(w)}{(w-E-i\epsilon)(w+E_B)^2}$$

$$= \frac{E+E_B}{g_{\rm NR}^2} + \frac{\mu R}{4\pi}\left(\frac{1}{R} - \sqrt{-2\mu E - i\epsilon}\right)^2$$

- Purely composite: $Z=0 \Rightarrow a_0 = \frac{1}{\sqrt{2\mu E_B}}, r_{e0} = 0$
- Purely elementary: $Z=1 \Rightarrow a_0 = 0, r_{e0} = -\infty$

- Classic example: deuteron as pn bound state. Exp.: $E_B = 2.2$ MeV, $a_{0[^3S_1]} = 5.4$ fm, $r_{e0[^3S_1]} = 1.8$ fm

$$a_{Z=1} \simeq 0 \text{ fm}, \qquad a_{Z=0} \simeq 4.3 \text{ fm}$$

- However, problematic for systems with positive effective range I. Matuschek, V. Baru, FKG, C. Hanhart, EPJA 57 (2021) 101; Y. Li, FKG, J.-Y. Pang, J.-J. Wu, PRD 105 (2022) L071502

  $r_{e0} > 0 \Rightarrow Z \notin (0,1)$, thus Z or $1 - Z = \sqrt{\frac{a_0}{a_0 + 2r_{e0}}}$ in Weinberg's relations loses a probability interpretation

  A generalization is suggested in Y. Li, FKG, J.-Y. Pang, J.-J. Wu, PRD 105 (2022) L071502

$$Z = \exp\left(\frac{1}{\pi}\int_0^\infty dE \frac{\delta_0(E)}{E - E_B}\right) \in [0,1]$$

  which reduces to Weinberg's relations for $r_{e0} < 0$.

NREFT at LO (1)

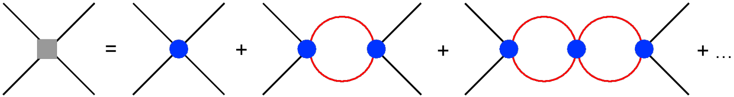
We consider a system of two particles of masses m_1, m_2

- in the near-threshold region, a momentum expansion for the interactions with the LO being a constant

$$\mathcal{L} = \sum_{i=1,2} \phi_i^\dagger \left(i\partial_0 - m_i + \frac{\nabla^2}{2m_i} \right) \phi_i - C_0\, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 + \ldots$$

  nonrelativistic propagator: $\dfrac{i}{p^0 - m_i - \boldsymbol{p}^2/(2m_i) + i\epsilon}$

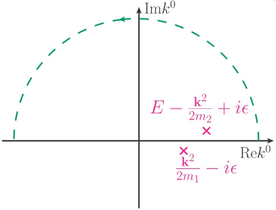
- to have a near-threshold bound state (hadronic molecule)



$$T_{\rm NR}(E) = C_0 + C_0\, G_{\sf NR}(E)\, C_0 + C_0\, G_{\sf NR}(E)\, C_0\, G_{\sf NR}(E)\, C_0 + \ldots$$
$$= \frac{1}{{C_0}^{-1} - G_{\sf NR}(E)}$$

- The loop integral is linearly divergent (E defined relative to $m_1 + m_2$), regularized with, e.g., a sharp cut

$$G_{\rm NR}(E) = i\int\frac{d^3\boldsymbol{k}dk^0}{(2\pi)^4}\left[\left(k^0 - \frac{\boldsymbol{k}^2}{2m_1} + i\epsilon\right)\left(E - k^0 - \frac{\boldsymbol{k}^2}{2m_2} + i\epsilon\right)\right]^{-1}$$



$$= -i2\mu(2\pi i)\int^\Lambda\frac{d^3\boldsymbol{k}}{(2\pi)^4}\frac{1}{2\mu E - \boldsymbol{k}^2 + i\epsilon}$$

$$= -\frac{\mu}{\pi^2}\left(\Lambda - \sqrt{-2\mu E - i\epsilon}\,\arctan\frac{\Lambda}{\sqrt{-2\mu E - i\epsilon}}\right)$$

$$= -\frac{\mu}{\pi^2}\Lambda + \frac{\mu}{2\pi}\sqrt{-2\mu E - i\epsilon} + \mathcal{O}\left(\Lambda^{-1}\right)$$

for real E, $\sqrt{-2\mu E - i\epsilon} = \sqrt{-2\mu E}\,\theta(-E) - i\sqrt{2\mu E}\,\theta(E)$

- Renormalization: $T_{\rm NR}$ is Λ-independent,

$$T_{\rm NR}(E) = \frac{1}{C_0^{-1} - G_{\rm NR}} = \Bigg(\underbrace{\frac{1}{C_0} + \frac{\mu}{\pi^2}\Lambda}_{\equiv 1/C_0^r} - \frac{\mu}{2\pi}\sqrt{-2\mu E - i\epsilon}\Bigg)^{-1} = \frac{2\pi/\mu}{2\pi/(\mu\, C_0^r) - \sqrt{-2\mu E - i\epsilon}}$$

- Other regularization can be used as well, equivalent to the sharp cutoff up to $1/\Lambda$ suppressed terms, e.g.
  - ☞ with a Gaussian regulator $\exp\left(-\boldsymbol{k}^2/\Lambda_{\rm G}^2\right)$, $\Lambda_{\rm G} = \sqrt{2/\pi}\Lambda$
  - ☞ with the power divergence subtraction (PDS) scheme in dimensional regularization by letting, $\Lambda_{\rm PDS} = 2\Lambda/\pi$ Kaplan, Savage, Wise (1998)

$$T_{\rm NR}(E) = \frac{2\pi/\mu}{2\pi/(\mu\, C_0^r) - \sqrt{-2\mu E - i\epsilon}} = \frac{2\pi/\mu}{2\pi/(\mu\, C_0^r) + i\, k}$$

- from matching to effective range expansion,

$$f_0^{-1}(k) = -\frac{2\pi}{\mu} T_{\rm NR}^{-1} = -\frac{1}{a_0} + \frac{1}{2} r_{e0} k^2 - i\, k + \mathcal{O}\left(k^4\right)$$

$2\pi/(\mu C_0^r) = 1/a_0$; higher terms are necessary to match both a and r_e

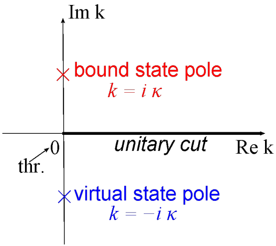
- pole below threshold at $E = -E_B$ with $E_B > 0$

$\kappa \equiv \left|\sqrt{2\mu E_B}\right|$

  - ☞ bound state pole, in the 1st Riemann sheet

    $\Rightarrow 2\pi/(\mu C_0^r) = \kappa$
  - ☞ virtual state pole, in the 2nd Riemann sheet

    $\Rightarrow 2\pi/(\mu C_0^r) = -\kappa$

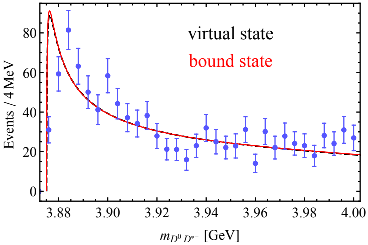


  - ☞ unable to get a resonance pole at LO with a single channel

Bound state and virtual state

- If the same binding energy, bound and virtual states cannot be distinguished above threshold ($E > 0$):

$$|T_{\rm NR}(E)|^2 \propto \left| \frac{1}{\pm\kappa + i\sqrt{2\mu E}} \right|^2 = \frac{1}{\kappa^2 + 2\mu E}$$

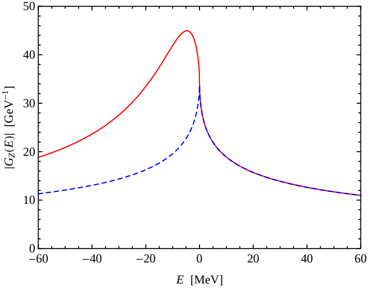


- Bound state and virtual state are different below threshold $(E < 0)$:
  - ☞ bound state: peaked below threshold

$$|T_{\rm NR}(E)|^2 \propto \frac{1}{(\kappa - \sqrt{-2\mu E})^2}$$

  - ☞ virtual state: a sharp cusp at threshold

$$|T_{\rm NR}(E)|^2 \propto \frac{1}{(\kappa + \sqrt{-2\mu E})^2}$$



Lower Fig.: bound state and virtual state with $E_B = 5$ MeV and a small width to the inelastic channel

For complexity of near-threshold line shapes, see X.-K. Dong, FKG, B.-S. Zou, PRL126(2021)152001

Coupling constant for S-wave bound state

$$T_{\rm NR}(E) = \frac{2\pi/\mu}{2\pi/(\mu\, C_0^r) - \sqrt{-2\mu E - i\epsilon}}$$

At LO, effective coupling strength for bound state

$$g_{\rm NR}^2 = \lim_{E\to -E_B}(E+E_B)\, T_{\rm NR}(E) = -\frac{2\pi}{\mu}\left(\frac{d}{dE}\sqrt{-2\mu E - i\epsilon}\right)^{-1}_{E=-E_B}$$

$$= \frac{2\pi}{\mu^2}\sqrt{2\mu E_B}$$

Recall the compositeness formula:

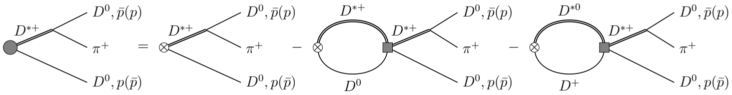
$$g_{\rm NR}^2 = (1-Z)\frac{2\pi}{\mu^2}\sqrt{2\mu E_B}$$

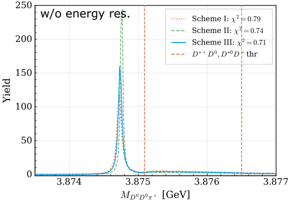
This means that the pole obtained at LO NREFT with only a constant contact term corresponds to a purely composite state ($Z = 0$)

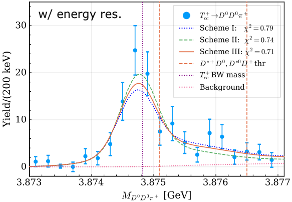
Range corrections: other components at shorter distances

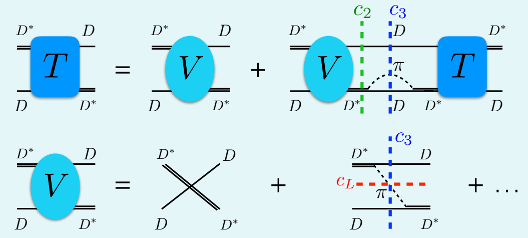
- energy/momentum-dependent interactions: higher order
- coupling to additional states/channels

- Consider $D^0\bar{D}^{*+}$-D^+D^{*0} coupled-channel system
- Contact term (two low-energy constants $C_{I=0}$ and $C_{I=1}$, assuming $C_{I=1}$ vanish) + one-pion exchange; 3-body effects ($2m_D + m_\pi \simeq m_D + m_{D^*}$)









- Pole w.r.t. $D^{*+}D^0$ th: $-356^{+39}_{-38} - i(28 \pm 1)\,\mathrm{keV}$
- Compositeness of $T_{cc}(3875)^+$:

  $X_{D^{*+}D^0} = 0.73 \pm 0.11, \quad X_{D^{*0}D^+} = 0.27 \pm 0.02$

Recent reviews on new hadrons (incomplete list)

Lots of new hadron resonances and resonance-like structures were found since 2003

- H.-X. Chen et al., *The hidden-charm pentaquark and tetraquark states*, Phys. Rept. 639 (2016) 1 [arXiv:1601.02092]
- A. Hosaka et al., *Exotic hadrons with heavy flavors — X, Y, Z and related states*, Prog. Theor. Exp. Phys. 2016, 062C01 [arXiv:1603.09229]
- R. F. Lebed, R. E. Mitchell, E. Swanson, *Heavy-quark QCD exotica*, Prog. Part. Nucl. Phys. 93 (2017) 143 [arXiv:1610.04528]
- A. Esposito, A. Pilloni, A. D. Polosa, *Multiquark resonances*, Phys. Rept. 668 (2017) 1 [arXiv:1611.07920]
- F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004 [arXiv:1705.00141]
- S. L. Olsen, T. Skwarnicki, *Nonstandard heavy mesons and baryons: Experimental evidence*, Rev. Mod. Phys. 90 (2018) 015003 [arXiv:1708.04012]
- M. Karliner, J. L. Rosner, T. Skwarnicki, *Multiquark states*, Ann. Rev. Nucl. Part. Sci. 68 (2018) 17 [arXiv:1711.10626]
- C.-Z. Yuan, *The XYZ states revisited*, Int. J. Mod. Phys. A 33 (2018) 1830018 [arXiv:1808.01570]
- N. Brambilla et al., *The XYZ states: experimental and theoretical status and perspectives*, Phys. Rept. 873 (2020) 154 [arXiv:1907.07583]
- F.-K. Guo, X.-H. Liu, S. Sakai, *Threshold cusps and triangle singularities in hadronic reactions*, Prog. Part. Nucl. Phys. 112 (2020) 103757 [arXiv:1912.07030]
- . . .

Thank you for your attention!

Bispinor fields for heavy mesons (1)

- $\frac{1}{2}(1+\not{v})$ projects onto the particle component of the heavy quark spinor.
- Convenient to introduce heavy mesons as bispinors:

$$H_a = \frac{1+\not{v}}{2}\left[P_a^{*\mu}\gamma_\mu - P_a\gamma_5\right], \qquad \bar{H}_a = \gamma_0 H_a^\dagger \gamma_0$$

$P = \{Q\bar{u}, Q\bar{d}, Q\bar{s}\}$: pseudoscalar heavy mesons, P^*: vector heavy mesons

For hadrons with arbitrary spin, see A. Falk, NPB378(1992)79

Charge conjugation:

☞ H_a destroys mesons containing a Q, but does not create mesons with a $\bar{Q}$

☞ Free to choose the phase convention for charge conjugation. If we use, e.g.,

$$P_a \xrightarrow{C} +P_a^{(\bar{Q})}, \qquad P_{a,\mu}^* \xrightarrow{C} -P_{a,\mu}^{*(\bar{Q})},$$

then the fields annihilating mesons containing a $\bar{Q}$ is ($\mathcal{C} = i\gamma^2\gamma^0$)

$$H_a^{(\bar{Q})} = \mathcal{C}\left[\frac{1+\not{v}}{2}\Big(\underbrace{-P_{a,\mu}^{*(\bar{Q})}}_{P_{a,\mu}^{*}\xrightarrow{C}}\gamma^\mu - \underbrace{P_a^{(\bar{Q})}}_{P_a\xrightarrow{C}}\gamma_5\Big)\right]^T \mathcal{C}^{-1}$$

$$= \left(+P_{a,\mu}^{*(\bar{Q})}\gamma^\mu - P_a^{(\bar{Q})}\gamma_5\right)\frac{1-\not{v}}{2}$$

Bispinor fields for heavy mesons (2)

- Free heavy-meson Lagrangian:

$$\mathcal{L}_{\text{free}} = -i\,\mathrm{Tr}\left[\bar{H}_a v_\mu \partial^\mu H_a\right] = 2i\,P_a^\dagger v_\mu \partial^\mu P_a - 2i\,P_{a\,\nu}^{*\dagger} v_\mu \partial^\mu P_a^{*\,\nu}$$

Tr: trace in the spinor space, a, b: indices in the light flavor space

- Notice that the mass dimension of H_a is 3/2.

Nonrelativistic normalization: $H_a \simeq \sqrt{M_H}\,H_a^{\text{rel.}}$

D-meson propagator:

$$\frac{i}{2v\cdot k + i\epsilon} \simeq M_H \times \underbrace{\frac{i}{p^2 - M_H^2 + i\epsilon}}_{=\frac{i}{2M_H v\cdot k + i\epsilon}\left[1+\mathcal{O}\left(k^2/M_H^2\right)\right]} \qquad (p = M_H v + k)$$

In some papers, the normalization factor is $\sqrt{2M_H}$, instead of $\sqrt{M_H}$. Then the corresponding propagator should be

$$\frac{i}{v\cdot k + i\epsilon} \quad \text{or} \quad \frac{i}{k^0 - \vec{k}^{\,2}/(2M_H) + i\epsilon}$$

Simplified two-component notation

The superfield for pseudoscalar and vector heavy mesons: (in this page, we use $^{(4)}$ to mean the usual 4-component notation)

$$H_a^{(4)} = \frac{1+\not{v}}{2}\left[P_a^{*\mu}\gamma_\mu - P_a\gamma_5\right]$$

In the rest frame of heavy meson, $v^\mu = (1, \mathbf{0})$. We take the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Simplifications: $\dfrac{1+\not{v}}{2} = \dfrac{1+\gamma^0}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$

$$H_a^{(4)} = \begin{pmatrix} 0 & -(P_a + \boldsymbol{P}_a^* \cdot \boldsymbol{\sigma}) \\ 0 & 0 \end{pmatrix}, \qquad \bar{H}_a^{(4)} = \begin{pmatrix} 0 & 0 \\ (P_a^\dagger + \boldsymbol{P}_a^{*\dagger} \cdot \boldsymbol{\sigma}) & 0 \end{pmatrix}$$

Thus, it is convenient to simply use the two-component notation

$$H_a = P_a + \boldsymbol{P}_a^* \cdot \boldsymbol{\sigma}, \qquad H_a^{(4)} \to -H_a, \qquad \bar{H}_a^{(4)} \to H_a^\dagger$$

Mass splittings among heavy mesons (1)

- Spin-dependent term ⇒ mass difference between vector and pseudoscalar mesons ($\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$)

$$\mathcal{L}_\Delta = \frac{\lambda_2}{m_Q}\mathrm{Tr}\left[\bar{H}_a^{(4)}\sigma_{\mu\nu}H_a^{(4)}\sigma^{\mu\nu}\right] = -\frac{2\lambda_2}{m_Q}\mathrm{Tr}\left[H_a^\dagger\sigma^i H_a\sigma^i\right]$$
$$= \frac{4\lambda_2}{m_Q}\left(\boldsymbol{P}_a^{*\,\dagger}\cdot\boldsymbol{P}_a^* - 3P_a^\dagger P_a\right) \qquad H_a = P_a + \boldsymbol{P}_a^*\cdot\boldsymbol{\sigma}$$

$$\Rightarrow \qquad M_{P_a^*} - M_{P_a} = -\frac{8\lambda_2}{m_Q}$$

- Thus, we expect

$$\frac{M_{B^*} - M_B}{M_{D^*} - M_D} \simeq \frac{m_c}{m_b} \simeq 0.3$$

measured values:

$$M_{D^*} - M_D \simeq 140\ \text{MeV}, \qquad M_{B^*} - M_B \simeq 46\ \text{MeV}$$

- At LO of nonrelativistic expansion, constant contact terms for S-wave interaction between a pair of heavy mesons

$$\mathcal{L}_{4H} = -\frac{1}{4}\,\mathrm{Tr}\left[H^{a\dagger}H_b\right]\mathrm{Tr}\left[\bar{H}^c\bar{H}_d^\dagger\right]\left(F_A\delta_a^b\delta_c^d + F_A^\lambda\vec{\lambda}_a^b\cdot\vec{\lambda}_c^d\right)$$
$$+\frac{1}{4}\,\mathrm{Tr}\left[H^{a\dagger}H_b\sigma^m\right]\mathrm{Tr}\left[\bar{H}^c\bar{H}_d^\dagger\sigma^m\right]\left(F_B\delta_a^b\delta_c^d + F_B^\lambda\vec{\lambda}_a^b\cdot\vec{\lambda}_c^d\right)$$

- Using the completeness relations for SU(N) generators T_m satisfying $\mathrm{Tr}\left[T_mT_n\right] = C\delta_{mn}$:

$$\delta_i^l\delta_k^j = \frac{1}{N}\delta_i^j\delta_k^l + \frac{1}{C}\left(T_m\right)_i^j\left(T_m\right)_k^l$$

Proof: Any $N\times N$ complex matrix can be expanded as $X = X_0 + X_mT_m$ ($m=1,\ldots,N^2-1$). Using $\mathrm{Tr}\left[T_mT_n\right] = C\delta_{mn}$, one gets

$$X_0 = \frac{1}{N}\,\mathrm{Tr}[X] = \frac{1}{N}X_i^i, \qquad X_m = \frac{1}{C}\,\mathrm{Tr}\left[XT_m\right] = \frac{1}{C}X_l^k\left(T_m\right)_k^l.$$

Then

$$X_l^k\delta_i^l\delta_k^j = \frac{1}{N}X_l^k\delta_k^l\delta_i^j + \frac{1}{C}X_l^k\left(T_m\right)_k^l\left(T_m\right)_i^j.$$

one finds that the two-trace terms can be rewritten in the form of single-trace terms:

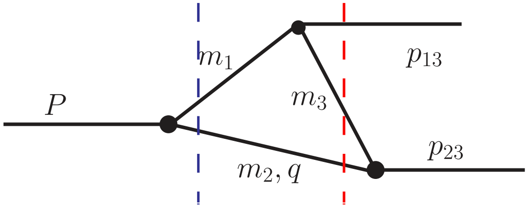
$$\begin{aligned}
\mathrm{Tr}\left[H^{a\dagger}H_a\right]\mathrm{Tr}\left[\bar{H}^b\bar{H}_b^\dagger\right] &= \left(H^{a\dagger}H_a\right)_l^i \delta_i^l \left(\bar{H}^b\bar{H}_b^\dagger\right)_j^k \delta_k^j \\
&= \left(H^{a\dagger}H_a\right)_l^i \left(\bar{H}^b\bar{H}_b^\dagger\right)_j^k \left[\frac{1}{2}\delta_k^l\delta_i^j + \frac{1}{2}\left(\sigma_m\right)_k^l\left(\sigma_m\right)_i^j\right] \\
&= \frac{1}{2}\,\mathrm{Tr}\left[H^{a\dagger}H_a\bar{H}^b\bar{H}_b^\dagger\right] + \frac{1}{2}\,\mathrm{Tr}\left[H^{a\dagger}H_a\sigma_m\bar{H}^b\bar{H}_b^\dagger\sigma_m\right], \\
\mathrm{Tr}\left[H^{a\dagger}H_a\sigma_m\right]\mathrm{Tr}\left[\bar{H}^b\bar{H}_b^\dagger\sigma_m\right] &= \left(H^{a\dagger}H_a\right)_l^k \left(\sigma_m\right)_k^l \left(\bar{H}^b\bar{H}_b^\dagger\right)_j^i \left(\sigma_m\right)_i^j \\
&= \left(H^{a\dagger}H_a\right)_l^k \left(\bar{H}^b\bar{H}_b^\dagger\right)_j^i \left(2\delta_i^l\delta_k^j - \delta_k^l\delta_i^j\right) \\
&= 2\,\mathrm{Tr}\left[H^{a\dagger}H_a\bar{H}^b\bar{H}_b^\dagger\right] - \mathrm{Tr}\left[H^{a\dagger}H_a\right]\mathrm{Tr}\left[\bar{H}^b\bar{H}_b^\dagger\right] \\
&= \frac{3}{2}\,\mathrm{Tr}\left[H^{a\dagger}H_a\bar{H}^b\bar{H}_b^\dagger\right] - \frac{1}{2}\,\mathrm{Tr}\left[H^{a\dagger}H_a\sigma_m\bar{H}^b\bar{H}_b^\dagger\sigma_m\right].
\end{aligned}$$

Triangle singularity

FKG, X.-H. Liu, S. Sakai, *Threshold cusps and triangle singularities in hadronic reactions*, Prog. Part. Nucl. Phys. 112 (2020) 103757

TS: some details (1)

M. Bayar, F. Aceti, FKG, E. Oset, PRD 94 (2016) 074039



Consider the scalar three-point loop integral

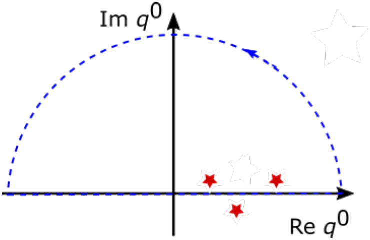
$$I = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{\left[(P-q)^2 - m_1^2 + i\epsilon\right]\left(q^2 - m_2^2 + i\epsilon\right)\left[(p_{23}-q)^2 - m_3^2 + i\epsilon\right]}$$

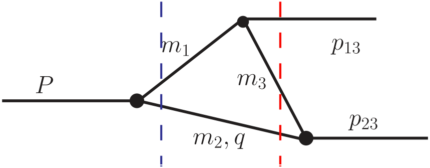
Rewriting a propagator into two poles:

$$\frac{1}{q^2 - m_2^2 + i\epsilon} = \frac{1}{\left(q^0 - \omega_2 + i\epsilon\right)\left(q^0 + \omega_2 - i\epsilon\right)} \quad \text{with} \quad \omega_2 = \sqrt{m_2^2 + \boldsymbol{q}^{\,2}}$$

Nonrelativistically, on the positive-energy poles

$$I \simeq \frac{i}{8 m_1 m_2 m_3} \int \frac{dq^0 d^3\boldsymbol{q}}{(2\pi)^4} \frac{1}{\left(P^0 - q^0 - \omega_1 + i\epsilon\right)\left(q^0 - \omega_2 + i\epsilon\right)\left(p_{23}^0 - q^0 - \omega_3 + i\epsilon\right)}$$





$$I \propto \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{[P^0 - \omega_1(q) - \omega_2(q) + i\,\epsilon][E_{23} - \omega_2(q) - \omega_3(\boldsymbol{p}_{23} - \boldsymbol{q}) + i\,\epsilon]}$$

$$\propto \int_0^\infty dq\, \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\,\epsilon} f(q)$$

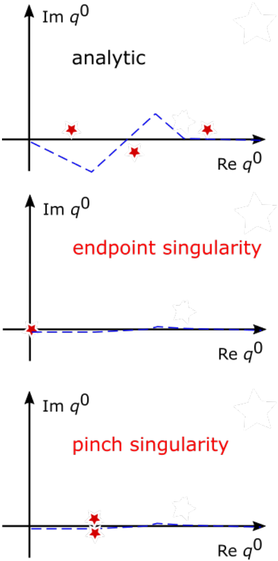
The second cut:

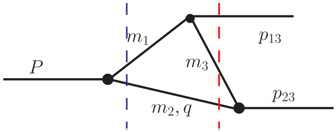
$$f(q) = \int_{-1}^{1} dz\, \frac{1}{E_{23} - \omega_2(q) - \sqrt{m_3^2 + q^2 + p_{23}^2 - 2p_{23}qz} + i\,\epsilon}$$

TS: some details (3)

Relation between singularities of integrand and integral

- singularity of integrand does not necessarily give a singularity of integral:
  integral contour can be deformed to avoid the singularity
- Two cases that a singularity cannot be avoided:
  - ☞ endpoint singularity
  - ☞ pinch singularity





$$I \propto \int_0^\infty dq \, \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\,\epsilon} f(q)$$

$$f(q) = \int_{-1}^{1} dz \, \frac{1}{A(q,z)} \equiv \int_{-1}^{1} dz \, \frac{1}{E_{23} - \omega_2(q) - \sqrt{m_3^2 + q^2 + p_{23}^2 - 2p_{23}qz} + i\,\epsilon}$$

Singularities of the **integrand** in the rest frame of initial particle:

- First cut: $M - \omega_1(l) - \omega_2(l) + i\,\epsilon = 0 \Rightarrow q_{\text{on}+} \equiv \frac{1}{2M}\sqrt{\lambda(M^2, m_1^2, m_2^2)} + i\,\epsilon$
- Second cut: $A(q, \pm 1) = 0 \Rightarrow$ endpoint singularities of $f(q)$

$$z = +1: \quad q_{a+} = \gamma\,(\beta\, E_2^* + p_2^*) + i\,\epsilon\,, \qquad q_{a-} = \gamma\,(\beta\, E_2^* - p_2^*) - i\,\epsilon\,,$$

$$z = -1: \quad q_{b+} = \gamma\,(-\beta\, E_2^* + p_2^*) + i\,\epsilon\,, \quad q_{b-} = -\gamma\,(\beta\, E_2^* + p_2^*) - i\,\epsilon$$

$$\beta = |\boldsymbol{p}_{23}|/E_{23}, \qquad \gamma = 1/\sqrt{1-\beta^2} = E_{23}/m_{23}$$

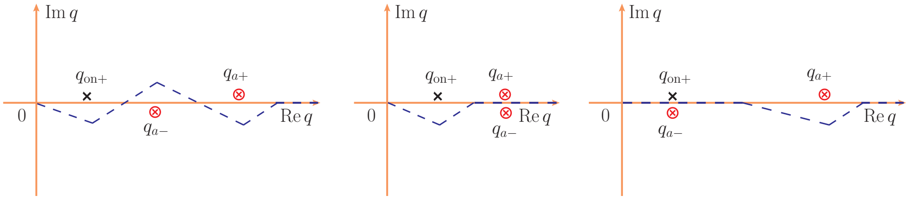
$E_2^*(p_2^*)$: energy (momentum) of particle-2 in the cmf of the (2,3) system

TS: some details (5)

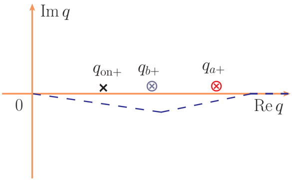
All singularities of the integrand:

$$q_{\rm on+}, \qquad q_{a+} = \gamma\,(\beta\,E_2^* + p_2^*) + i\,\epsilon, \qquad q_{a-} = \gamma\,(\beta\,E_2^* - p_2^*) - i\,\epsilon,$$

$$q_{b+} = -q_{a-}, \qquad q_{b-} = -q_{a+} < 0 \ (\text{for } \epsilon = 0)$$

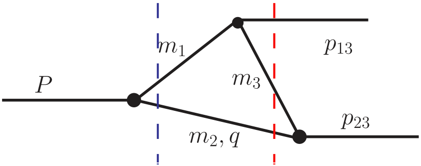


(a) (b) (c)



2-body threshold singularity at $m_{23} = m_2 + m_3$

triangle singularity at $q_{\rm on+} = q_{a-}$



Rewrite $q_{a-} = p_2 - i\,\epsilon, \quad p_2 \equiv \gamma\,(\beta\,E_2^* - p_2^*)$

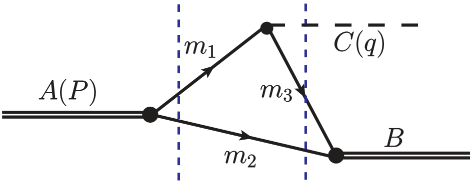
Kinematics for $p_2 > 0$, which is relevant to triangle singularity:

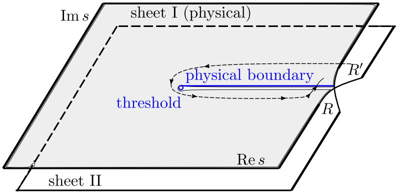
- $p_3 = \gamma\,(\beta\,E_3^* + p_2^*) > 0 \Rightarrow$
  particles 2 and 3 move in the same direction in the rest frame of initial particle
- velocities in the rest frame of the initial particle: $v_3 > \beta > v_2$

$$v_2 = \beta\,\frac{E_2^* - p_2^*/\beta}{E_2^* - \beta\,p_2^*} < \beta\,, \qquad v_3 = \beta\,\frac{E_3^* + p_2^*/\beta}{E_3^* + \beta\,p_2^*} > \beta$$

particle 3 moves faster than particle 2 in the rest frame of initial particle

TS: Coleman-Norton theorem





- Coleman–Norton theorem: S. Coleman and R. E. Norton, Nuovo Cim. 38 (1965) 438

  The singularity is on the physical boundary if and only if the diagram can be interpreted as a classical process in space-time.

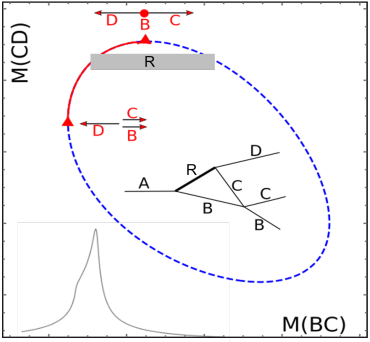
  ☞ physical boundary: upper edge (lower edge) of the unitary cut in the first (second) Riemann sheet

- Translation:

  ☞ all three intermediate states can go on shell

  ☞ $\boldsymbol{p}_2 \parallel \boldsymbol{p}_3$, m_3 can catch up with the m_2 to rescatter

TS: kinematical region



- For fixed $m_{B,C}, m_A$ and m_D, TS on the physical boundary only when

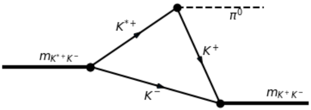
$$m_R^2 \in \left[\frac{m_A^2 m_C + m_D^2 m_B}{m_B + m_C} - m_B m_C, \; (m_A - m_B)^2 \right]$$

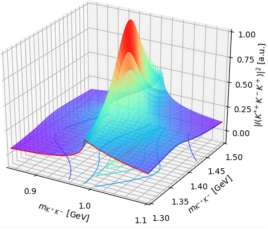
- TS on the physical boundary only when

$$m_A^2 \in \left[(m_R + m_B)^2, (m_R + m_B)^2 + \frac{m_B}{m_C} \left[(m_R - m_C)^2 - m_D^2 \right] \right],$$

$$m_{BC}^2 \in \left[(m_B + m_C)^2, (m_B + m_C)^2 + \frac{m_B}{m_R} \left[(m_R - m_C)^2 - m_D^2 \right] \right]$$

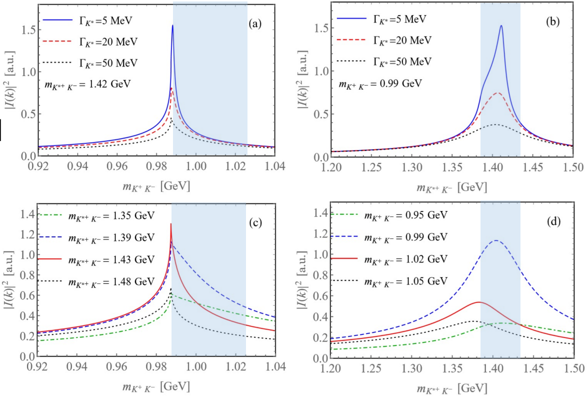
TS: invariant mass spectra



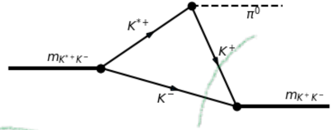


- TS structure is sensitive to energy
- While a resonance would persist independent of energy

TS only when $m_{K\bar{K}} \in [987,1026]$ MeV



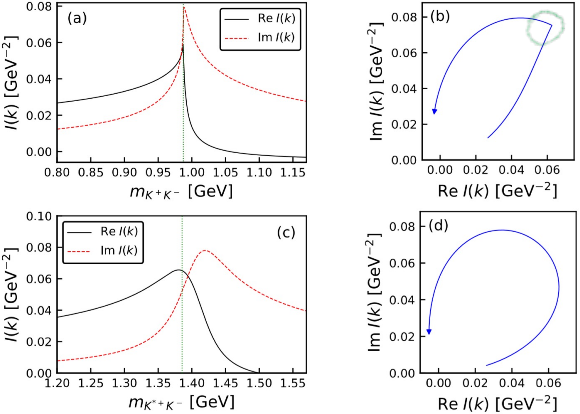
- Phase motion of triangle diagram in the presence of a TS



$K\overline{K}$ threshold

$m_{K^*\overline{K}} = 1.42$ GeV

$m_{K\overline{K}} = 0.99$ GeV



The argand plot is counterclockwise, resembling that of a resonance