Maximum-entropy "hydrodynamics"

Chandrodoy Chattopadhyay

North Carolina State University

Foundations and applications of relativistic hydrodynamics

Arcetri, Firenze, April 16

Collaborators: Jean-Paul Blaizot (CEA, Saclay), Derek Everett (OSU), Ulrich Heinz (OSU), Sunil Jaiswal (OSU), Thomas Schaefer (NCSU)

Motivation

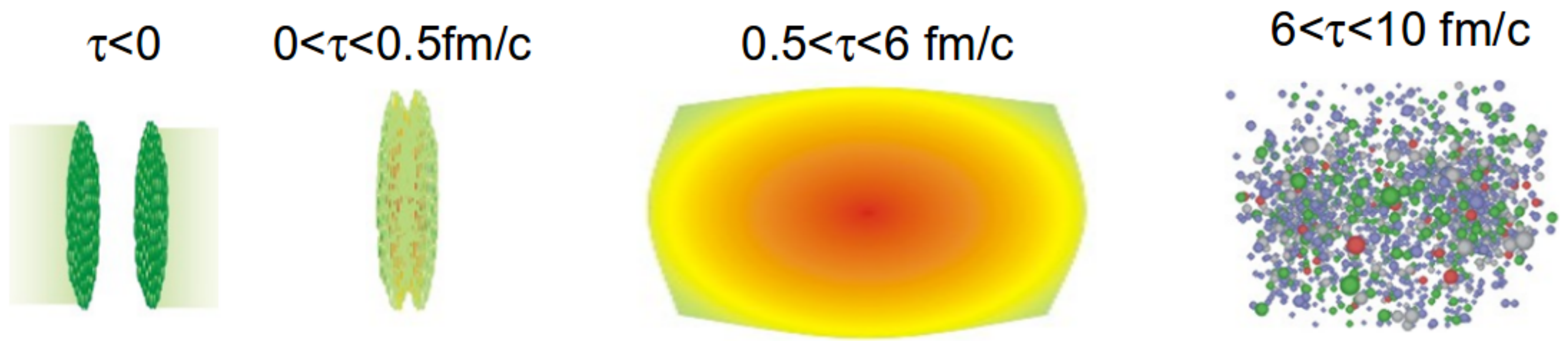


Fig. by Steffen A. Bass

- Hydro: Description of $(T^{\mu\nu}, N^{\mu})$ using macroscopic variables (T, u^{μ}, μ) and their gradients; accompanied by transport coefficients (η, ζ, σ).

  Should be distinguished from Israel-Stewart type hydro (ISH) [Muller '67, Israel, Stewart '76] where dissipative fluxes are promoted to dynamical degrees of freedom.

- ISH have been remarkably successful in describing intermediate stages of heavy-ion collisions.

  [Heinz et al, Romatschke et al, Dusling & Teaney, Song et al, and several others]

- ISH derived from kinetic theory works even when a fluid is not close to equilibrium.

  [Heller et al, Romatschke, Strickland, Denicol, Noronha, Blaizot and others.]

- However, applicability is sensitive to truncation scheme of moment equations. How to choose an appropriate truncation scheme?

IS-type hydro from kinetics

De Groots, Van Leeuwen, Van Weert

- Consider a system of weakly interacting classical particles; description via kinetic theory using single particle distribution function $f(x,p)$.

- Evolution of $f(x,p)$ governed by the Boltzmann equation,

$$p^\mu \partial_\mu f = C[f]$$ Collision kernel denotes interactions

- Conserved currents $(T^{\mu\nu}, N^\mu)$ appearing in hydro are moments of $f(x,p)$. For example,

$$T^{\mu\nu} = \int_p p^\mu\, p^\nu\, f(x,p) = e\, u^\mu\, u^\nu - (P+\Pi)\ \Delta^{\mu\nu} + \pi^{\mu\nu}$$

projector $\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$

Notation: $\int_p \equiv \int d^3p/[(2\pi)^3 E_p]$

For a fluid in local equilibrium $f \to f_{eq} = \exp(-u \cdot p/T + \mu/T)$, then $T^{\mu\nu} \to T^{\mu\nu}_{ideal}$, $N^\mu \to N^\mu_{ideal}$

- Off-equilibrium parts of conserved currents stem from $\delta f \equiv f - f_{eq}$

IS-type hydro from kinetics

Denicol, Koide, Rischke PRL (2010)

Denicol, Niemi, Molnar, Rischke PRD (2012)

- Conservation eqs: $\partial_\mu T^{\mu\nu} = 0$. Evolution equations for (T, u^μ) coupled to $(\Pi, \pi^{\mu\nu})$
- The bulk and shear stresses are $\Pi = -\frac{1}{3}\Delta_{\mu\nu}\int_p p^\mu p^\nu \delta f \qquad \pi^{\mu\nu} = \int_p p^{\langle\mu} p^{\nu\rangle} \delta f$
- Apply coming time-derivative $u^\mu \partial_\mu$ on both sides of above def. and use the Boltzmann equation

$$p^\mu \partial_\mu f = C[f]$$

to get (exact) evolution equations for the shear and bulk stresses.

Notation: $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$; double-symmetric, traceless projector orthogonal to u^μ

$$\Delta^{\mu\nu}_{\alpha\beta} = \left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right)/2 - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3$$

IS-type hydro from kinetics

- Consider a massive Boltzmann gas. Also take a simplistic collisional kernel given by the relaxation-time approximation (RTA) Anderson & Witting '74

$$C[f] \approx -\frac{u \cdot p}{\tau_R}\left(f - f_{eq}\right)$$

τ_R is the time-scale for establishment of local equilibrium.

- One then obtains a relaxation-type evolution of Π Denicol, Niemi, Molnar, Rischke PRD (2012)

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\alpha_1\,\theta + \alpha_2\,\Pi\,\theta + \alpha_3\,\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{m^2}{3}\rho^{\mu\nu}_{(-2)}\,\sigma_{\mu\nu} + \frac{m^2}{3}\nabla_\mu\rho^{\mu}_{(-1)} + \frac{m^2}{9}\,\rho_{(-2)}\,\theta$$

Standard definitions: $\dot{\Pi} = u^\mu \partial_\mu \Pi$ (time-derivative), $\nabla_\mu = \Delta^\alpha_\mu \partial_\alpha$ (spatial-derivative), $\theta = \partial_\mu u^\mu$ (expansion-rate), $\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$ (velocity stress tensor)

IS-type hydro from kinetics

- However the equation

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\alpha_1\,\theta + \alpha_2\,\Pi\,\theta + \alpha_3\,\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{m^2}{3}\rho^{\mu\nu}_{(-2)}\,\sigma_{\mu\nu} + \frac{m^2}{3}\,\nabla_\mu\rho^{\mu}_{(-1)} + \frac{m^2}{9}\,\rho_{(-2)}\,\theta$$

is not closed due to couplings to ρ-tensors.

- The ρ-tensors are higher-order ("non-hydro") moments of $f(x,p)$. For example,

$$\rho^{\mu}_{(-1)} = \Delta^{\mu}_{\alpha}\int_p \left(u\cdot p\right)^{-1}\,p^{\alpha}\,f, \qquad \rho^{\mu\nu}_{(-2)} = \int_p \left(u\cdot p\right)^{-2}\,p^{\langle\mu}\,p^{\nu\rangle}\,f$$

- Similar feature exists for shear stress evolution. Needs truncation, i.e., to express δf in terms of components of $T^{\mu\nu}$.

Standard truncations I: Grad

[Grad, Mueller, Israel & Stewart]

- Method of Grad, Israel and Stewart: Expand $f(x,p)$ in powers of the particle momenta p^μ

$$\frac{\delta f(x,p)}{f_{eq}} \equiv \phi(x,p) \approx a + b_\mu\, p^\mu + c_{\mu\nu} p^\mu p^\nu + \cdots$$

(Now truncate at second order)

- In the absence of conserved charge currents,

$$\phi(x,p) \approx \mathscr{A} + c_{\langle\mu\rangle}\, \left(u \cdot p\right)\, p^{\langle\mu\rangle} + \mathscr{C}\, \left(u \cdot p\right)^2 + c_{\langle\mu\nu\rangle} p^{\langle\mu} p^{\nu\rangle}$$

- Determine these 10 coefficients using components of energy-momentum tensor

$$T^{\mu\nu} = \int_p p^\mu\, p^\nu\, f_{eq} \left(1 + \phi\right) \quad \Longrightarrow \quad \phi = \frac{p^\mu p^\nu}{2\,(e+P)\,T^2} \left(\pi_{\mu\nu} + \frac{2}{5}\,\Pi\,\Delta_{\mu\nu}\right)$$

Dusling, Teaney '08

Standard truncations II: Chapman-Enskog

- δf motivated by Chapman-Enskog like expansion of a simplified collision kernel

$$p^\mu \partial_\mu f = -\frac{u \cdot p}{\tau_R} \delta f \quad \Longrightarrow \quad \delta f \approx -\frac{\tau_R}{u \cdot p} p^\mu \partial_\mu f_{eq} \quad = \tau_R \beta \mathscr{A} \frac{\theta}{u \cdot p} + \tau_R \beta \frac{p_\mu p_\nu \sigma^{\mu\nu}}{u \cdot p}$$

(Expansion in velocity gradients)

- Using this δf relate bulk and shear stresses to the strains: $\Pi = -\tau_R \beta_\Pi \theta \qquad \pi^{\mu\nu} = 2\tau_R \beta_\pi \sigma^{\mu\nu}$

- Re-express δf in terms of the stresses using first-order results

$$\phi = -\frac{\beta}{\beta_\Pi}\left(3c_s^2 \left(u \cdot p\right)^2 + p_{\langle\mu\rangle} p^{\langle\mu\rangle}\right) \frac{\Pi}{u \cdot p} + \frac{\beta}{2\beta_\pi} \frac{p_\mu p_\nu \pi^{\mu\nu}}{u \cdot p} .$$

Bhalerao, A. Jaiswal et al ’14

A. Jaiswal, Ryblewski, Strickland ‘14

Standard truncations III: Romatschke-Strickland

- In heavy-ion collisions medium expands rapidly along beam (longitudinal) axis at early times. Creates large anisotropies between longitudinal and transverse pressures.

- Attempts to handle large momentum space anisotropies by expanding $f(x, p)$ around a locally anisotropic distribution

$$f = f_{RS} + \delta f \qquad \text{where} \qquad f_{RS}^{LRF} = \exp\left(-\frac{\sqrt{p_T^2 + (1+\xi)p_z^2}}{\Lambda_{RS}}\right)$$

Romatschke & Strickland ‘03

- Gives rise to anisotropic hydrodynamics (aHydro), viscous aHydro and its variants.

Strickland, Florkowski, Ryblewski, Heinz, Martinez, McNelis and several others…

Why a new truncation scheme?

- Grad assumes δf to be quadratic in momenta (ad-hoc). Chapman-Enskog δf should not be valid far from equilibrium (gradient-expansion).

- Both become negative (unphysical) at large momenta. Resulting macroscopic framework breaks down in certain flow profiles.

- The aHydro ansatz does not become negative and can handle large shear deformations at early stages of HIC.

  But: its form is ad-hoc. Not possible to describe large bulk viscous pressures.

  May not be possible to model arbitrary flow profiles.

- We aim for a truncation scheme that (i) leads to a framework which may work both near and far from equilibrium (ii) does not invoke uncontrolled assumptions about microscopic physics.

The ‘least-biased’ distribution

E. Jaynes, Phys. Rev. 106, 620 (1957)

- We want to re-construct δf solely using quantities appearing in $T^{\mu\nu}$, i.e., $(e, u^{\mu}, \pi^{\mu\nu}, \Pi)$

- This is, in principle, impossible. However, what is our best guess?

- The least biased distribution that uses all of, and only the information provided by $T^{\mu\nu}$ is the one that maximizes the non-equilibrium entropy

$$s[f] = -\int dP \, (u \cdot p) \, (f \, \log f - f)$$

subject to constraints that f(x,p) satisfies,

$$\int_p (u \cdot p)^2 \, f = e, \qquad -\frac{1}{3}\int_p p_{\langle\mu\rangle} p^{\langle\mu\rangle} f = P + \Pi, \qquad \int_p p^{\langle\mu} p^{\nu\rangle} f = \pi^{\mu\nu}$$

The basic idea

Consider a system in a macrostate specified by (E, V, N). The system can be in a variety of microstates consistent with the macrostate.

One may, in general, assign any probability distribution to these microstates.

But the probability distribution where all such microstates are assumed equi-probable is the "least-biased" one.

Such a distribution maximizes the Shannon entropy $S = -\sum_i p_i \ln p_i$

The maximum-entropy distribution

Introduce Lagrange multipliers,

$$s[f] = -\int_p (u\cdot p) f (\log f - 1)$$

$$+ \Lambda \left[e - \int_p (u\cdot p)^2 f \right] + \lambda_\Pi \left[P + \Pi + \int_p p_{\langle\mu\rangle} p^{\langle\mu\rangle} f \right] + \gamma_{\alpha\beta} \left[\pi^{\alpha\beta} - \int_p p^{\langle\alpha} p^{\beta\rangle} f \right]$$

Functional derivative w.r.t. f: $\dfrac{\delta s[f]}{\delta f} = 0$

The maximum entropy solution: C.C., Heinz, Schaefer, PRC 108 (2023), 034907, Everett, C.C., Heinz, PRC (2021), 064902

$$\boxed{f_{\mathrm{ME}} = \exp\left[-\Lambda\,(u\cdot p) + \frac{\lambda_\Pi}{u\cdot p} p_{\langle\alpha\rangle} p^{\langle\alpha\rangle} - \frac{\gamma_{\langle\alpha\beta\rangle}}{u\cdot p} p^{\langle\alpha} p^{\beta\rangle} \right]}$$

In the absence of non-eq. fluxes, $\pi^{\mu\nu} = \Pi = 0$, we recover the Boltzmann distribution.

The near equilibrium limit of f_{ME}

- Expand f_{ME} around equilibrium:

$$f_{ME} \approx f_{eq}\left[1-\left(c_\lambda\lambda_\Pi + c_{\mu\nu}\gamma^{\mu\nu}\right)(u\cdot p) + \lambda_\Pi \frac{p_{\langle\mu\rangle}p^{\langle\mu\rangle}}{u\cdot p}\right] - \gamma_{\mu\nu}\frac{p^{\langle\mu}p^{\nu\rangle}}{u\cdot p}$$

- Plug δf_{ME} in definition for shear and bulk stresses,

$$\pi^{\mu\nu} = \int_p p^{\langle\mu}p^{\nu\rangle}\,\delta f_{ME} \qquad \Pi = -\frac{1}{3}\int_p p_{\langle\mu\rangle}p^{\langle\mu\rangle}\,\delta f_{ME}$$

and invert. δf_{ME} to linear order in non-equilibrium stresses is:

$$\delta f_{ME} = f_{eq}\left[-\frac{\beta}{\beta_\Pi}\left(3c_s^2\,(u\cdot p)^2 + p_{\langle\mu\rangle}p^{\langle\mu\rangle}\right)\frac{\Pi}{u\cdot p} + \frac{\beta}{2\beta_\pi}\frac{p_\mu p_\nu \pi^{\mu\nu}}{u\cdot p}\right]$$

- Matches exactly with the Chapman-Enskog like expansion of RTA Boltzmann eq.!

Features of Max-Ent distribution

In the fluid rest-frame
$$f_{\mathrm{ME}} = \exp\left[-\Lambda\, E_p + \frac{\lambda_\Pi}{E_p}\,\vec{p}^{\,2} - \frac{\gamma_{\langle ij\rangle}}{E_p}\,p^{\langle i}p^{j\rangle}\right]$$

Plays role similar to an inverse temperature

Isotropic deviation from equilibrium

Anisotropic deviation from equilibrium

- Positive-definite for all momenta
- Non-linear dependence on shear and bulk stresses
- Reduces to the Chapman-Enskog δf in the limit of small viscous stresses.

Max-Ent like idea pursued before: in non-relativistic context by Levermore '96, "dissipative-like" theories by Calzetta, Cantarutti, Peralta-Ramos '19, '23

Non-linear inversion for Lagrange multipliers

- The multipliers must be matched to $T^{\mu\nu}$. The full (non-linear) problem requires an inversion of 7 parameters $(\Lambda, \lambda_\Pi, \gamma_{ij})$; numerically intractable.

- Simplification: To match shear stress tensor:

$$\pi^{ij} = \Delta^{ij}_{kl} \int dP\, p^k\, p^l \exp\left(-\Lambda E_p - \frac{\lambda_\Pi}{E_p} p^2\right) \exp\left(-\frac{\gamma_{rs} p^r p^s}{E_p}\right)$$

One can show $\quad \pi = \Gamma - \frac{1}{3} I\, \mathrm{tr}(\Gamma) \qquad \Gamma \equiv \sum_i c_i \gamma^i$

- The shear tensor and γ commutes, $[\pi, \gamma] = 0$; simultaneously diagonalizable.

Simplifying non-linear inversion

- π^{ij} is symmetric; has real eigenvalues and admits orthogonal eigenvectors. Can be diagonalized by spatial rotation.

- Diagonalize shear tensor: $\pi \to \pi_D = R^T \pi R$. This diagonalizes γ.

- 3 out of 5 independent degrees of freedom in the matrix γ are fixed using eigenvectors of π

- Only two-dimensional root finding required to obtained $\gamma_D = \mathrm{diag}\left(\gamma_1, \gamma_2, -(\gamma_1 + \gamma_2)\right)$ in terms of eigenvalues of π

The Max-Ent framework

C.C., Heinz, Schaefer, PRC 108 (2023), 034907

- The Max-Ent framework: To evolve components of $T^{\mu\nu} = e\, u^\mu\, u^\nu - (P + \Pi)\, \Delta^{\mu\nu} + \pi^{\mu\nu}$

$$\dot{e} = -\,(e + P + \Pi)\, \nabla_\mu u^\mu + \pi^{\mu\nu}\, \nabla_{(\mu} u_{\nu)}$$ (energy density evolution)

$$(e + P + \Pi)\, \dot{u}^\mu = \nabla^\mu P + \cdots$$ (velocity evolution)

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = 2\,\eta\, \nabla^{\langle\mu}\, u^{\nu\rangle} - \frac{4}{3}\,\pi^{\mu\nu}\, \nabla_\mu u^\mu \cdots - 2\, \rho^{\mu\nu\alpha\beta}_{(-2)}\, \nabla_\alpha u_\beta$$ (shear evolution)

Similar eq. for bulk pressure

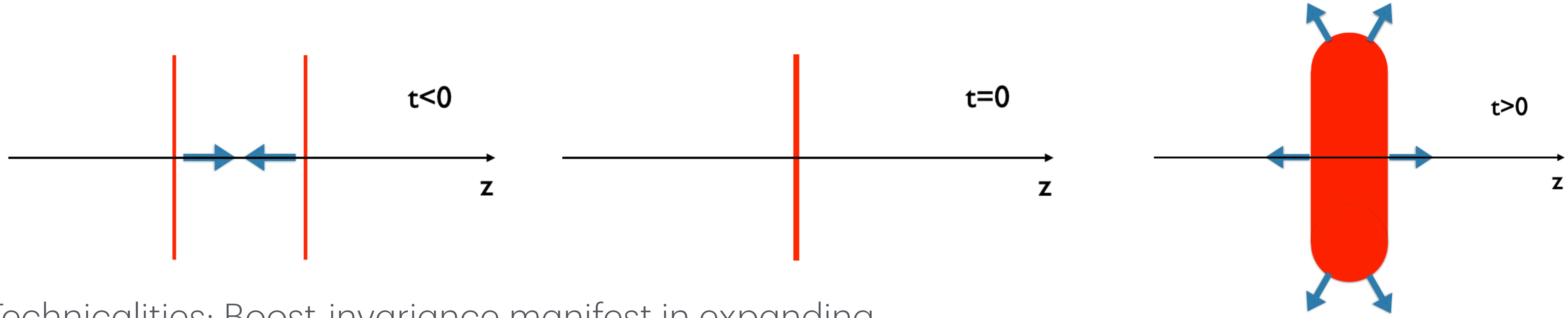
- To compute the higher moments, $\rho^{\mu\nu\alpha\beta}_r \equiv \int_p \left(u \cdot p\right)^{-r}\, p^{\langle\mu} \cdots p^{\beta\rangle}\, f$

Replace $f \to f_{\mathrm{ME}} = \exp\left[-\Lambda\, \left(u \cdot p\right) + \frac{\lambda_\Pi}{u \cdot p}\, p_{\langle\alpha\rangle} p^{\langle\alpha\rangle} - \frac{\gamma_{\langle\alpha\beta\rangle}}{u \cdot p}\, p^{\langle\alpha} p^{\beta\rangle}\right]$

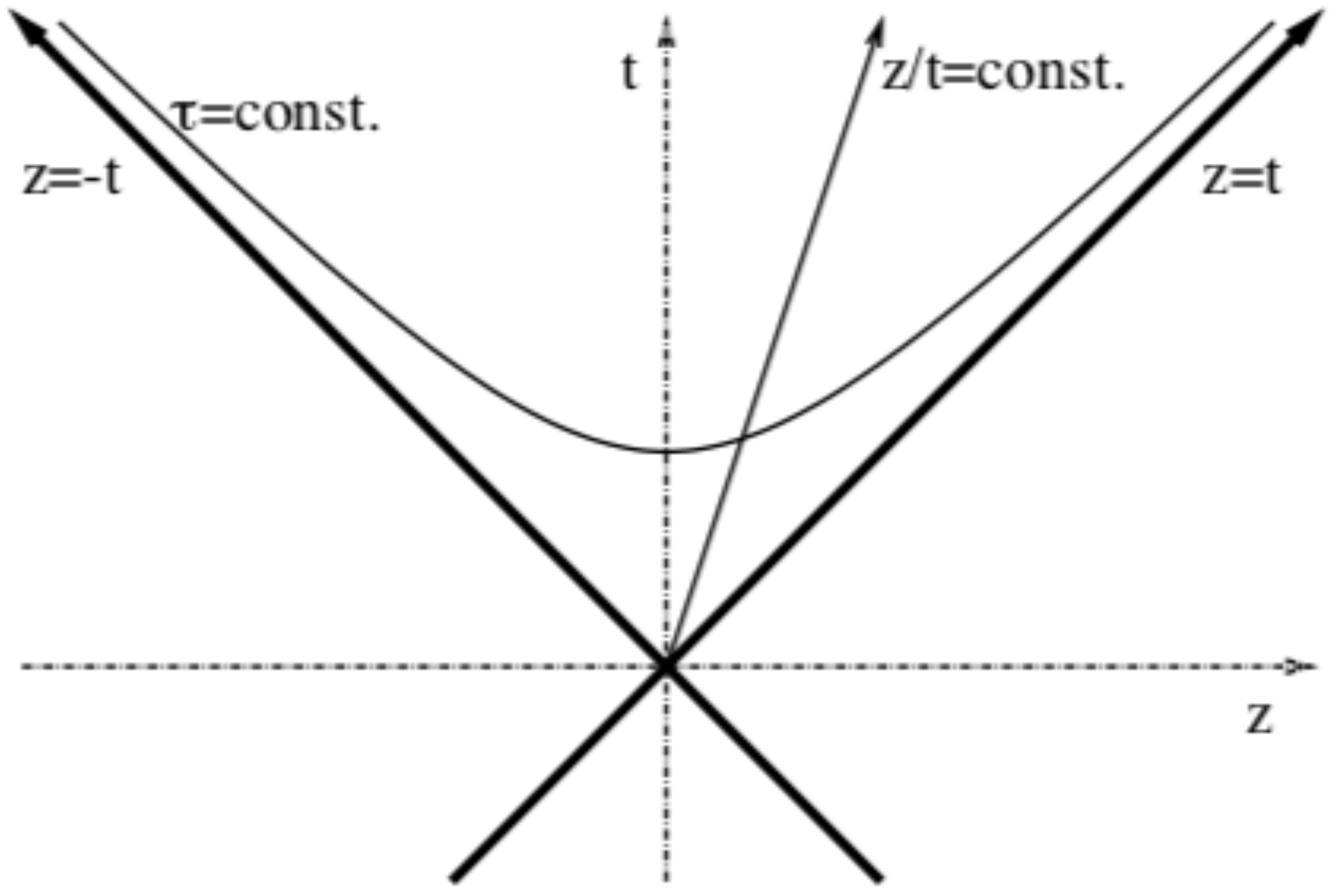
Tests of the Maximum-Entropy framework

- Test 1: Energy momentum evolution in Bjorken Flow for far-off-equilibrium initializations

Early-time dynamics of QGP: Bjorken flow



Technicalities: Boost-invariance manifest in expanding coordinates $\tau = \sqrt{t^2 - z^2}, \quad \eta = \tanh^{-1}(z/t)$



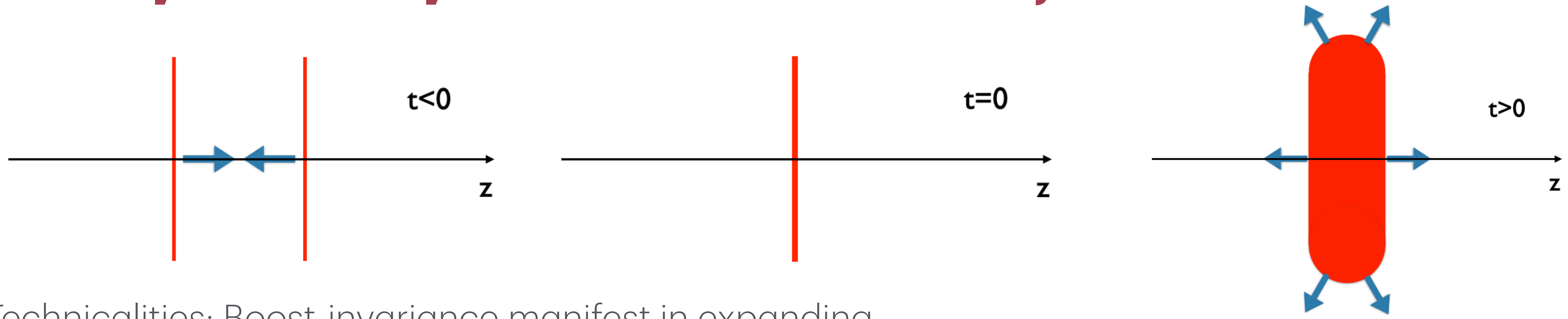
The energy momentum tensor is diagonal:

$$T^{\mu\nu} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & P_T & 0 & 0 \\ 0 & 0 & P_T & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix}$$

The net transverse and longitudinal pressures:

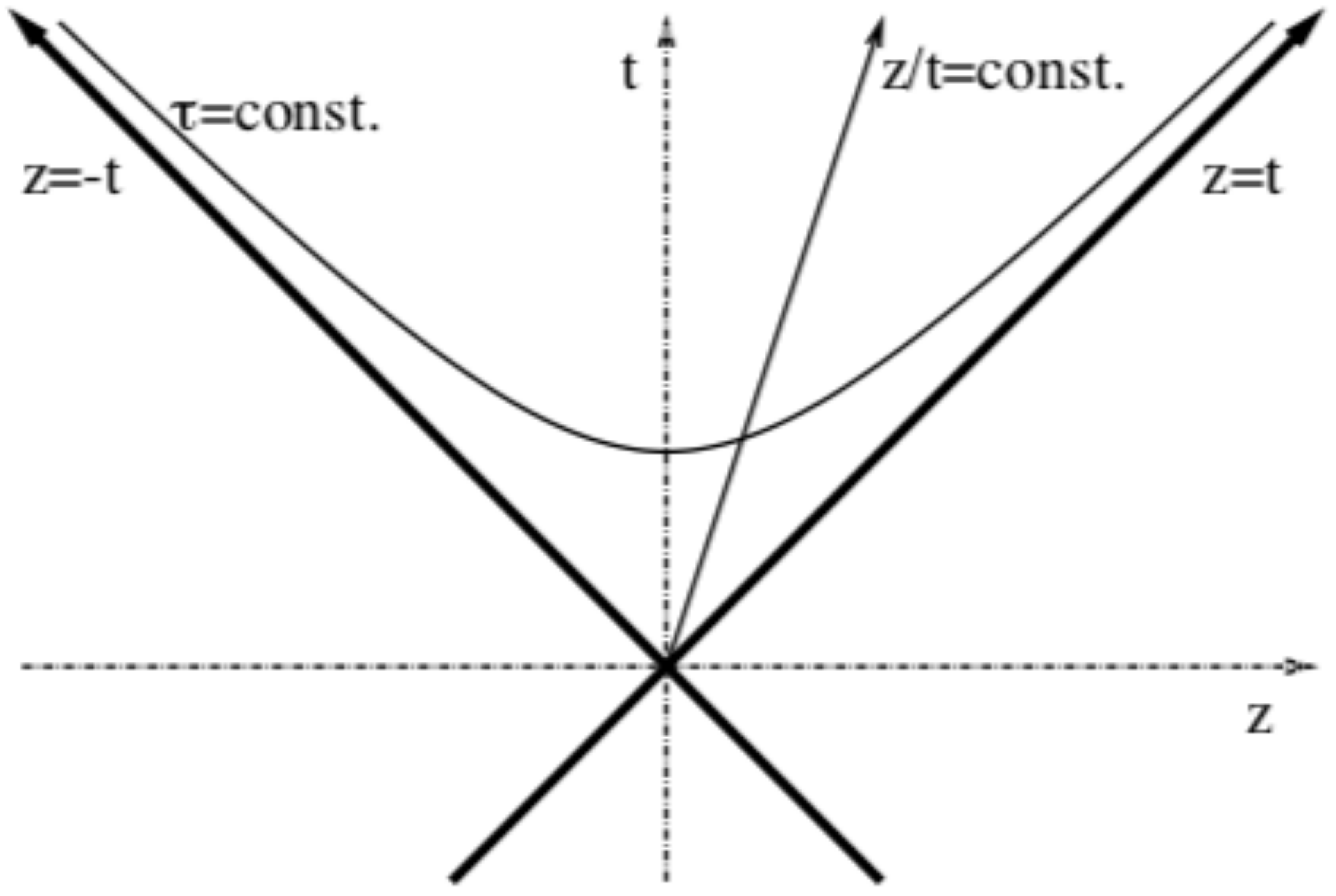
$$P_T = P + \Pi + \frac{\pi}{2}$$
$$P_T = P + \Pi - \pi$$

Early-time dynamics of QGP: Bjorken flow



Technicalities: Boost-invariance manifest in expanding coordinates $\tau = \sqrt{t^2 - z^2}, \quad \eta = \tanh^{-1}(z/t)$

Fluid expansion rate: $\dfrac{1}{\tau}$



$$\frac{de}{d\tau} = -\frac{1}{\tau}\,(e + P - \pi)$$

Energy density drops because of expansion, work done by pressure

$$\tau_R \frac{d\pi}{d\tau} + \pi = \frac{4}{3}\frac{\eta}{\tau} + \cdots$$

Shear relaxes to its Navier-Stokes limit

Bjorken flow: Max-Ent evolution equations

- Using kinetic equation for distribution function: $$\frac{\partial f}{\partial \tau} - \frac{p^z}{\tau}\frac{\partial f}{\partial p^z} = -\frac{1}{\tau_R}\left(f - f_{eq}\right)$$

- Derive evolution equations of energy density and effective pressures

$$\frac{de}{d\tau} = -\frac{1}{\tau}\left(e + P_L\right)$$

$$\mathcal{A}_L = -3P_L + \int_p \frac{1}{E_p^2} p_z^4 f$$

$$\frac{dP_L}{d\tau} = -\frac{P_L - P}{\tau_R} + \frac{\mathcal{A}_L}{\tau}, \qquad \frac{dP_T}{d\tau} = -\frac{P_T - P}{\tau_R} + \frac{\mathcal{A}_T}{\tau}$$

$$\mathcal{A}_T = -P_T + \frac{1}{2}\int_p \frac{1}{E_p^2} p_z^2 p_T^2 f$$

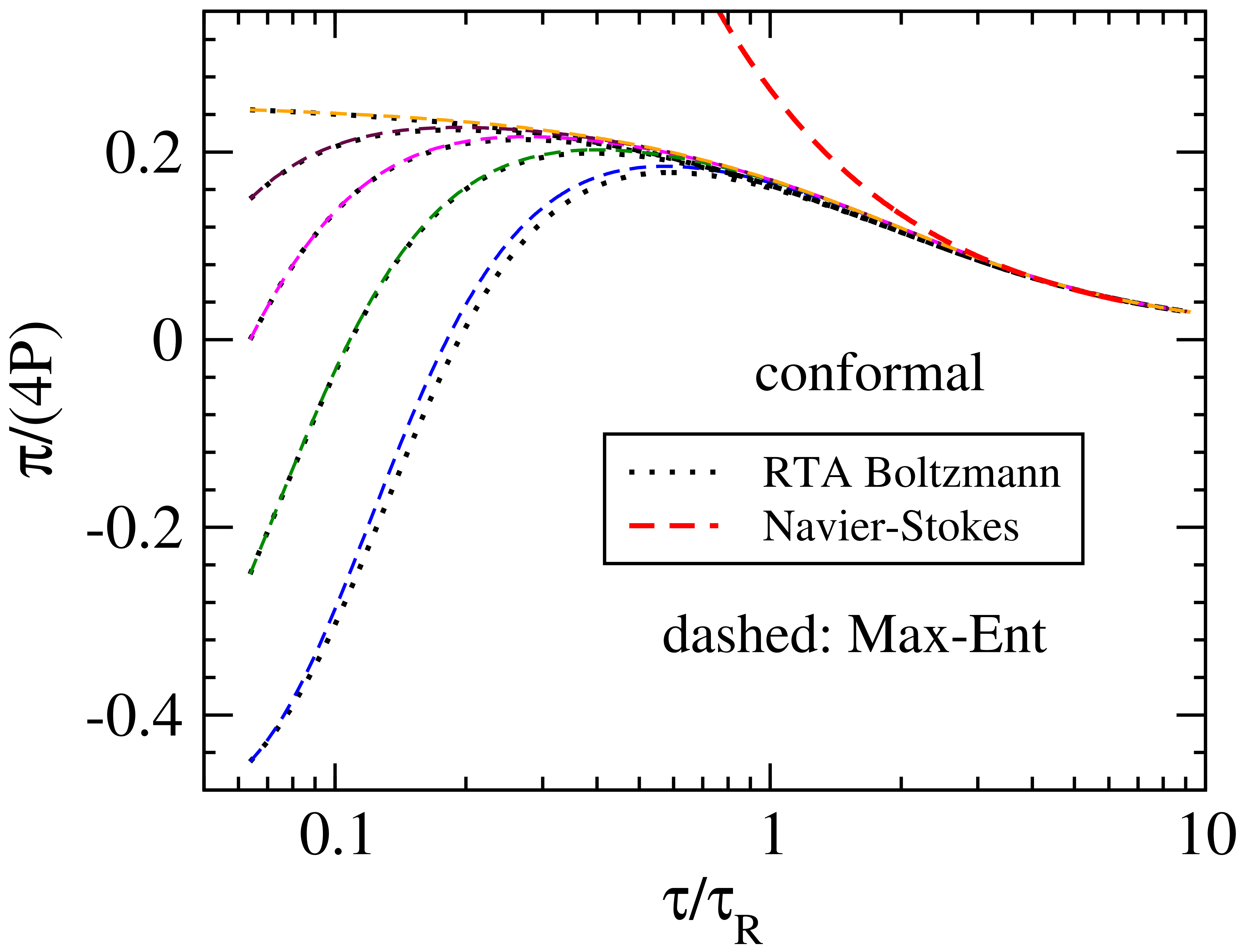
- Use (e, P_L, P_T) to construct f_{ME} and compute $(\mathcal{A}_L, \mathcal{A}_T)$ to close system of equations.

  Same complexity as solving hydro equations. $$f_{ME} = \exp\left(-\Lambda E_p - \lambda_\Pi \frac{\vec{p}^2}{E_p} - \gamma\,\frac{p_T^2/2 - p_z^2}{E_p}\right)$$

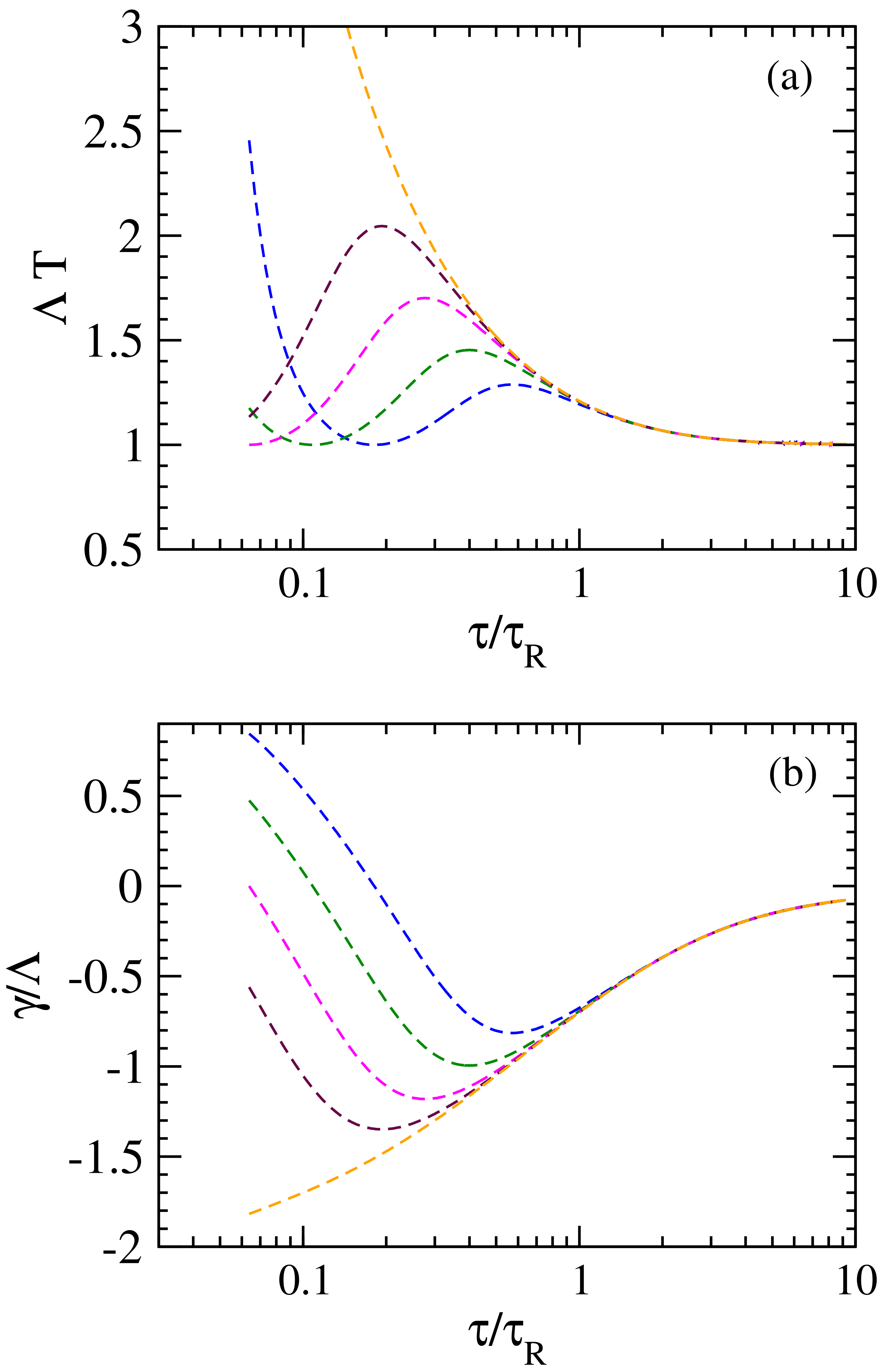
Conformal dynamics: Max-Ent

Massless particles $e = 3P$, $\Pi = 0$

$\tau_R = 5(\eta/s)/T$



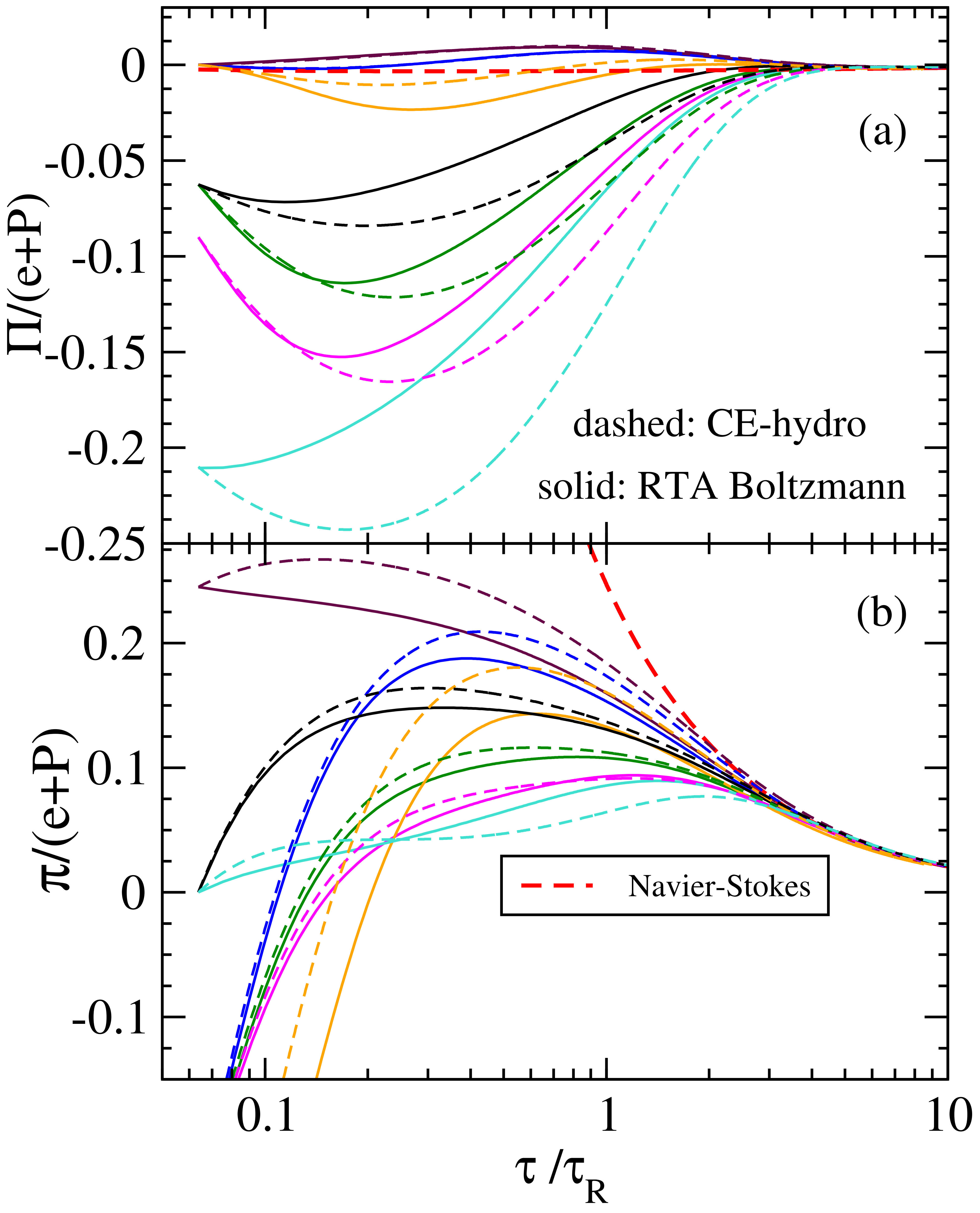
Lagrange multipliers of Max-Ent

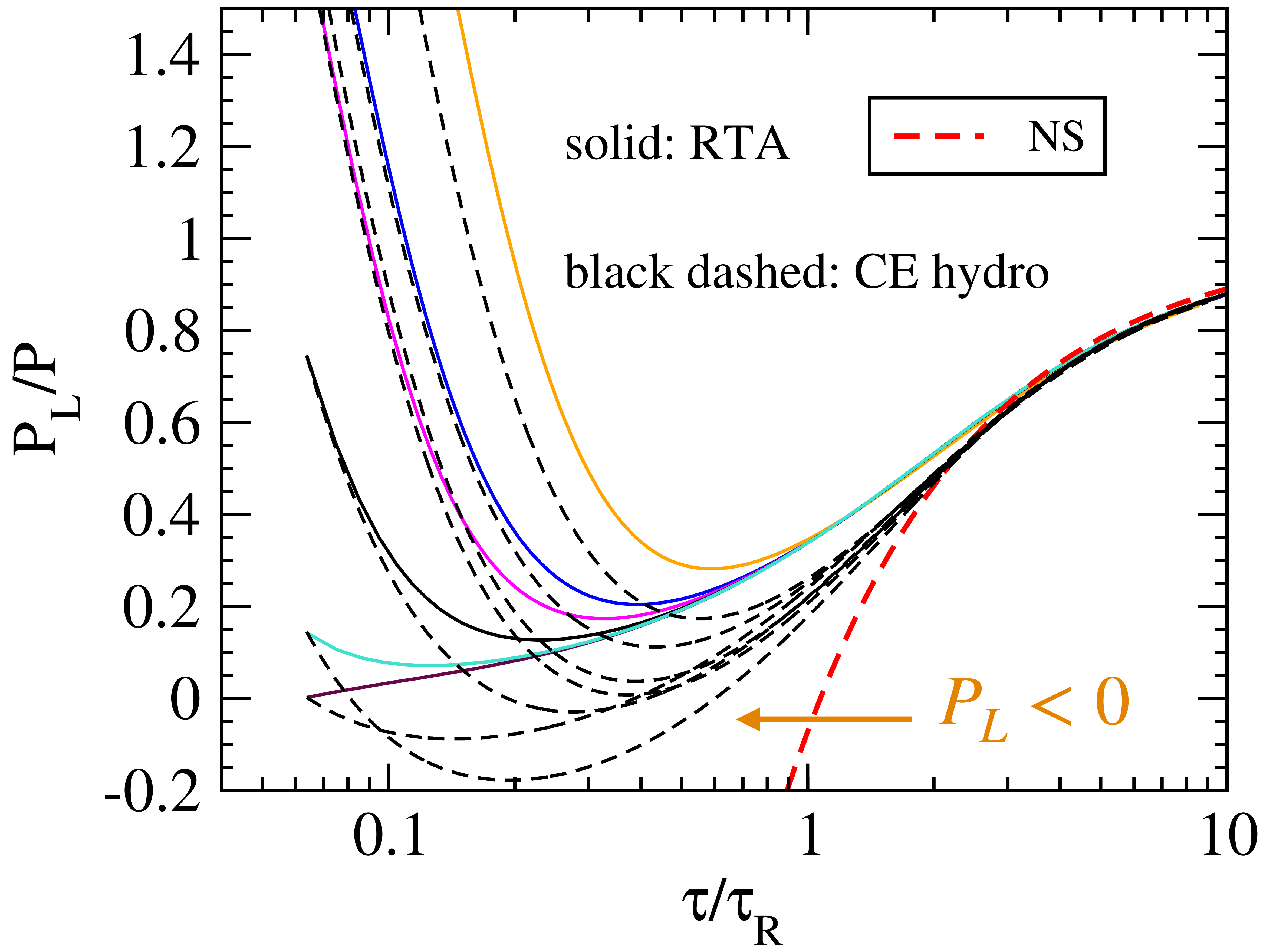


Good agreement between Max-Ent and exact solution of RTA Boltzmann even far-off-equilibrium

Non-conformal second-order hydro

S. Jaiswal, C.C., et al, PRC 105, 024911 (2022)

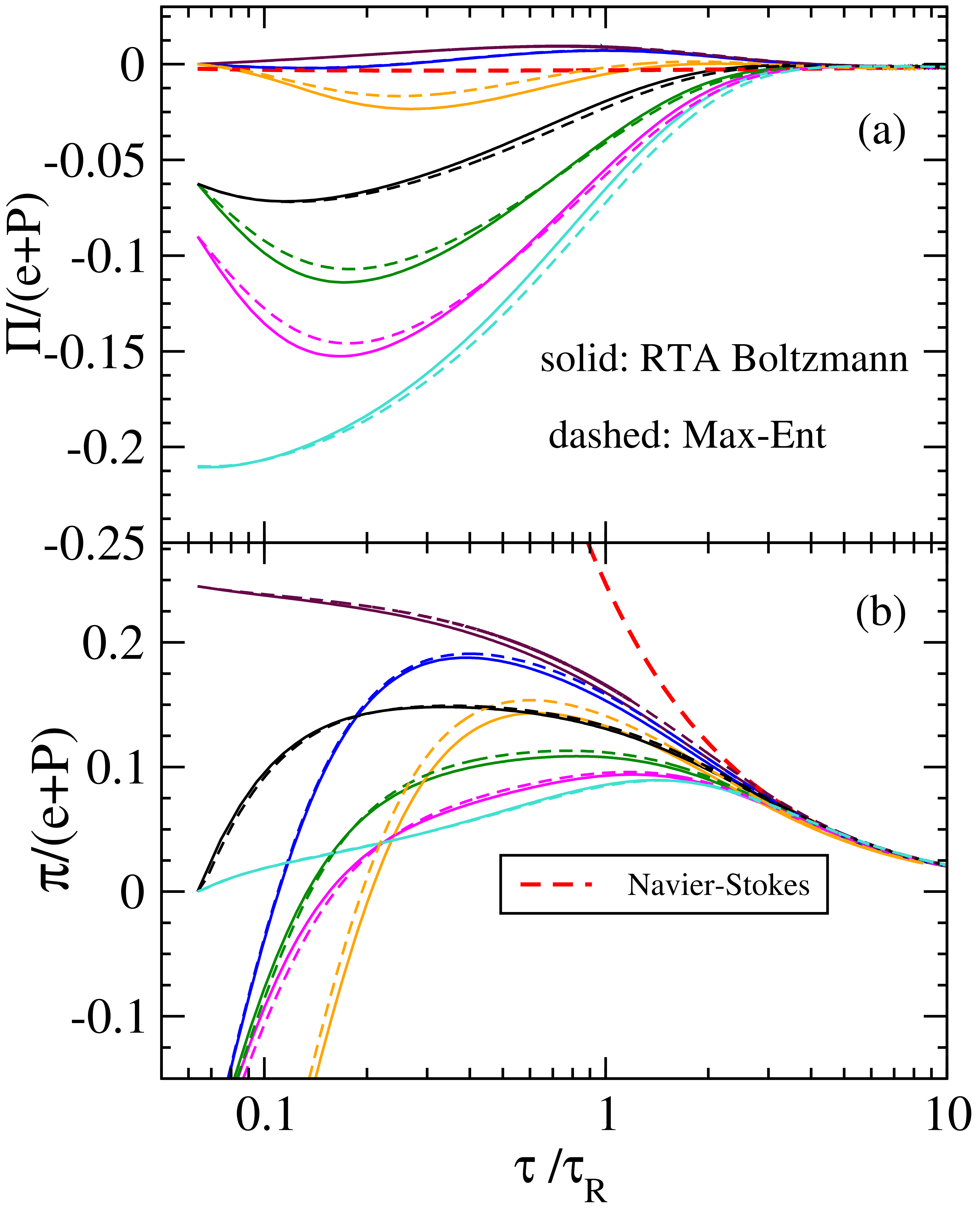


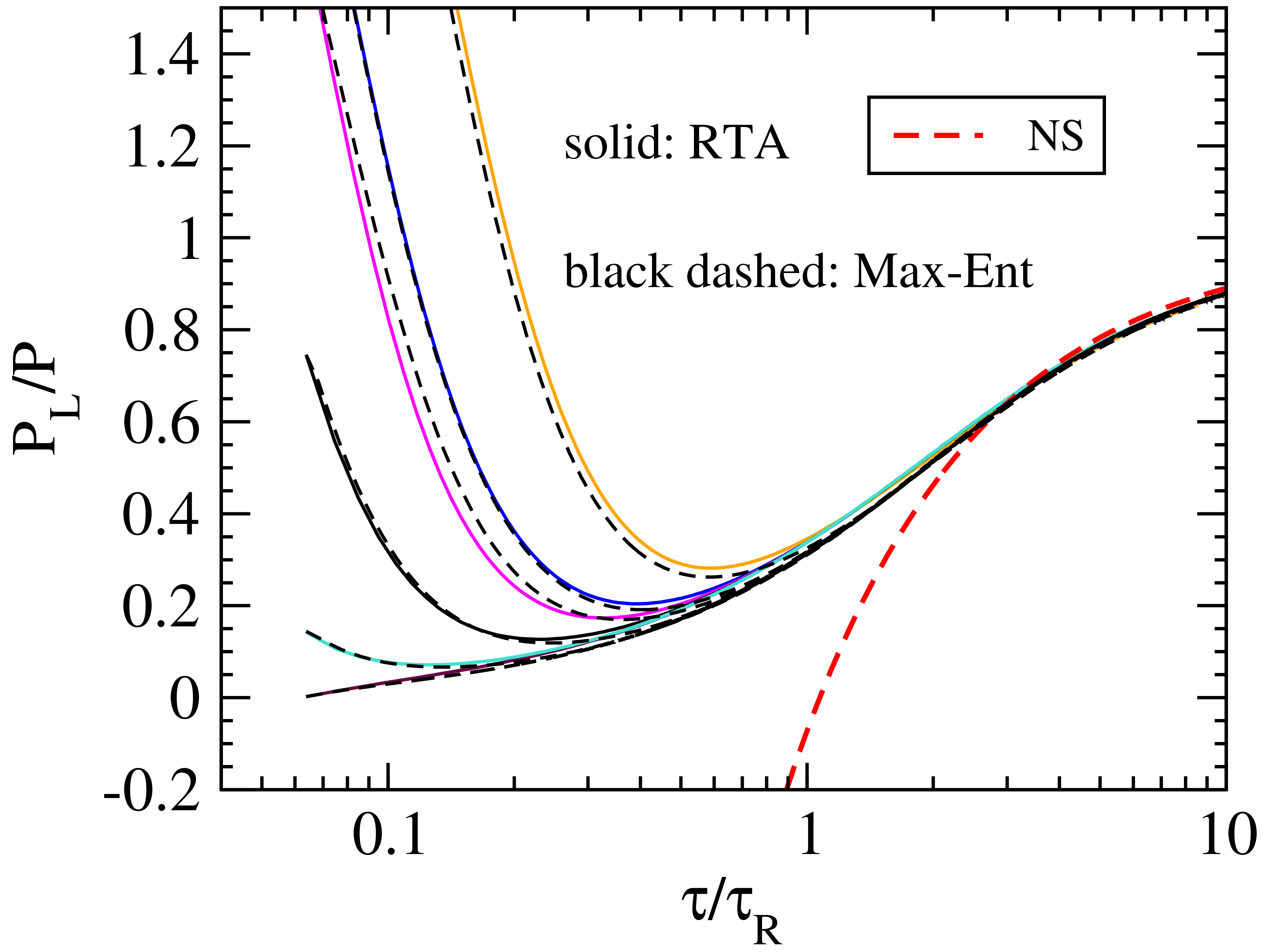


- Standard hydro is not in good agreement with kinetic theory at large Knudsen numbers.
- Does not describe early time universality accurately

Max-Ent framework

C.C., Heinz, Schaefer, PRC 108 (2023), 034907



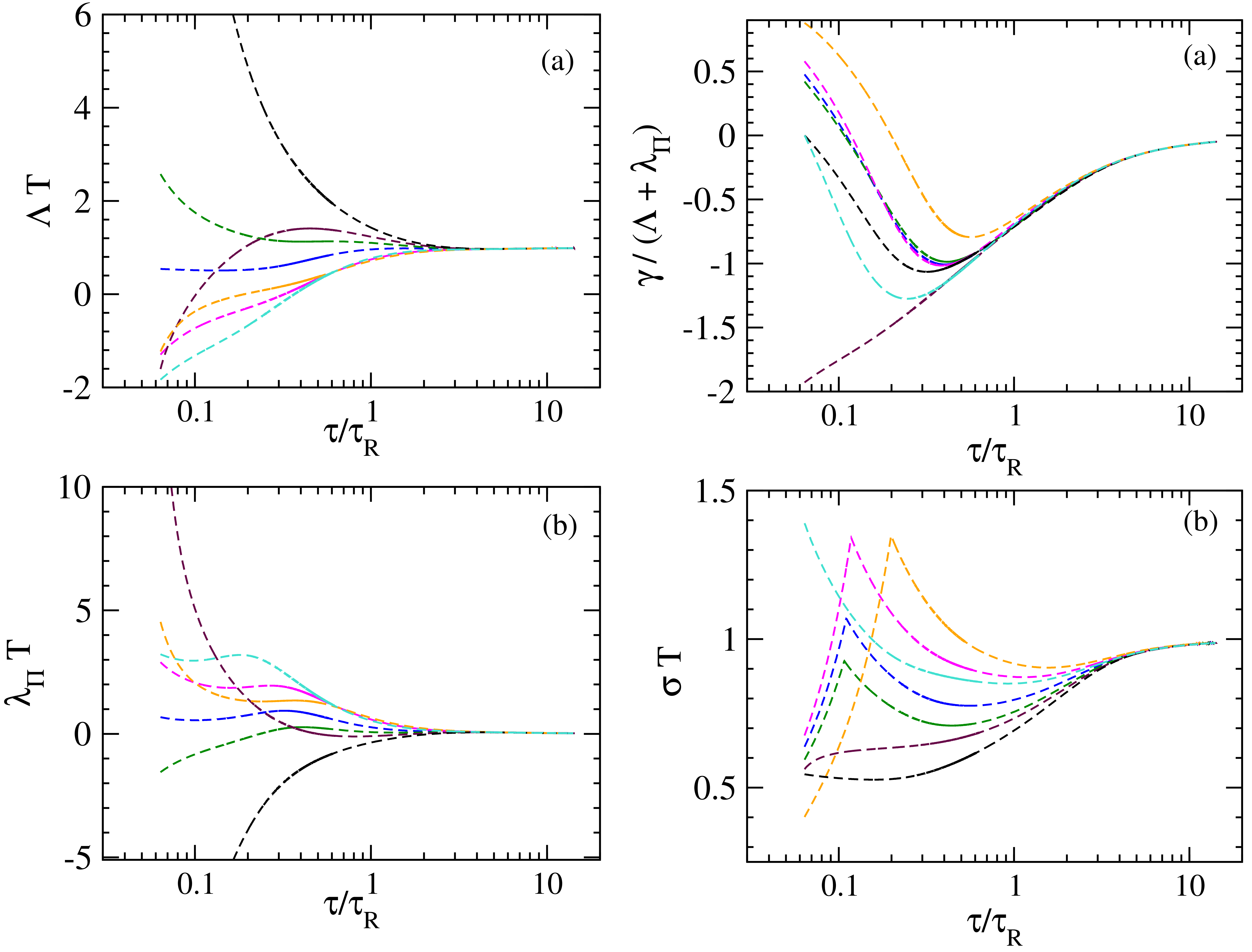


- Max-Ent is in good agreement with kinetic theory at large Knudsen numbers.
- Accurately describes early time universality accurately

Evolution of Lagrange multipliers

- In far-off-equilibrium regimes, $\Lambda < 0$!!
- Should not be identified with an inverse temperature at early times.
- The quantity $\sigma \equiv \Lambda + \lambda_\Pi - |\min(\gamma/2, -\gamma)|$ should is positive.

Ensures $f_{ME} \rightarrow 0$ at large momenta



Large negative bulk viscous pressure and $\Lambda < 0$

- The total isotropic pressure in kinetics: $P + \Pi = \dfrac{1}{3}\displaystyle\int_p \frac{\vec{p}^2}{E_p} f$

- $\Pi \sim -P$ can be attained by populating low momentum states with large number of particles, $f \sim A\,\delta(|\vec{p}|)/\vec{p}^2$

- At low momenta $f_{ME} \approx \exp(-\Lambda\, m)$. Enhancement of occupation of low momentum modes facilitated by $\Lambda < 0$.

Tests of the Maximum-Entropy framework

- Test 2: Energy momentum evolution in Gubser flow

Gubser flow

S.S. Gubser, PRD 82 085027 (2010) S.S. Gubser and A. Yarom, Nucl. Phys. B (2010)

Bjorken flow assumed that the fluid expands only in the longitudinal direction, $\theta_L = \frac{1}{\tau},\ \theta_\perp = 0$

Gubser fluid expands both in longitudinal as well transverse directions. Longitudinal expansion rate $\tau_L = 1/\tau$,.

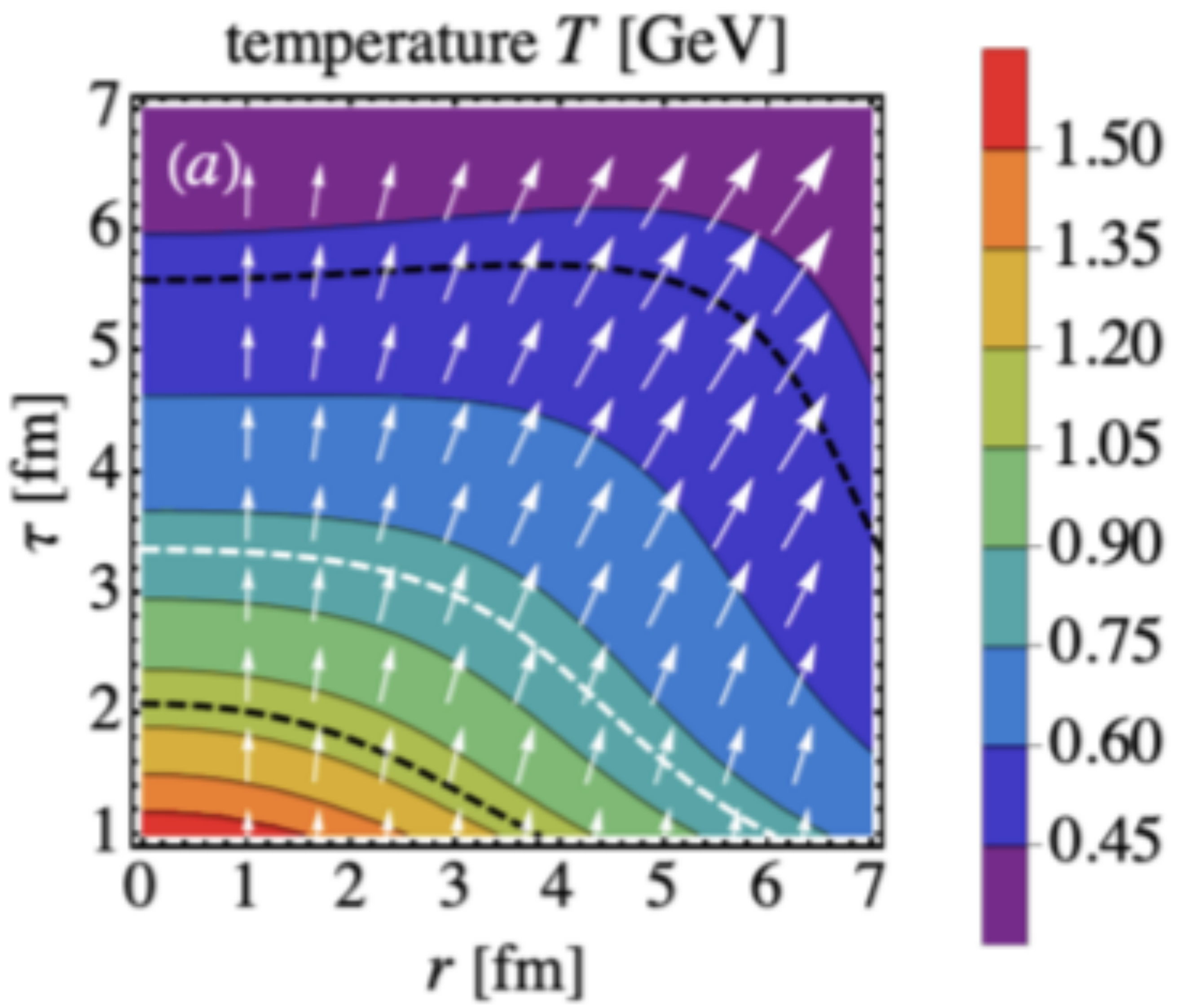
At late times, transverse expansion dominates: $\theta_\perp/\theta_L \sim 4$: $\frac{dT}{d\tau} = -\frac{5}{3}\frac{T}{\tau} \implies T \sim \tau^{-5/3}$

Knudsen number $Kn \propto \tau_R/\tau \sim \tau^{2/3}$ grows, medium does not thermalize.

The flow: $v^z = z/t,\ u^\phi = 0,\ u^r \neq 0$. Re-scale metric $ds^2 \to d\hat{s}^2 = ds^2/\tau^2$, followed by coordinate transformation

$$\rho = -\sinh^{-1}\left(\frac{1 - q^2\tau^2 + q^2r^2}{2q\tau}\right) \qquad \theta = -\tan^{-1}\left(\frac{2qr}{1 + q^2\tau^2 - q^2r^2}\right)$$

Here $\hat{u}^\mu = (1,\vec{0})$ and Weyl re-scaled quantities are used: $e(\tau, r) = \hat{e}(\rho)/\tau^4$



L. Du et al

Evolution equations: Gubser flow

- The energy-momentum tensor has 2 independent components $(\hat{e}, \hat{P}_T)$. Their evolution are given by:

$$\frac{d\hat{e}}{d\rho} = -2\tanh\rho\left(\hat{e} + \hat{P}_T\right)$$

$$\frac{d\hat{P}_T}{d\rho} = -\frac{1}{\hat{\tau}_R}\left(\hat{P}_T - \hat{P}\right) - 2\tanh\rho\ \hat{\zeta}_\perp$$

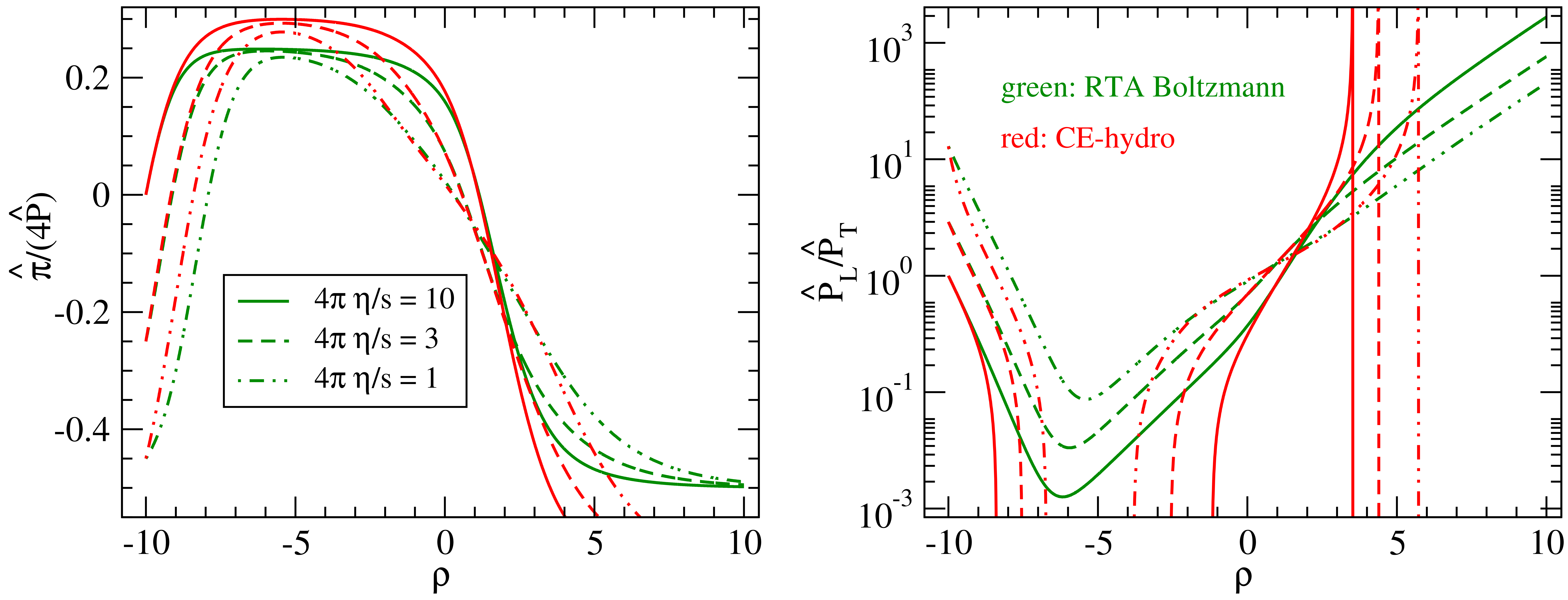
- Similar to Bjorken case, the equations are not closed as: $\hat{\zeta}^\perp = 2\hat{P}_T - \frac{1}{4}\int_{\hat{p}} \left(\hat{p}^\rho\right)^2 \left(\frac{\hat{p}_\Omega}{\cosh\rho}\right)^4 f$

Here $\hat{p}_\Omega = \sqrt{\hat{p}_\theta^2 + \hat{p}_\phi^2/\sin^2\theta}$ and $\hat{p}^\rho = \sqrt{\hat{p}_\Omega^2/\cosh^2\rho + \hat{p}_\eta^2}$

As before, truncate using $f \to f_{ME}$

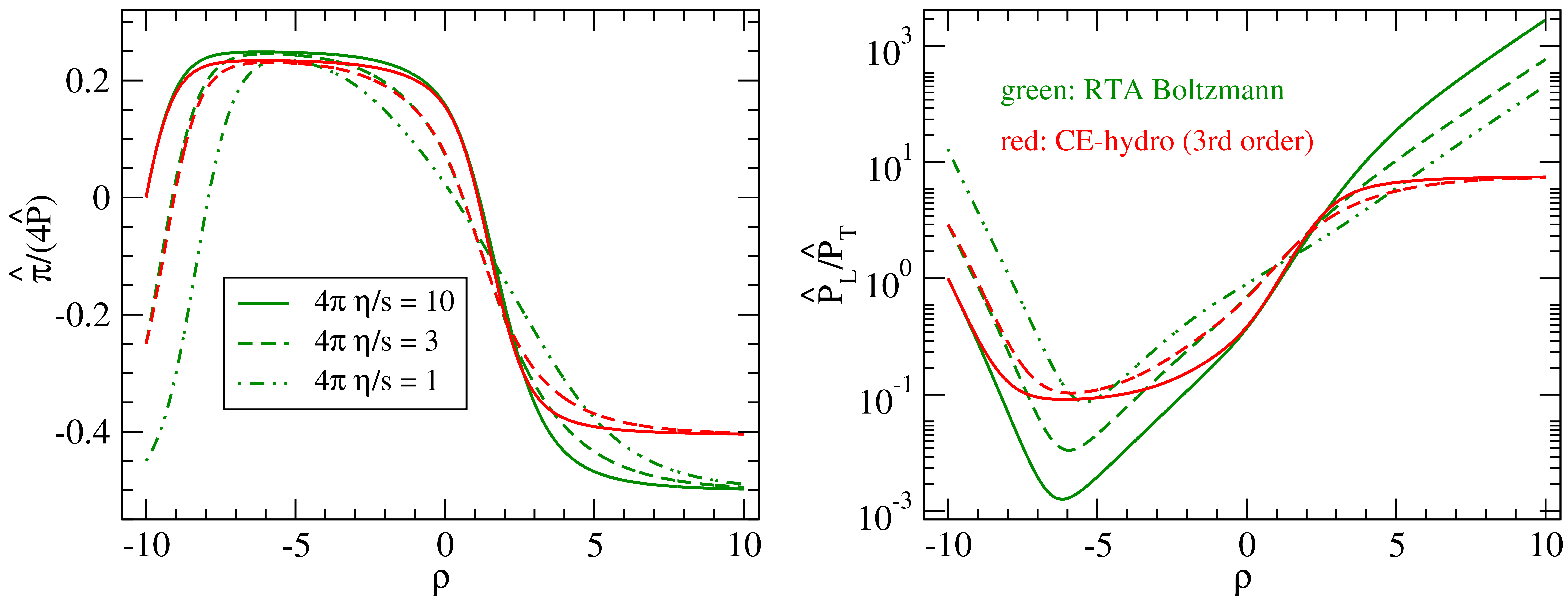
$$f_{ME} = \exp\left(-\hat{\Lambda}\hat{p}^\rho - \frac{\hat{\gamma}}{\hat{p}^\rho}\left(\frac{\hat{p}_\Omega^2}{\cosh\rho^2} - \hat{p}_\eta^2\right)\right)$$

Standard second-order hydro



- Rapid transverse expansion in Gubser flow at late times prevents system from thermalizing. Fluid approaches transverse free-streaming $\hat{P}_T \to 0$.
- Standard hydro breaks down. Transverse and longitudinal pressures become negative.

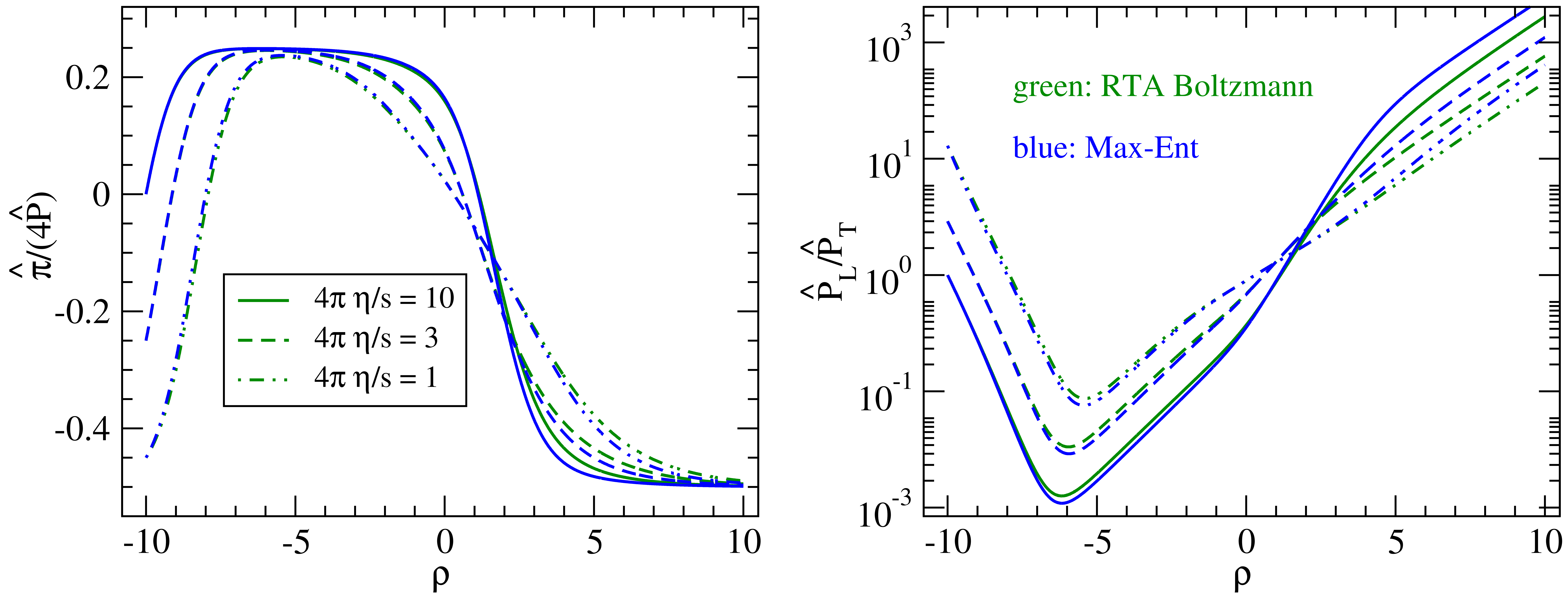
Third-order hydro



- Third-order CE yields incorrect asymptotic value of $\hat{\pi}/(4\hat{P}) \approx -0.4$
- For initializations $\hat{\pi}/(4\hat{P}) \lesssim -0.4$, third-order CE equations become numerically unstable.

Max-Ent framework

C.C., Heinz, Schaefer, PRC 108 (2023), 034907



- Max-Ent framework describes the far-off-equilibrium regimes satisfactorily. Effective pressures remain positive.

Tests of the Maximum-Entropy framework

- Test 3: Energy momentum evolution in a finite slab

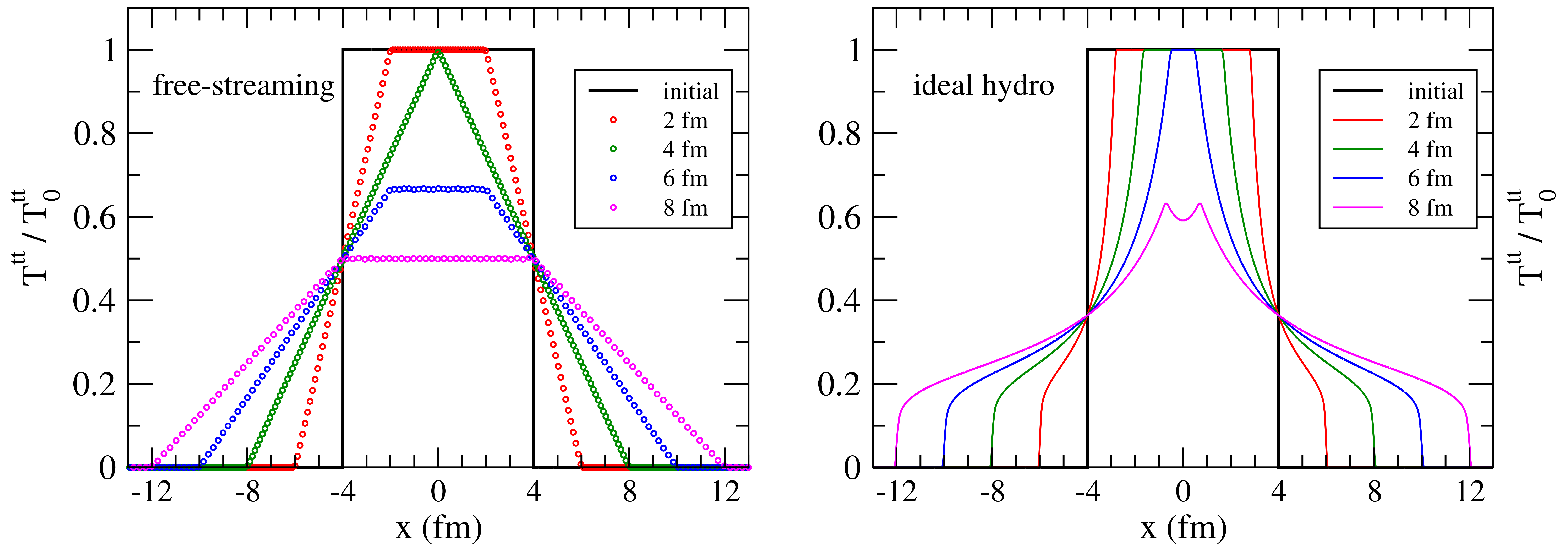
In heavy-ion collisions, the matter expands $\sim$ boost-invariantly along the beam direction.

In contrast, in transverse plane the matter is $\sim$ at rest initially with finite extent. The simplest problem to understand is the relativistic expansion of a finite slab into vacuum.

In preparation (Blaizot, C.C., Jaiswal, Schaefer)

Finite slab using hydro and free-streaming

- Consider a slab of matter that is finite along x. Take two extreme limits: non-interacting particles $\tau_R \to \infty$ and ideal hydro $\tau_R \to 0$.

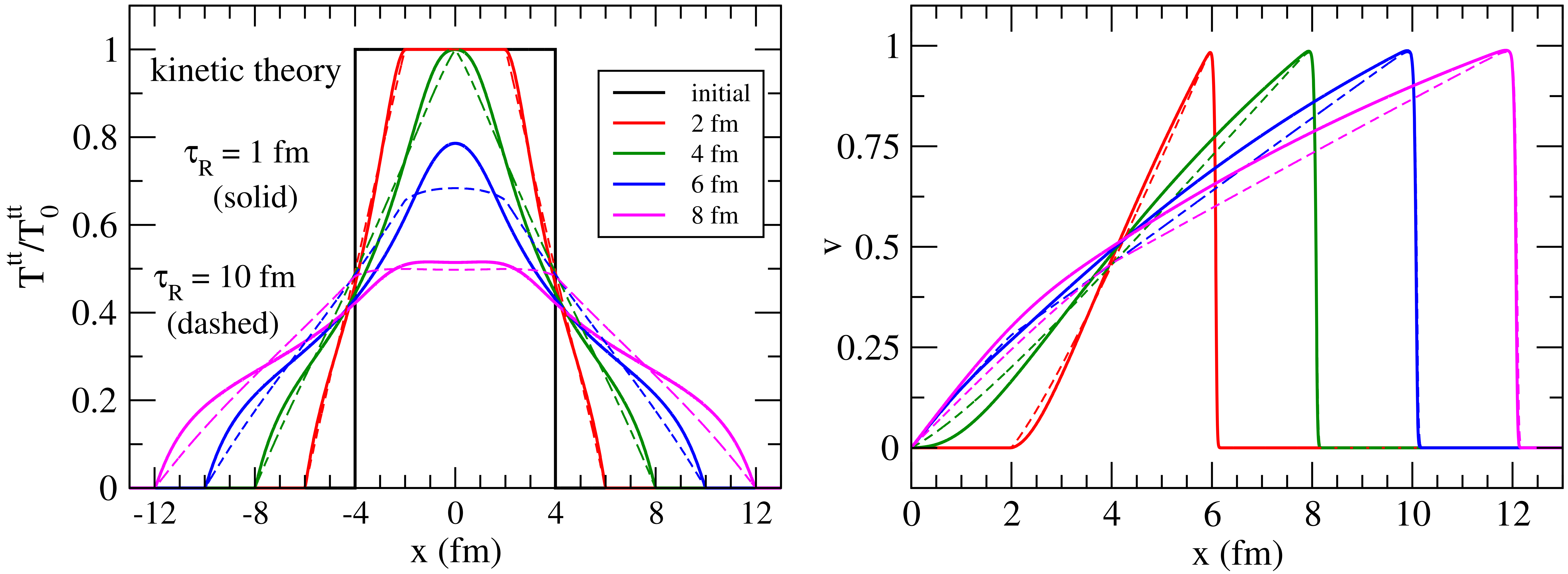


- In both cases, a rarefaction wave travels inward: propagation speed = c for free-streaming, and $c_s = 1/\sqrt{3}$ for ideal hydro. Shock front absent in free-streaming kinetics.

Finite slab: kinetic theory

- Now solve kinetics at finite τ_R:

$$\frac{\partial f}{\partial t} + v_p^x \frac{\partial f}{\partial x} = -\frac{u \cdot v_p}{\tau_R}\left(f - f_{eq}\right)$$



- Kinetics smoothly interpolates between two extremes: no surprises.

Max-Ent framework for finite slab

- Now use the max-ent approach: We have conservation equations

$$\partial_t T^{00} + \partial_x T^{0x} = 0 \qquad \partial_t T^{0x} + \partial_x T^{xx} = 0$$

The evolution of T^{xx} involves higher order moments. Use kinetics.

- If particles are massless, and we are only interested in $T^{\mu\nu}$ evolution, solve energy-weighted moments

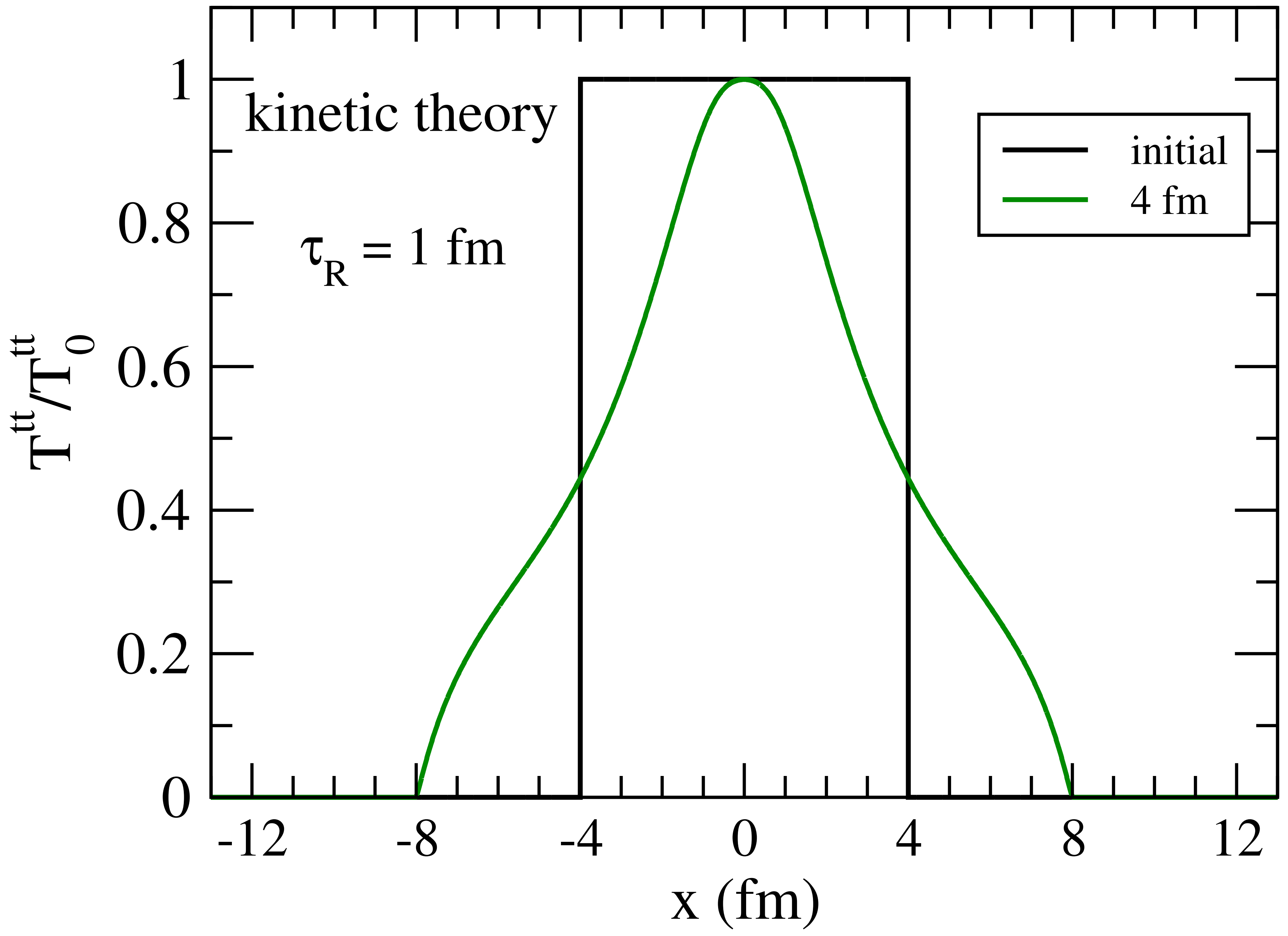
Kurkela, Wiedemann, Wu (2018)

$$\frac{\partial F}{\partial t} + \vec{v}_p \cdot \vec{\nabla} F = -\frac{u \cdot v_p}{\tau_R}\left(F - F_{eq}\right) \qquad F \equiv \int \frac{dp}{2\pi^2} p^3 f(t, \vec{x}, \vec{p}) \qquad F_{eq} \equiv \frac{3T^4/\pi^2}{\left(u \cdot v_p\right)^4}$$

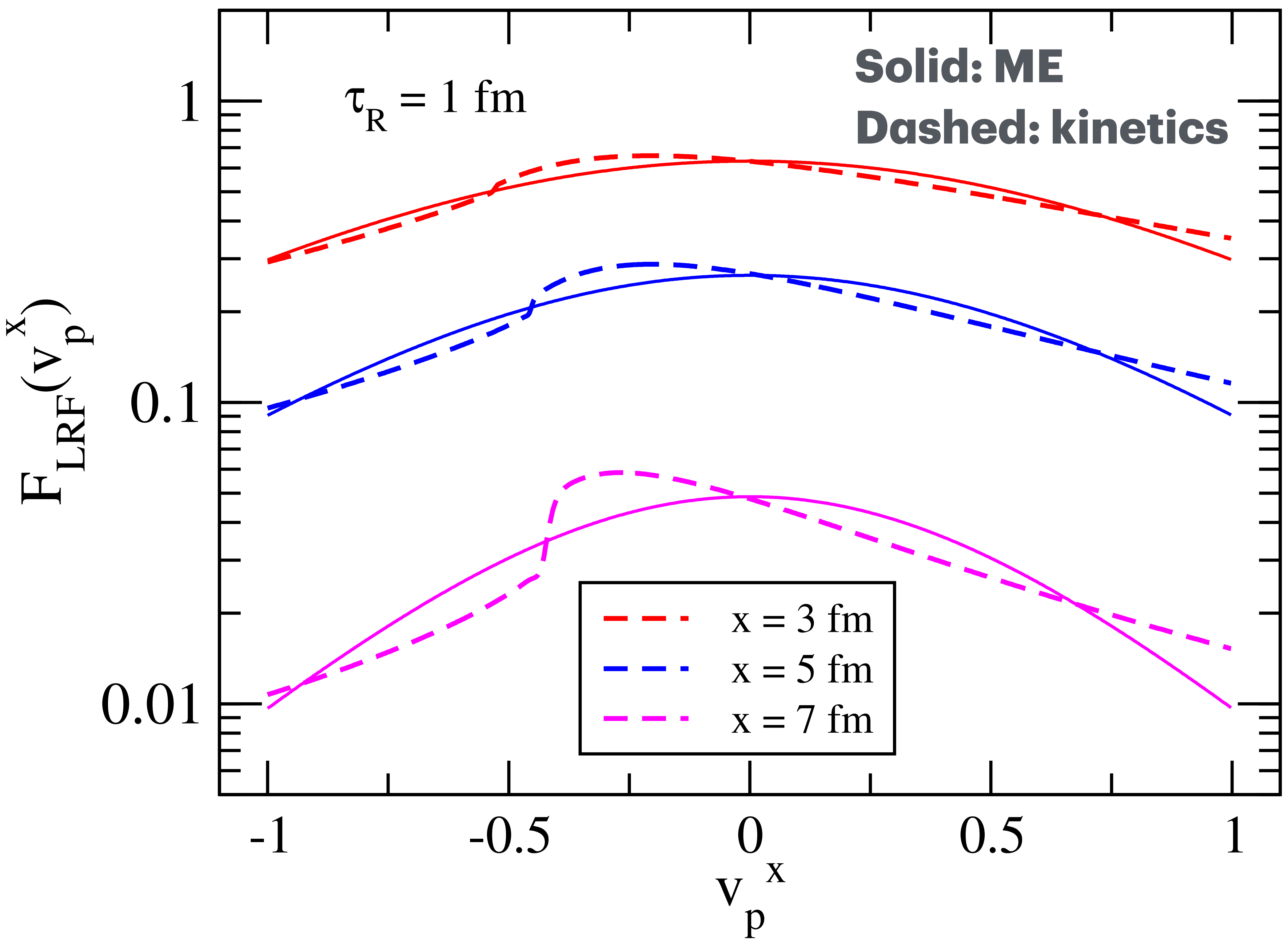
- Evolution of T^{xx} is thus $\partial_t T^{xx} + \partial_x \mathcal{K}^{xxx} = C_F^{xx}$ $\qquad \mathcal{K}^{xxx} = \int_{\Omega_p} \left(v_p^x\right)^3 F$

- Use maximum entropy distribution: $F_{ME} = \dfrac{3}{\pi^2}\left[\dfrac{u \cdot v_p}{\Lambda_{\mu\nu} v_p^\mu v_p^\nu}\right]^4$

Max-Ent framework in finite slab



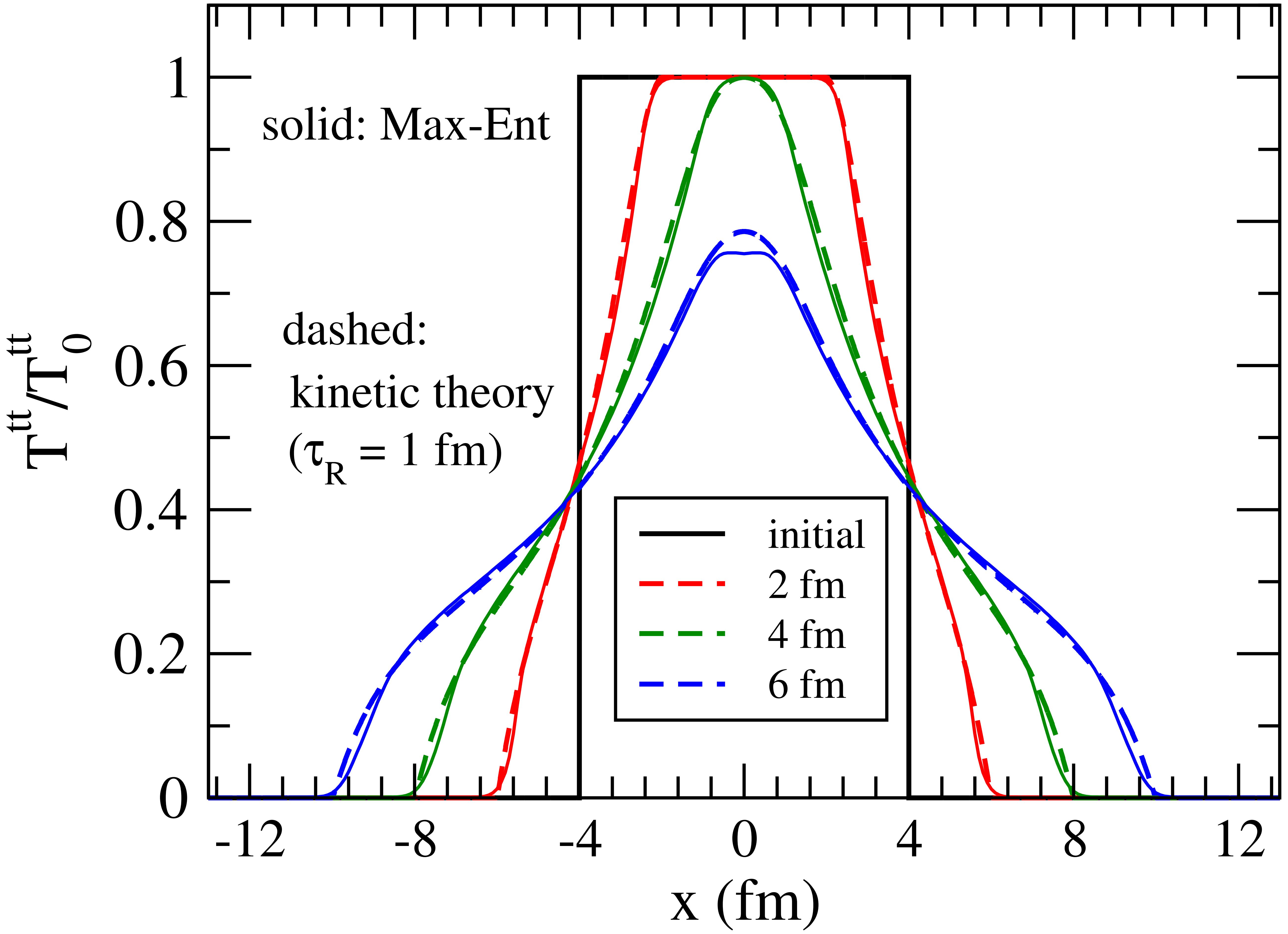
Angular distribution of energy density in fluid rest frame



Max-Ent captures average properties of exact microscopic distribution.

Max-Ent framework in finite slab

In preparation (Blaizot, C.C., Jaiswal, Schaefer)



Max-Ent describes energy density evolution reasonably well (compared to kinetics).

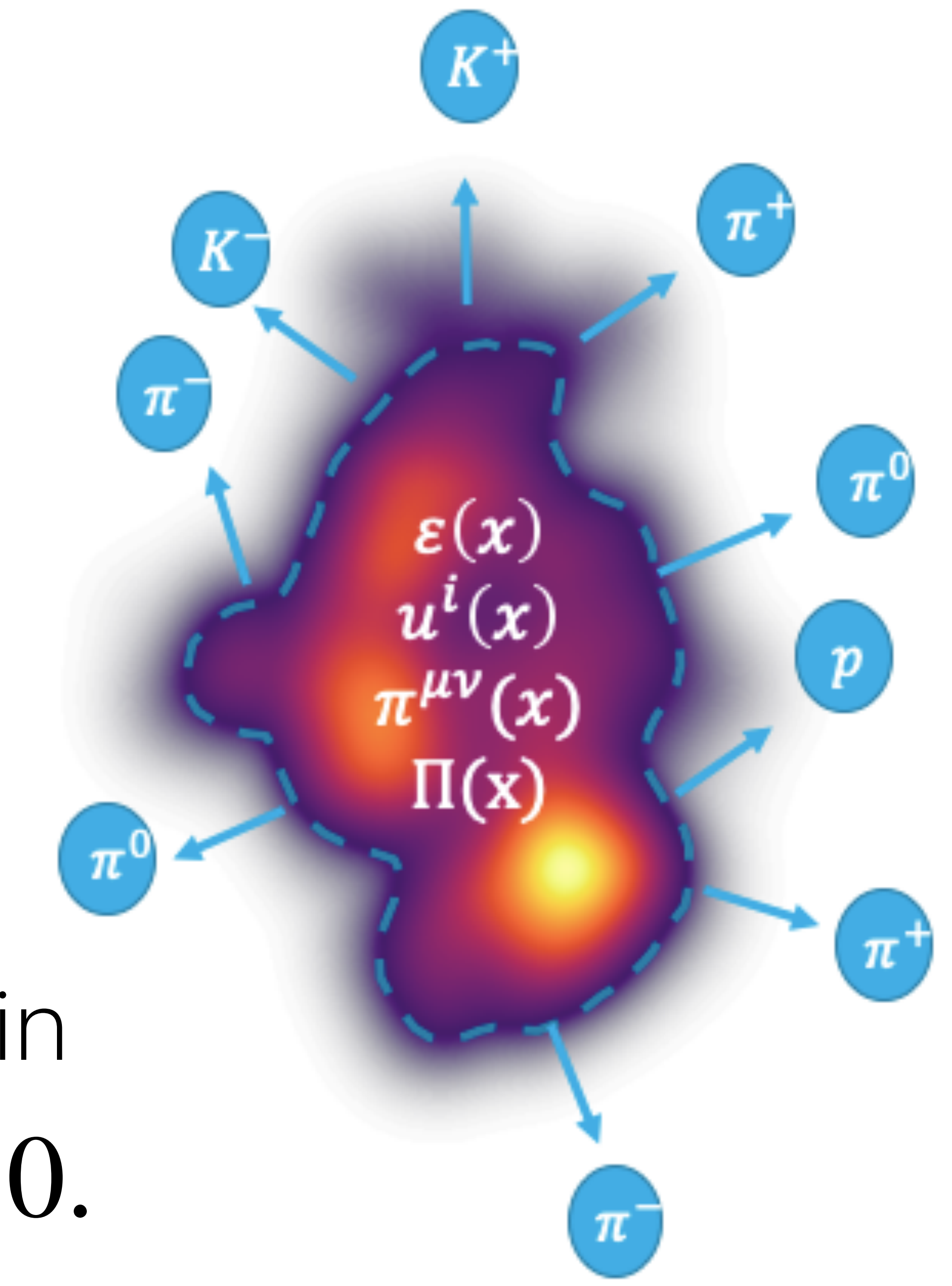
Freeze-out in heavy-ion collisions

Everett, C.C., Heinz PRC '21

- Hydrodynamics ceases to be valid when the Knudsen number $Kn \sim \partial \cdot u/T \gtrsim 1$.
  Need a change in language: convert hydrodynamic fields to particles

$$E_p \frac{dN_p}{d^3p} = \int d\Sigma_\mu \, p^\mu \, f(x,p)$$

  - Information available at freeze-out from the preceding hydro evolution: $\left(e, u^\mu, \Pi, \pi^{\mu\nu}\right)$. They provide constraints on moments of $f(x,p)$



- Viscous corrections in δf present significant sources of uncertainty in extraction of (η, ζ). Also, breaks down at large momenta: $f(x,p) < 0$.

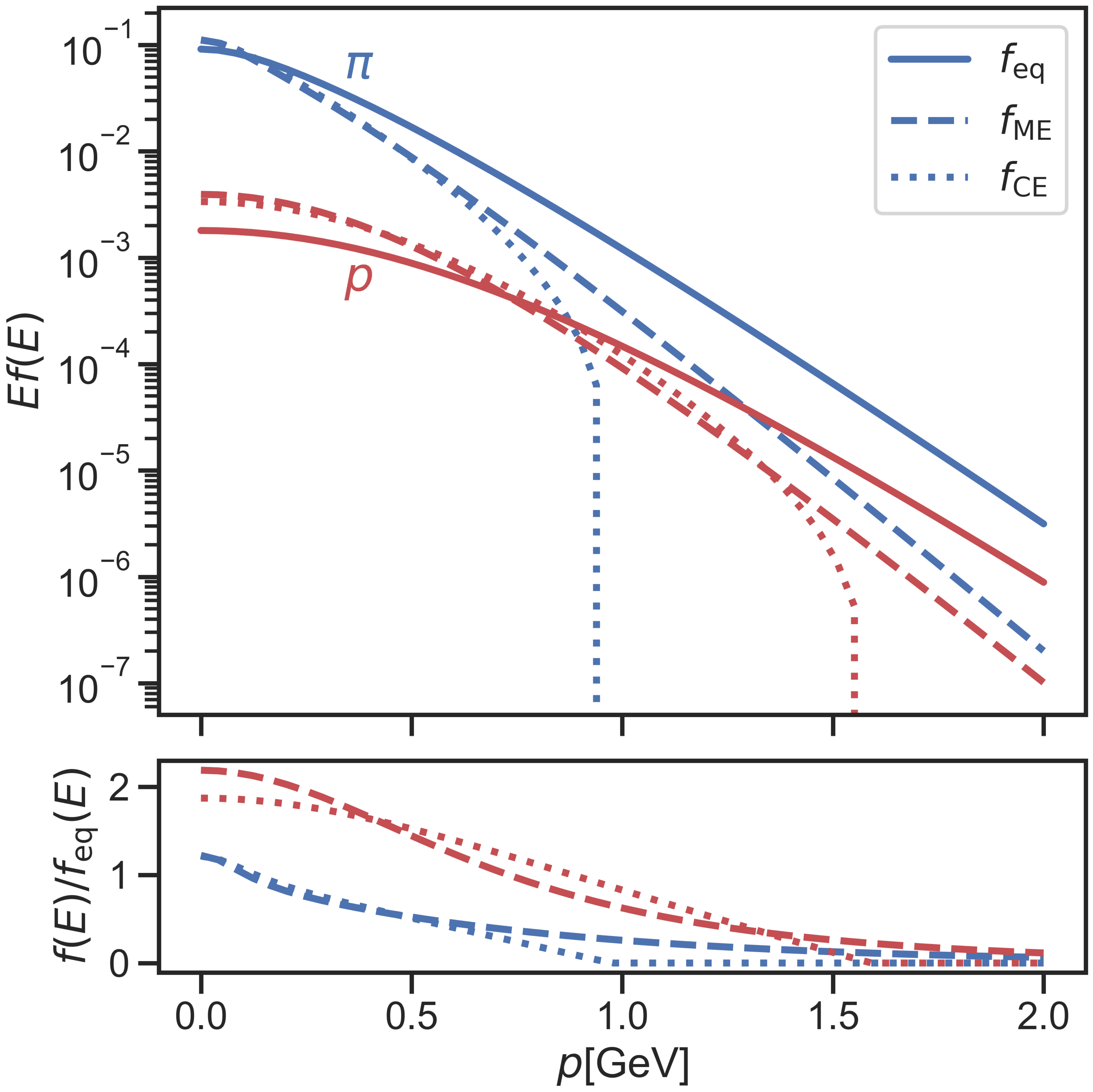
- Use the maximum-entropy distribution.

Summary

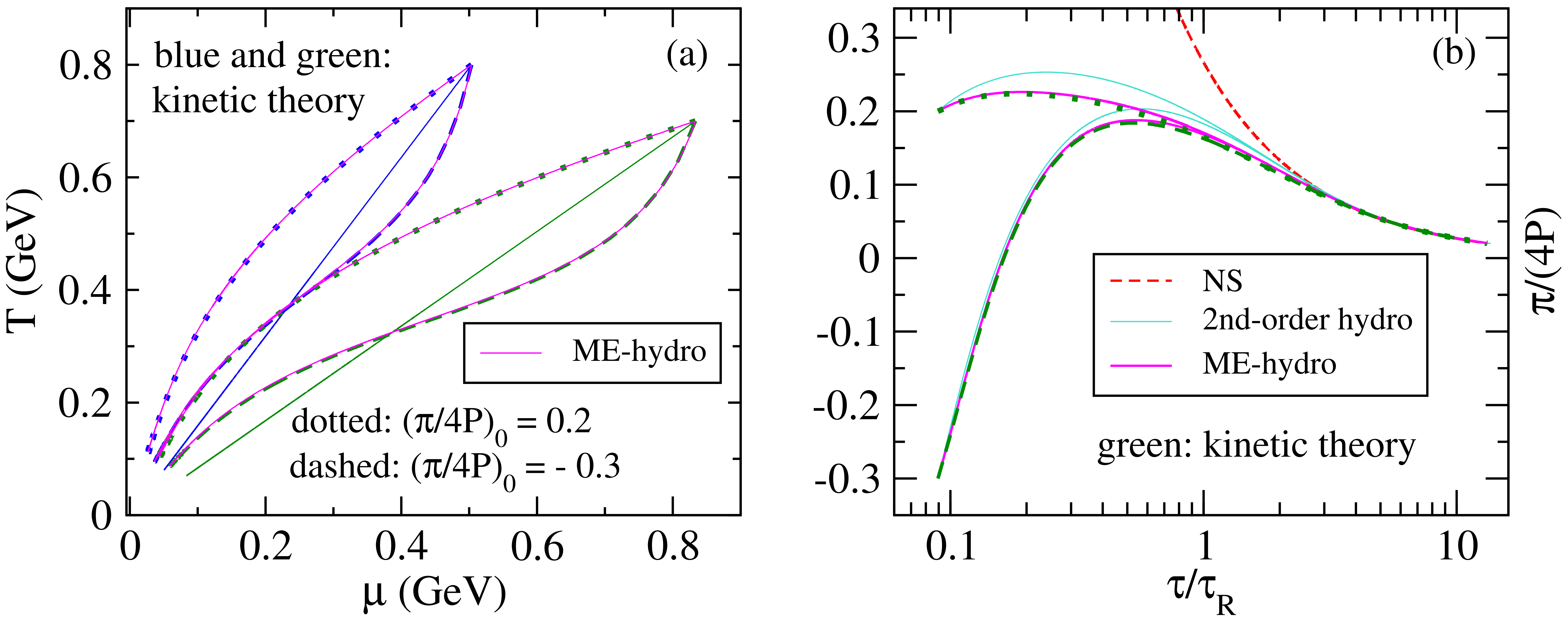
- Israel-Stewart like hydrodynamic theories can capture certain features of kinetic theory even when the system is not close to local equilibrium.
- Presented the derivation of a far-off-equilibrium macroscopic theory using a maximum-entropy distribution
    - This scheme does not introduce ad-hoc assumptions about flow being modeled; uses information contained in conserved currents only.
    - Max-Ent accurately describes kinetic theory evolution of $T^{\mu\nu}$ in both far and near-equilibrium regimes of Bjorken and Gubser flows. It models nicely the expansion of a finite slab of matter.
    - The description of $T^{\mu\nu}$ within this approach for more general flow profiles remain to be explored.

Thank you!

Backup: Max-Ent for freeze-out



Backup: Inclusion of chemical potential



Backup: entropy

- The canonical entropy $S = -\sum_i p_i \ln(p_i)$ for a continuous distribution:

$$S = -\int \frac{d^{3N}x\, d^{3N}p}{N!} \rho \ln(\rho),$$

where,

$$\rho(x_1, \cdots, x_N, p_1, \cdots, p_N) = \frac{\exp(-\beta H_N(x_1, \cdots, x_N, p_1, \cdots, p_N))}{Z(T, V, N)}$$

- Due to weak interaction,

$$H_N = \sum_i H_i, \quad Z(T, V, N) = Z(T, V, 1)^N = V^N n^N / N!,$$

where n is number density. Thus,

$$S = -\frac{\beta V^N}{Z(T, V, 1)^N} \int d^{3N}p\, H(p) \exp(-\beta H(p)) - \ln(Z(T, V, N))$$

Backup

- For large N, $\ln(Z(T, V, N)) \approx N$. Thus,

$$S = V \int d^3p \, (\beta \, H(p) \, f_{eq} + f_{eq}) \, ,$$

and the entropy density:

$$s = - \int d^3p \, f_{eq} \, (\ln(f_{eq}) - 1) \, .$$

- Out of equilibrium, replace $f_{eq} \to f$. Relativistic version,

$$s = - \int dP \, (u \cdot p) \, f \, (\ln(f) - 1) \, .$$

Backup: Gubser symmetries [R. Loganayagam (2008)]

- Equations of hydro are Lorentz covariant: admits rotationally and boost-invariant solutions.
- Hydro equations also have conformal invariance: should admit conformally invariant solutions.
- Under a conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$
- Weyl covariant derivative $\mathcal{D}_\mu T^{\mu\nu} \to e^{-w\phi} \tilde{\mathcal{D}}_\mu \tilde{T}^{\mu\nu}$ if $T^{\mu\nu} \to e^{-w\phi} \tilde{T}^{\mu\nu}$
- Using definition of $\mathcal{D}$ one can show

$$\mathcal{D}_\mu T^{\mu\nu} = d_\mu T^{\mu\nu} + A^\nu T^\mu_\mu$$

where $A^\mu = \dot{u}^\mu - (\theta/3) u^\mu$

- Hydro equations are conformal if $T^\mu_\mu = m^2 \int dP\, f = 0$.

Backup: Gubser symmetries S.S. Gubser, PRD 82 085027 (2010)

- Instead of translational invariance (whose generators are $\xi_i = \frac{\partial}{\partial x^i}$), Gubser uses invariance under the group $SO(3)_q$ whose generators are $\partial/\partial\phi$, $\partial/\partial\eta$, and

$$\xi_i = \frac{\partial}{\partial x^i} + q^2 \left[2x^i x^\mu \frac{\partial}{\partial x^\mu} - x^\mu x_\mu \frac{\partial}{\partial x^i} \right], \quad (i = 1, 2)$$

  $1/q \approx$ transverse size

- These generators are easy to understand in $dS_3 \times R$

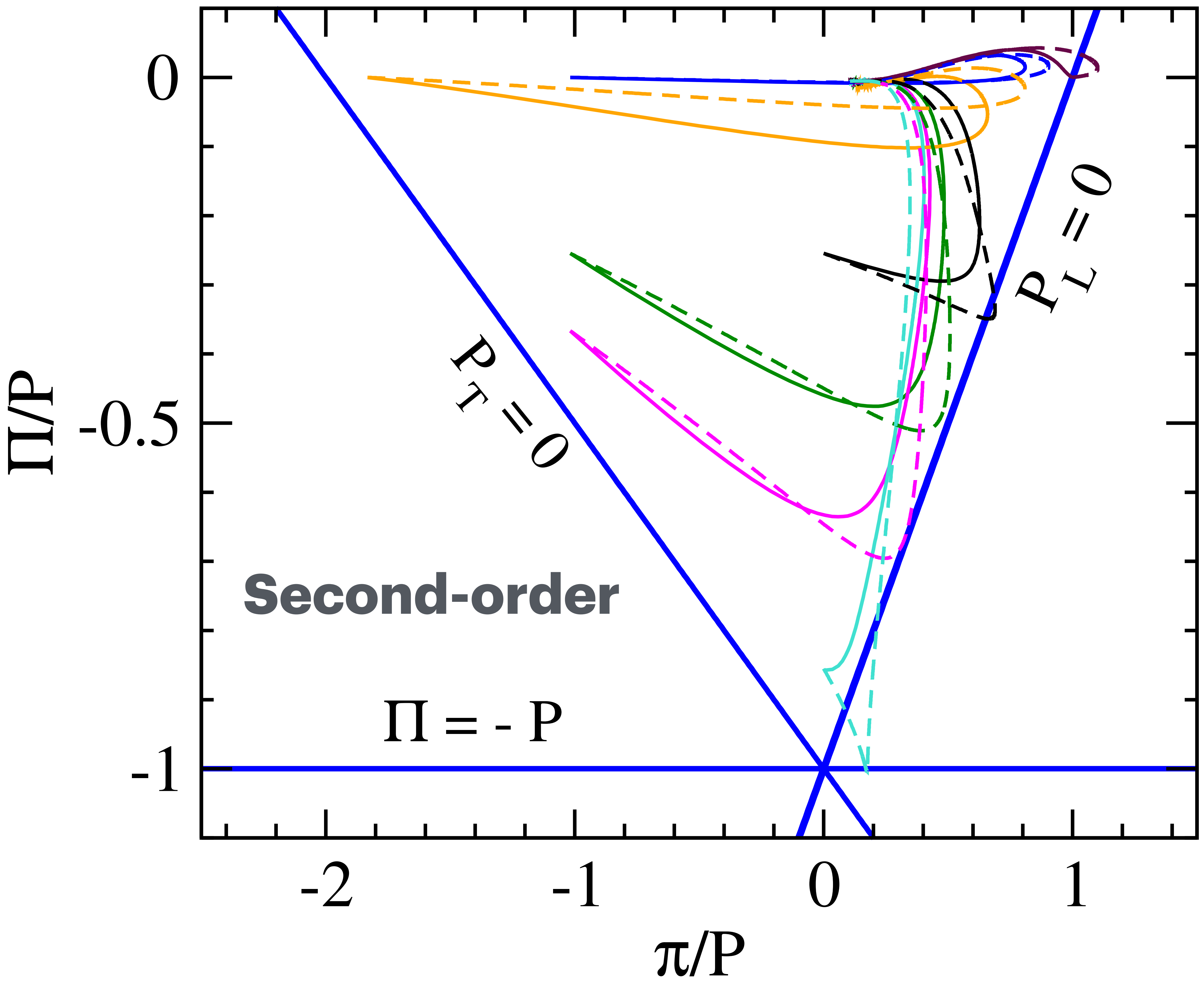
$$d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} = d\rho^2 - \cosh^2\rho \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) - d\eta^2,$$

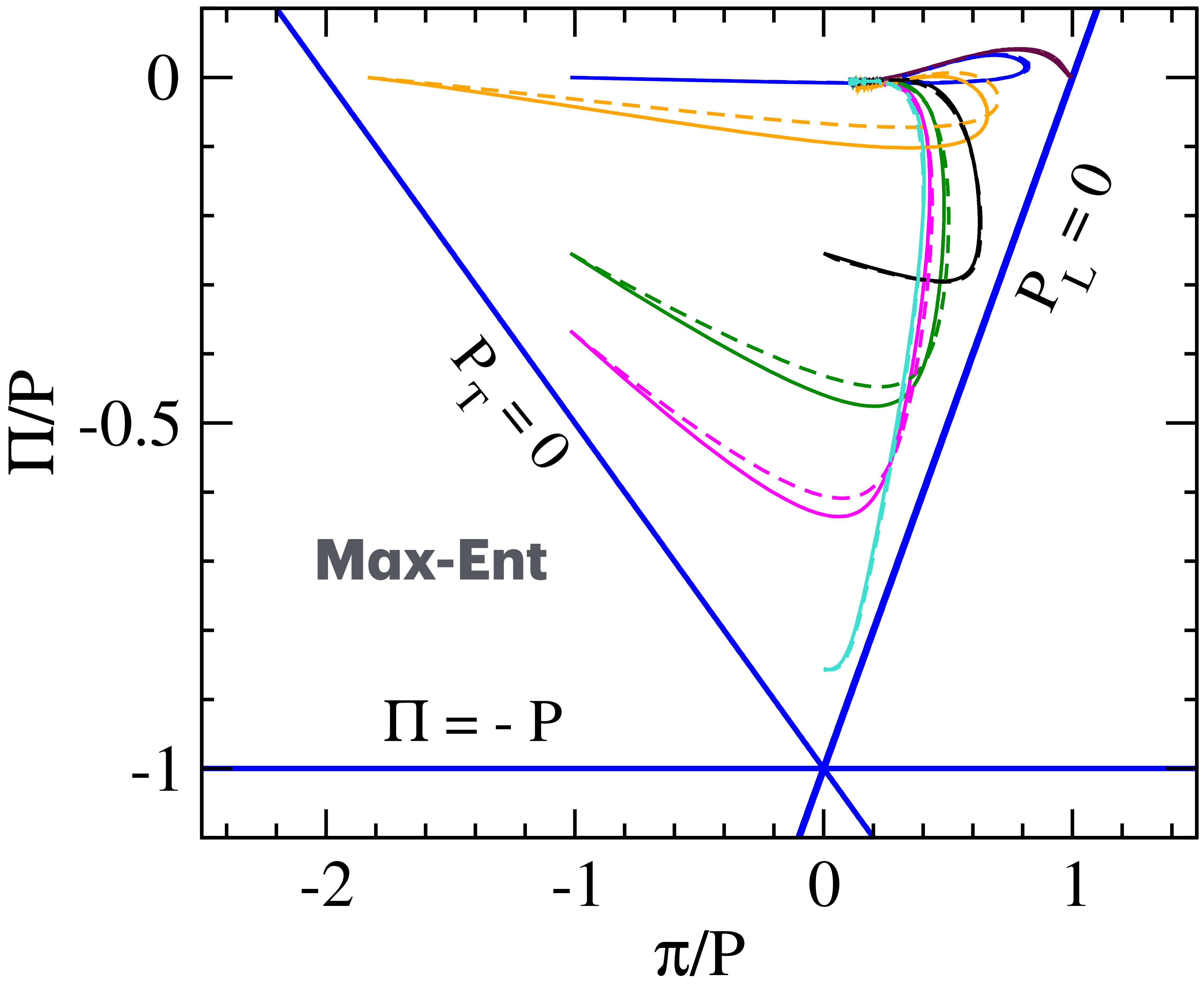
  where they correspond to rotations in (θ, ϕ):

$$\rho = -\sinh^{-1}\left(\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau} \right), \quad \theta = \tan^{-1}\left(\frac{2qr}{1 + q^2\tau^2 - q^2 r^2} \right),$$

- The only time-like four vector invariant under these transformations $[\xi, \hat{u}] = 0$ is $\hat{u}^\mu = (1, 0, 0, 0)$.

Kinetic theory bounds





Standard hydro breaks bounds on positive effective pressures. Max-Ent preserves them.