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Hydrodynamics without matching conditions

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Foundations and Applications of Relativistic Hydrodynamics

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What you will see

- ✔ Basics of fluid dynamics, Matching conditions
- ✔ First order hydrodynamics from kinetic theory
- ✔ Hydrodynamics without an equilibrium reference state (from kinetic theory)
- ✔ Conclusions

What you will see

- ✔ Basics of fluid dynamics, Matching conditions

- ✔ First order hydrodynamics from kinetic theory

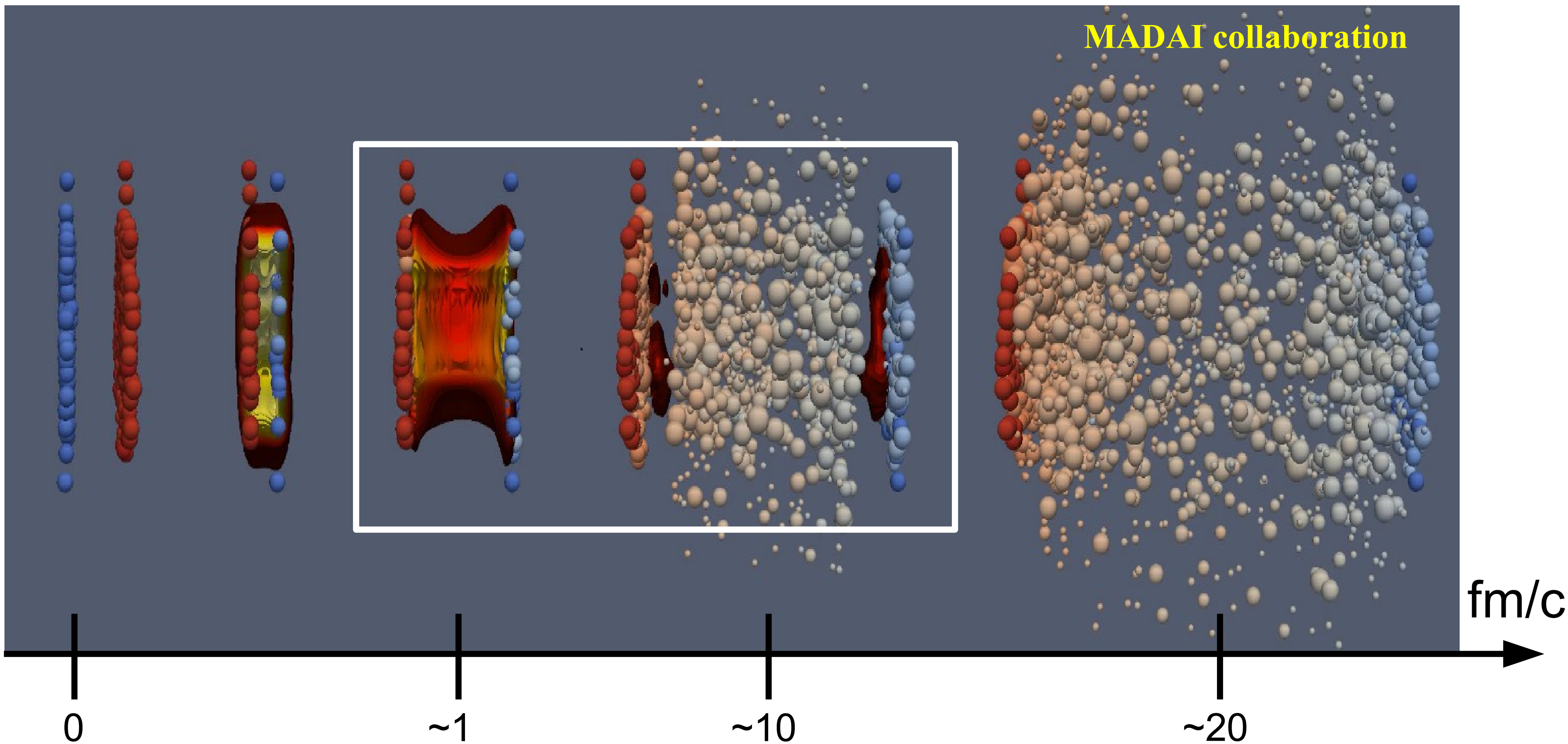
- ✔ Hydrodynamics without an equilibrium reference state (from kinetic theory)

Fluid dynamics for dilute gases

- ✔ Conclusions

Current Theoretical Description of HIC

Empirical: Fluid-dynamical modeling of heavy ion collisions works well at RHIC and LHC energies



Main assumption: (transient) fluid dynamics can be applied at very early times ~1 fm

Fluid dynamics

Effective theory describing the dynamics of a system over long-times and long-distances

Conservation laws

+

Equation of state

+

constitutive/dynamical relations

Validity of fluid dynamics

- **proximity to (local) equilibrium**

- **"small" gradients**

Separation of scales $\rightarrow$ **macroscopic:** L **microscopic:** ℓ

Knudsen number: $K_N \sim \dfrac{\ell}{L} \ll 1$

Ideal fluid dynamics

<u>*Conservation laws*</u>

$$\partial_\mu N^\mu_{(0)} = 0$$ **net-charge conservation**

$$\partial_\mu T^{\mu\nu}_{(0)} = 0$$ **energy-momentum conservation**

<u>*local thermodynamic equilibrium:*</u> existence of a local reference frame (rest frame) where

$$T^{\mu\nu}_{\mathrm{RF}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

EoS

$$N^\mu_{\mathrm{RF}} = (n, 0, 0, 0)^T$$
$$S^\mu_{\mathrm{RF}} = (s, 0, 0, 0)^T$$

defines a velocity field

$$N^\mu = \Lambda^\mu_\alpha(\mathbf{u})\, N^\alpha_{\mathrm{RF}} \,,$$
$$S^\mu = \Lambda^\mu_\alpha(\mathbf{u})\, S^\alpha_{\mathrm{RF}} \,,$$
$$T^{\mu\nu} = \Lambda^\mu_\alpha(\mathbf{u})\, \Lambda^\nu_\beta(\mathbf{u})\, T^{\alpha\beta}_{\mathrm{RF}} \,,$$

Ideal fluid dynamics

Conservation laws

$$\partial_\mu N^\mu_{(0)} = 0$$ **net-charge conservation**

$$\partial_\mu T^{\mu\nu}_{(0)} = 0$$ **energy-momentum conservation**

local thermodynamic equilibrium: existence of a local reference frame (rest frame) where

$$N^\mu_{\mathrm{ideal}} \equiv N^\mu_{(0)} = n u^\mu \ ,$$

$$S^\mu_{\mathrm{ideal}} \equiv S^\mu_{(0)} = s u^\mu \ ,$$

$$T^{\mu\nu}_{\mathrm{ideal}} \equiv T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} p \ ,$$

4-velocity: flow of energy, net-charge, entropy

EoS: $$s_0 \equiv s_0(\varepsilon, n) \quad \beta_0 = \left.\frac{\partial s}{\partial \varepsilon}\right|_n \ , \qquad \alpha_0 = \left.\frac{\partial s}{\partial n}\right|_\varepsilon \ ,$$

Disspative fluid dynamics

Conservation laws

$$\partial_\mu N^\mu = 0$$

net-charge conservation

$$\partial_\mu T^{\mu\nu} = 0$$

energy-momentum conservation

Corrections to equilibrium

$$N^\mu = n u^\mu + \nu^\mu,$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu},$$

net-charge diffusion 4-current

isotropic pressure

energy diffusion 4-current

Shear stress tensor

Projection operator: $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$

Fictitious equilibrium must be defined

Special directions

$$T^{\mu\nu} \mathsf{u}_\nu \equiv \mathsf{eu}^\mu$$

follows the flow of energy

don't have to be thermal

$$N^\mu \equiv \mathsf{nv}^\mu$$

follows the flow of particles

$$\mathsf{u}^\mu - \mathsf{v}^\mu \approx \frac{1}{\mathsf{e} + \mathsf{p}} Q^\mu$$

heat flow

Definition of “equilibrium state”

Landau and Eckart picture

$$\varepsilon \equiv u_\nu u_\mu T^{\mu\nu}, \longrightarrow$$ definition of **energy density**

$$n \equiv u_\mu N^\mu. \longrightarrow$$ definition of **net-charge density**

$$s_0 \equiv s_0(n, \varepsilon),$$ definition of an eq. entropy density

$$\beta_0 = \left.\frac{\partial s}{\partial \varepsilon}\right|_n, \qquad \alpha_0 = \left.\frac{\partial s}{\partial n}\right|_s, \qquad p_0 = -\varepsilon + T_0 s_0 + \mu_0 n.$$

Definition of velocity

$$u_\mu T^{\mu\nu} = \varepsilon u^\nu \quad \text{or} \quad N^\mu = n u^\mu \quad \text{or} \ldots$$

Fictitious equilibrium state

- **We also want to explore other matching conditions ...**

Generalized "first order theory" only works for other matching conditions

Fábio S. Bemfica, Marcelo M. Disconzi, Jorge Noronha, Phys.Rev.D 98 (2018) 10, 104064

Introduce a reference local equilibrium state

$$n \equiv n_0(\alpha, \beta) + \delta n, \qquad \varepsilon \equiv \varepsilon_0(\alpha, \beta) + \delta\varepsilon,$$

$$P \equiv P_0(\alpha, \beta) + \Pi,$$

“Usual” approach

Chapman-Enskog expansion (gradient expansion)

$$\underbrace{k^\mu \partial_\mu f_{\mathbf{k}}}_{1/L} = \underbrace{C\left[f\right]}_{1/\lambda}$$

1/L **1/λ** → mean free-path

Search for asymptotic solution with **L >> λ**

Hydrodynamic regime

"Usual" approach

Chapman-Enskog expansion (gradient expansion)

Perturbative solution of the Boltzmann eq.

$$\epsilon E_{\mathbf{p}} D f_{\mathbf{p}} + \epsilon p^{\mu} \nabla_{\mu} f_{\mathbf{p}} = C[f_{\mathbf{p}}] \quad \begin{cases} f_{\mathbf{p}} = \sum_{i=0}^{\infty} \epsilon^i f_{\mathbf{p}}^{(i)}, \\ D f_{\mathbf{p}} = \sum_{i=0}^{\infty} \epsilon^i D^{(i)} f_{\mathbf{p}}, \end{cases}$$

Can be solved analytically for
self-interacting $\lambda\varphi^4$ scalar field theory
(classical, massless limits)

$$\sigma_T(s) = \frac{\lambda^2}{32\pi s} \equiv \frac{g}{s}$$

Linearized Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} \approx \hat{L}\phi_{\mathbf{k}}$$

near local equilibrium

self-interacting $\lambda\varphi^4$ scalar field theory

$$\sigma_T(s) = \frac{\lambda^2}{32\pi s} \equiv \frac{g}{s}$$

$$\hat{L}\phi_{\mathbf{k}} = \frac{g}{2}\int dK' dP dP' f_{0\mathbf{k}'} (2\pi)^5 \delta^{(4)}(k + k' - p - p')(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'})$$

Eigenfunctions and Eigenvalues can be calculated exactly

(*massless*, *classical* limits)

$$\hat{L}\left[L_{n\mathbf{p}}^{(2\ell+1)} p^{\langle\mu_1}\cdots p^{\mu_\ell\rangle}\right] = \chi_{n\ell} L_{n\mathbf{p}}^{(2\ell+1)} p^{\langle\mu_1}\cdots p^{\mu_\ell\rangle},$$

$$\chi_{n\ell} = -\frac{g}{2} I_{0,0}\left(\frac{n+\ell-1}{n+\ell+1} + \delta_{n0}\delta_{\ell 0}\right),$$

gsd, J. Noronha, 2209.10370

All hydrodynamic theories *and* their transport coefficients can be derived (or obtained) analytically

Relativistic Navier-Stokes theory

Perturbative solution

$$\epsilon E_{\mathbf{p}} D f_{\mathbf{p}} + \epsilon p^{\mu} \nabla_{\mu} f_{\mathbf{p}} = C[f_{\mathbf{p}}] \quad \begin{cases} f_{\mathbf{p}} = \sum_{i=0}^{\infty} \epsilon^i f_{\mathbf{p}}^{(i)}, \\ D f_{\mathbf{p}} = \sum_{i=0}^{\infty} \epsilon^i D^{(i)} f_{\mathbf{p}}, \end{cases}$$

0-th order: $C[f_{\mathbf{k}}^{(0)}] = 0$ local equilibrium, $f_{\mathbf{p}}^{(0)} = f_{0\mathbf{p}}$

temperature and chemical potential appear

Relativistic Navier-Stokes theory

Perturbative solution

$$\epsilon E_{\mathbf{p}} D f_{\mathbf{p}} + \epsilon p^{\mu} \nabla_{\mu} f_{\mathbf{p}} = C[f_{\mathbf{p}}]$$

$$\begin{cases} f_{\mathbf{p}} = \sum_{i=0}^{\infty} \epsilon^{i} f_{\mathbf{p}}^{(i)}, \\ D f_{\mathbf{p}} = \sum_{i=0}^{\infty} \epsilon^{i} D^{(i)} f_{\mathbf{p}}, \end{cases}$$

0-th order: $C[f_{\mathbf{k}}^{(0)}] = 0$ local equilibrium, $f_{\mathbf{p}}^{(0)} = f_{0\mathbf{p}}$

temperature and chemical potential appear

1-st order: $$\frac{1}{4} L_{1\mathbf{p}}^{(3)} p_{\langle\mu\rangle} \nabla^{\mu} \alpha - \beta p_{\langle\mu} p_{\nu\rangle} \sigma^{\mu\nu} = \hat{L} \phi_{\mathbf{p}}$$

$$f_{\mathbf{k}}^{(1)} = f_{0\mathbf{k}} (1 + \phi_{\mathbf{k}})$$

Relativistic Navier-Stokes theory

0-th order: $C[f^{(0)}_{\mathbf{k}}] = 0$ local equilibrium, $f^{(0)}_{\mathbf{p}} = f_{0\mathbf{p}}$

temperature and chemical potential appear

1-st order: $$\frac{1}{4} L^{(3)}_{1\mathbf{p}} p_{\langle\mu\rangle} \nabla^\mu \alpha - \beta p_{\langle\mu} p_{\nu\rangle} \sigma^{\mu\nu} = \hat{L}\phi_{\mathbf{p}}$$

$$\phi_{\mathbf{p}} = \phi^{\text{hom}}_{\mathbf{p}} + \hat{L}^{-1}\left[\frac{1}{4} L^{(3)}_{1\mathbf{p}} p_{\langle\mu\rangle} \nabla^\mu \alpha - \beta p_{\langle\mu} p_{\nu\rangle} \sigma^{\mu\nu}\right]$$

$a + b_\mu p^\mu \longrightarrow a = 0$ and $b^\mu = z\nabla^\mu \alpha / (4\chi_{11})$

matching conditions

$$\int dP\, E^z_{\mathbf{p}} p^{\langle\mu\rangle} \delta f_{\mathbf{p}} = 0,$$

z = 0 → Eckart

z = 1 → Landau

Relativistic Navier-Stokes theory

Matching conditions and homogeneous solution

Combine equilibrium solution with homogenous contribution

$$\hat{f}_{0\mathbf{k}} = f_{0\mathbf{k}}\left(1+\phi_{\mathbf{k}}^{\mathrm{hom}}\right) = f_{0\mathbf{k}}\left(1+\underline{a+b_\mu k^\mu}\right)$$

"small" correction

$$\approx f_{0\mathbf{k}}\exp\left(a+b_\mu k^\mu\right)$$

$$= \exp\left(\alpha+\beta_\mu k^\mu\right)\exp\left(a+b_\mu k^\mu\right)$$

$$= \exp\left[\underline{\alpha+a}+\left(\underline{\beta_\mu+b_\mu}\right)k^\mu\right]$$

effective thermal potential   effective thermal 4-velocity

Relativistic Navier-Stokes theory

Relativistic Boltzmann equation

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C[f]$$

self-interacting $\lambda\varphi^4$ scalar field theory

classical, massless limits

$$\sigma_T(s) = \frac{\lambda^2}{32\pi s} \equiv \frac{g}{s}$$

Constitutive relations

$$\delta n = 0, \quad \nu^{\mu} = z\frac{3}{g\beta^2}\nabla^{\mu}\alpha,$$

$$\delta\varepsilon = 0, \quad h^{\mu} = (z-1)\frac{12}{g\beta^3}\nabla^{\mu}\alpha,$$

$$\pi^{\mu\nu} = \frac{96}{g\beta^3}\sigma^{\mu\nu}.$$

matching condition

$$\int dP\, E_{\mathbf{p}}^{z} p^{\langle\mu\rangle}\delta f_{\mathbf{p}} \equiv 0,$$

$z = 0 \rightarrow$ Eckart

$z = 1 \rightarrow$ Landau

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Is there a way to derive hydrodynamics without introducing a fictitious equilibrium state?

Boltzmann

Relativistic Boltzmann equation

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C\left[f\right]$$

momentum distribution

collision term

collision term – elastic 2-to-2 collisions

$$C\left[f\right] = \frac{1}{\nu}\int dK'dPdP'W_{\mathbf{kk}'\to\mathbf{pp}'}\left(f_{\mathbf{p}}f_{\mathbf{p}'}\tilde{f}_{\mathbf{k}}\tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}}f_{\mathbf{k}'}\tilde{f}_{\mathbf{p}}\tilde{f}_{\mathbf{p}'}\right)$$

$$\tilde{f}_{\mathbf{k}} \equiv 1 - af_{\mathbf{k}}$$

transition rate

$$W_{\mathbf{kk}'\to\mathbf{pp}'} = s\sigma(s,\Theta)\left(2\pi\right)^{6}\delta^{(4)}\left(k^{\mu}+k'^{\mu}-p^{\mu}-p'^{\mu}\right)$$

cross section – microscopic information

self-interacting $\lambda\varphi^4$ scalar field theory

$$\sigma_T(s) = \frac{\lambda^2}{32\pi s} \equiv \frac{g}{s}$$

Moments of the Boltzmann equation

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

$$\begin{aligned}
\int_k k^\mu \partial_\mu f_{\mathbf{k}} &= \int_k C\left[f\right] = 0 \\
\int_k k^\nu k^\mu \partial_\mu f_{\mathbf{k}} &= \int_k k^\nu C\left[f\right] = 0 \\
\int_k k^\alpha k^\nu k^\mu \partial_\mu f_{\mathbf{k}} &= \int_k k^\alpha k^\nu C\left[f\right] \\
\int_k k^\alpha k^\beta k^\nu k^\mu \partial_\mu f_{\mathbf{k}} &= \int_k k^\alpha k^\beta k^\nu C\left[f\right] \\
&\vdots
\end{aligned}$$

Moments of the Boltzmann equation

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

$$J^{\mu_1\cdots\mu_n} = \int_k k^{\mu_1}\cdots k^{\mu_n} f_{\mathbf{k}}$$

$$C^{\mu_1\cdots\mu_n} = \int_k k^{\mu_1}\cdots k^{\mu_n} C\left[f\right]$$

can calculate it exactly for this interaction

Conservation laws

$$\partial_\mu J^\mu = 0,$$

$$\partial_\mu J^{\mu\nu} = 0,$$

$$\partial_\mu J^{\mu\nu\alpha} = C^\alpha,$$

$$\partial_\mu J^{\mu\nu\alpha\beta} = C^{\alpha\beta},$$

$$\vdots$$

Non-hydro d.o.f.

Moments of the Boltzmann equation

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

$$\begin{aligned} J^{\mu_1 \cdots \mu_n} &= \int_k k^{\mu_1} \cdots k^{\mu_n} f_{\mathbf{k}} \\ C^{\mu_1 \cdots \mu_n} &= \int_k k^{\mu_1} \cdots k^{\mu_n} C\left[f\right] \end{aligned}$$

Conservation laws

$$\begin{aligned} \partial_\mu J^\mu &= 0, \\ \partial_\mu J^{\mu\nu} &= 0, \end{aligned}$$

$$\begin{aligned} \partial_\mu J^{\mu\nu\alpha} &= C^\alpha, \\ \partial_\mu J^{\mu\nu\alpha\beta} &= C^{\alpha\beta}, \\ &\vdots \end{aligned}$$

can calculate it exactly for this interaction, e.g.

$$C^{\mu\nu} = \frac{g}{3} J^\mu J^\nu - \frac{g}{6} \mathcal{A} J^{\mu\nu} - \frac{g}{12} g^{\mu\nu} J_\lambda J^\lambda$$

on-hydro d.o.f.

Moments of the Boltzmann equation

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

$$\mathcal{A} = \int dK \; f_{\mathbf{k}}.$$

$$\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu J^{\mu\nu} &= 0 \\
\partial_\mu J^{\mu\nu\alpha} &= \frac{g}{3}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) - \frac{g}{6}\mathcal{A} J^{\nu\alpha} \\
\partial_\mu J^{\mu\nu\alpha\lambda} &= \frac{g}{4} 3 J^{(\nu} J^{\lambda\alpha)} - \frac{g}{12} 3 J_\rho J^{\rho(\alpha} g^{\lambda\nu)} - \frac{g}{4}\mathcal{A} J^{\nu\alpha\lambda} \\
&\vdots
\end{aligned}$$

Moments of the Boltzmann equation

Turn this into a perturbative problem
(*similar to Chapman-Enskog*)

$$
\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu J^{\mu\nu} &= 0 \\
\varepsilon\, \partial_\mu J^{\mu\nu\alpha} &= \frac{g}{3}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) - \frac{g}{6}\mathcal{A} J^{\nu\alpha} \\
\varepsilon\, \partial_\mu J^{\mu\nu\alpha\lambda} &= \frac{g}{4} 3 J^{(\nu} J^{\lambda\alpha)} - \frac{g}{12} 3 J_\rho J^{\rho(\alpha} g^{\lambda\nu)} - \frac{g}{4}\mathcal{A} J^{\nu\alpha\lambda} \\
&\vdots
\end{aligned}
$$

Moments of the Boltzmann equation

Relativistic Boltzmann equation $$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

rewrite as

$$J^{\nu\alpha} = \frac{2}{\mathcal{A}}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) - \frac{6}{g\mathcal{A}}\partial_\mu J^{\mu\nu\alpha}$$

$$J^{\nu\alpha\lambda} = \frac{1}{\mathcal{A}} 3 J^{(\nu} J^{\lambda\alpha)} - \frac{1}{\mathcal{A}} J_\rho J^{\rho(\alpha} g^{\lambda\nu)} - \frac{4}{g\mathcal{A}}\partial_\mu J^{\mu\nu\alpha\lambda}$$

$$\vdots$$

Moments of the Boltzmann equation

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

rewrite as

$$J^{\nu\alpha} = \frac{2}{\mathcal{A}}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) - \frac{6}{g\mathcal{A}}\partial_\mu J^{\mu\nu\alpha}$$

$$J^{\nu\alpha\lambda} = \frac{1}{\mathcal{A}} 3 J^{(\nu} J^{\lambda\alpha)} - \frac{1}{\mathcal{A}} J_\rho J^{\rho(\alpha} g^{\lambda\nu)} - \frac{4}{g\mathcal{A}}\partial_\mu J^{\mu\nu\alpha\lambda}$$

$$\vdots$$

and iterate

Moments of the Boltzmann equation

Relativistic Boltzmann equation

$$k^{\mu}\partial_{\mu} f_{\mathbf{k}} = C\left[f\right]$$

Gradient expansion

$$J^{\nu\alpha} = \frac{2}{\mathcal{A}}\left(J^{\nu}J^{\alpha} - \frac{g^{\nu\alpha}}{4}J_{\lambda}J^{\lambda}\right)$$

$$\underbrace{-\frac{6}{g\mathcal{A}}\partial_{\mu}\left[\frac{1}{\mathcal{A}}3J^{(\nu}J^{\mu\alpha)} - \frac{1}{\mathcal{A}}J_{\rho}J^{\rho(\alpha}g^{\mu\nu)}\right]}_{\mathcal{O}\left(g^{-1}\partial\right)} + \mathcal{O}\left(g^{-2}\partial^{2}\right)$$

Leading order

$$J^{\nu\alpha} \approx \frac{2}{\mathcal{A}}\left(J^{\nu}J^{\alpha} - \frac{g^{\nu\alpha}}{4}J_{\lambda}J^{\lambda}\right)$$

Gradient expansion

Relativistic
Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

Gradient expansion

$$\begin{aligned} J^{\nu\alpha} &= \frac{2}{\mathcal{A}}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) \\ &\quad - \frac{6}{g\mathcal{A}}\partial_\mu\left[\frac{1}{\mathcal{A}} 3 J^{(\nu} J^{\mu\alpha)} - \frac{1}{\mathcal{A}} J_\rho J^{\rho(\alpha} g^{\mu\nu)}\right] + \mathcal{O}\left(g^{-2}\partial^2\right) \end{aligned}$$

Leading order

$$J^{\nu\alpha} \approx \frac{2\mathrm{n}^2}{\mathcal{A}} \mathrm{v}^\nu \mathrm{v}^\alpha - \frac{\mathrm{n}^2}{2\mathcal{A}} g^{\nu\alpha} \qquad \mathrm{v}^\mu \equiv N^\mu / \sqrt{N_\lambda N^\lambda},$$

Truncated gradient expansion

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

"Ideal fluid"

$$\mathcal{A} = \int dK\; f_{0\mathbf{k}} = \frac{\exp\alpha}{2\pi^2} T^2$$

$$\mathsf{n} = \int dK\; E_{\mathbf{k}} f_{0\mathbf{k}} = \frac{\exp\alpha}{\pi^2} T^3$$

$$\Longrightarrow \frac{3\mathsf{n}^2}{2\mathcal{A}} = 3\frac{\exp\alpha}{\pi^2} T^4 = \varepsilon$$

Leading order

$$J^{\nu\alpha} \approx \frac{2\mathsf{n}^2}{\mathcal{A}} \mathsf{v}^\nu \mathsf{v}^\alpha - \frac{\mathsf{n}^2}{2\mathcal{A}} g^{\nu\alpha} \qquad \mathsf{v}^\mu \equiv N^\mu / \sqrt{N_\lambda N^\lambda},$$

Truncated gradient expansion

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

Gradient expansion

$$J^{\nu\alpha} = \frac{2}{\mathcal{A}}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) - \frac{6}{g\mathcal{A}}\partial_\mu\left[\frac{1}{\mathcal{A}} 3 J^{(\nu} J^{\mu\alpha)} - \frac{1}{\mathcal{A}} J_\rho J^{\rho(\alpha} g^{\mu\nu)}\right] + \mathcal{O}\left(g^{-2}\partial^2\right)$$

First-order

$$\frac{g}{3}\left(J^\nu J^\alpha - \frac{g^{\nu\alpha}}{4} J_\lambda J^\lambda\right) - \frac{g}{6}\mathcal{A} J^{\nu\alpha} = \partial_\mu\left[\frac{3}{\mathcal{A}} J^{(\alpha} J^{\mu\nu)} - \frac{1}{\mathcal{A}} J_\rho J^{\rho(\alpha} g^{\mu\nu)}\right]$$

Equations of motion for *A*, *J*^{i} and *J*^{ij}

Truncated gradient expansion

Traditional projections

$$\Pi^{\lambda\chi} = \Delta^{\lambda\chi}_{\nu\alpha} T^{\nu\alpha}, \;\; N^\mu = nU^\mu$$

$$\begin{aligned}\Delta^{\lambda\chi}_{\nu\alpha} U^\mu \partial_\mu \Pi^{\nu\alpha} + \frac{g\mathcal{A}^2}{6n}\Pi^{\lambda\chi} = &\frac{4e}{3}\Delta^{\lambda\chi}_{\nu\alpha}\nabla^\nu U^\alpha - 2\Delta^{\lambda\chi}_{\nu\alpha} h^\nu \dot{U}^\alpha \\ &-2\Delta^{\lambda\chi}_{\nu\alpha}\Pi^{\mu\nu}\nabla_\mu U^\alpha + \frac{2}{3}\Delta^{\lambda\chi}_{\nu\alpha}\nabla^\nu h^\alpha \\ &+\frac{2}{3n}\Delta^{\lambda\chi}_{\nu\alpha} h^\alpha \nabla^\nu n + \frac{2}{3}\mathcal{A}\Delta^{\lambda\chi}_{\nu\alpha} h^\alpha \nabla^\nu \left(\frac{1}{\mathcal{A}}\right) + \frac{1}{\mathcal{A}}\Pi^{\lambda\chi}\dot{\mathcal{A}}\end{aligned}$$

Very similar to Israel-Stewart theory

Truncated gradient expansion

Traditional projection: shear-stress

$$\Pi^{\lambda\chi} = \Delta^{\lambda\chi}_{\nu\alpha} T^{\nu\alpha}, \; N^\mu = n U^\mu$$

Relax. term

Navier-Stokes term

$$\Delta^{\lambda\chi}_{\nu\alpha} U^\mu \partial_\mu \Pi^{\nu\alpha} + \frac{g\mathcal{A}^2}{6n}\Pi^{\lambda\chi} = \frac{4e}{3}\Delta^{\lambda\chi}_{\nu\alpha}\nabla^\nu U^\alpha - 2\Delta^{\lambda\chi}_{\nu\alpha} h^\nu \dot{U}^\alpha$$
$$-2\Delta^{\lambda\chi}_{\nu\alpha}\Pi^{\mu\nu}\nabla_\mu U^\alpha + \frac{2}{3}\Delta^{\lambda\chi}_{\nu\alpha}\nabla^\nu h^\alpha$$
$$+\frac{2}{3n}\Delta^{\lambda\chi}_{\nu\alpha} h^\alpha \nabla^\nu n + \frac{2}{3}\mathcal{A}\Delta^{\lambda\chi}_{\nu\alpha} h^\alpha \nabla^\nu \left(\frac{1}{\mathcal{A}}\right) + \frac{1}{\mathcal{A}}\Pi^{\lambda\chi}\dot{\mathcal{A}}$$

Very similar to Israel-Stewart theory

Here, Israel-Stewart theory appears as a first-order theory

Sad ending

Truncated gradient expansion

Traditional projection: scalar component

Project the equation with $U_\nu U_\alpha$ $(U^\mu = N^\mu/n)$

$$\dot{e} + \frac{g\mathcal{A}^2}{2n}\left(e - \frac{3n^2}{2\mathcal{A}}\right) = \frac{8}{3}e\partial_\mu U^\mu + 7e\frac{\dot{\mathcal{A}}}{\mathcal{A}} - \frac{5}{n}h^\mu\nabla_\mu n + \frac{6}{\mathcal{A}}h^\mu\nabla_\mu\mathcal{A} + 5h^\alpha\dot{U}_\alpha + \Pi^{\mu\rho}\partial_\mu U_\rho$$

Eckart energy density

Eckart particle density

Conservation law projected in U_μ

$$U_\nu\partial_\mu T^{\mu\nu} = \dot{e} + \frac{4}{3}e\partial_\mu U^\mu + \partial_\mu h^\mu - h^\mu\dot{U}_\mu - \Pi^{\mu\nu}\partial_\mu U_\nu = 0$$

Unstable!

Conclusions

- Usual hydrodynamic formulations depend on a choice of equilibrium frame

- Matching conditions can significantly affect the magnitude of some transport coefficients

- Using the method of moments, it is possible to implement a gradient expansion for the conserved currents – without a reference frame

But equations are unstable ...