About the Second Law of Thermodynamics for fluids and fields

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Content

Fluid instabilities

(In)famous hydrodynamic instabilities: Lorentz-Abraham-Dirac (EM-hydro(?)), Jeans (gravity-hydro), Bobylev (kinetic-hydro), Eckart (relativistic-hydro), turbulence, ..., Korteweg (gradient-hydro).
Van der Waals: gradient of density is a thermodynamic variable. Surface tension and capillarity. Korteweg (1901): dynamics?

Balances of mass and momentum, gradient expansion of pressure

$$\dot{\rho} + \rho \nabla \cdot \boldsymbol{v} = 0,$$
$$\rho \dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P} = \boldsymbol{0}$$
$$\boldsymbol{P}_{Kort} = \left(p - \alpha \Delta \rho - \beta (\nabla \rho)^2\right) \boldsymbol{I} - \delta \nabla \rho \circ \nabla \rho - \gamma \nabla^2 \rho.$$

$\alpha, \beta, \gamma, \delta$ are density dependent material parameters.

Instable. Second law? 🕮Eckart fluids 1940, 🕮Dunn and Serrin (ARMA, 1985). Weakly nonlocal fluid.

Quantum to hydro

Schrödinger (superfluid) equation:

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\Delta\psi - V\psi = 0$$

Madelung transformation:

$$\psi = Re^{iS},$$

ρ density (probability or superfluid), $R=\sqrt{\rho}$, S velocity potential: $\boldsymbol{v}=\frac{\hbar}{m}\nabla S$.

$$\frac{i\hbar}{2\rho}\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})\right)\psi-\left(m\frac{\hbar}{m}\frac{\partial S}{\partial t}+m\frac{v^2}{2}-\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-V\right)\psi=0$$

Continuity and Bernoulli equations of classical rotation free fluids. The gradient of the second one

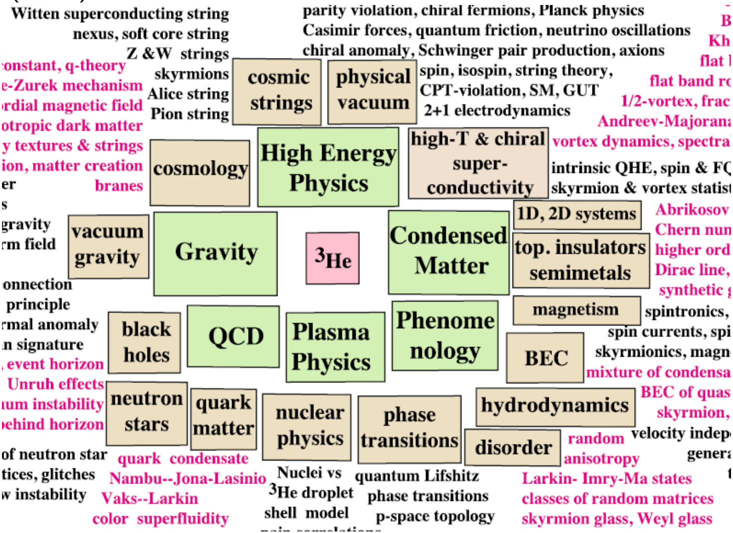
$$\dot{\boldsymbol{v}}+\nabla(U_Q(\rho,\nabla\rho,\nabla^2\rho)+V)=0 \quad \to \rho\dot{\boldsymbol{v}}+\nabla\cdot P_Q(\rho,\nabla\rho,\nabla^2\rho)+\rho\nabla V=0$$

$U_Q(\rho,\nabla\rho,\nabla^2\rho)=-\frac{\hbar^2}{2m^2}\frac{\Delta R}{R}$ is the Bohm potential.

potential flow??, U_Q??, P_Q??

However: Dirac (Takabayashi, 1957), QFT (Jackiw et al. 2004), ...

Volovik (2021), truncated version



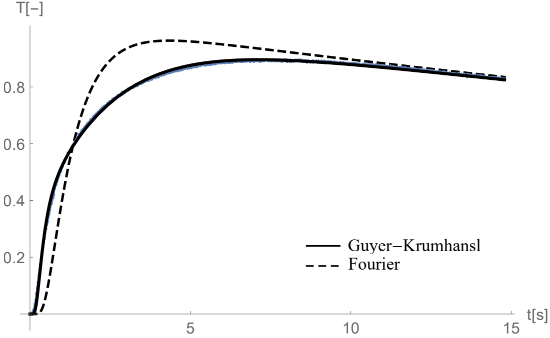
heavy-ion physics: Koide-Kodama (2012)

Rheology and Extended Thermodynamics

Rheology: fluids (tar droplet), solids (rocks, ASR). Method: **internal variables**.

Evolution equation is constructed with thermodynamics.

Kinetic theory is helpful: Extended Irreversible Thermodynamics, phonon hydrodynamics, ...



Heat pulse propagation through a metal foam. Internal variable: heat flux.

Fields and weak nonlocality

Szücs-VP, arXiv:2504.07296

Non-equilibrium thermodynamics – entropy inequality

Thermodynamic state variables: (e, ρ, φ)

Balances of mass, momentum and internal energy:

$$\dot{\rho} + \rho \nabla \cdot \boldsymbol{v} = 0,$$
$$\rho \dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P} = \boldsymbol{0},$$
$$\rho \dot{e} + \nabla \cdot \boldsymbol{J}_E = 0,$$
$$\dot{\varphi} + f = 0.$$

Entropy inequality:

$$\boxed{\rho \dot{s} + \nabla \cdot \boldsymbol{J} \geq 0}$$

Constitutive functions: $\boldsymbol{P}, \boldsymbol{J}_E, \boldsymbol{J}, f, s$. Constitutive state space?
Weakly nonlocal extension: $(e, \rho, \varphi) \quad \longrightarrow \quad (e, \rho, \nabla\rho, \varphi, \nabla\varphi)$

Non-equilibrium thermodynamics – Liu procedure

Constitutive state space: $(e, \nabla e, \rho, \nabla\rho, \nabla^2\rho, (\boldsymbol{v}), \nabla\boldsymbol{v}, \varphi, \nabla\varphi, \nabla^2\varphi)$

Balances of mass, momentum and internal energy:

$$\dot{\rho} + \rho\nabla\cdot\boldsymbol{v} = 0,$$
$$\rho\dot{\boldsymbol{v}} + \nabla\cdot\boldsymbol{P} = \boldsymbol{0},$$
$$\rho\dot{e} + \nabla\cdot\boldsymbol{J}_E = 0,$$
$$\dot{\varphi} + f = 0.$$

Constitutive functions: $\boldsymbol{P}, \boldsymbol{J}_E, \boldsymbol{J}, f$

Methods: Liu procedure (constrained inequality, with Lagrange-Farkas multipliers) or divergence separation.

Process direction space:

$$(\dot{e}, (\nabla e)^{\cdot}, \nabla^2 e, \dot{\rho}, (\nabla\rho)^{\cdot}, (\nabla^2\rho)^{\cdot}, \nabla^3\rho, \dot{\boldsymbol{v}}, (\nabla^2\boldsymbol{v})^{\cdot}, \dot{\varphi}, (\nabla\varphi)^{\cdot}, (\nabla^2\varphi)^{\cdot}, \nabla^3\varphi)$$

Thermodynamic potential: specific entropy s

Follows:

$\rightarrow$ thermodynamic state variables: $(e, \rho, \nabla\rho, \varphi, \nabla\varphi)$

Second Law restrictions

A convenient simplification:

$$s(e, \boldsymbol{v}, \rho, \nabla\rho, \varphi, \nabla\varphi) = s(\underbrace{e - \frac{v^2}{2} - \frac{\varepsilon(\rho, \nabla\rho, \varphi, \nabla\varphi)}{\rho}}_{=u_{\text{fl}}}, \rho)$$

u is the specific internal energy, ε is the density of *perfect energy*.

$$\mathrm{d}u_{\text{fl}} = T\mathrm{d}s - p_{\text{fl}}\mathrm{d}v = T\mathrm{d}s + \frac{p_{\text{fl}}}{\rho^2}\mathrm{d}\rho$$

Entropy production (density rate):

$$\frac{1}{T}\left[\frac{\partial\varepsilon}{\partial\varphi} - \nabla\cdot\left(\frac{\partial\varepsilon}{\partial\left(\nabla\varphi\right)}\right)\right]f +$$

$$\left[\mathrm{J}_E - \boldsymbol{P}\cdot\boldsymbol{v} - \frac{\varrho}{2}\left(\frac{\partial\varepsilon}{\partial\left(\nabla\varrho\right)}\left(\nabla\cdot\boldsymbol{v}\right) + \frac{\partial\varepsilon}{\partial\left(\nabla\varrho\right)}\cdot\nabla\boldsymbol{v}\right) - \frac{\partial\varepsilon}{\partial\left(\nabla\varphi\right)}f\right]\cdot\nabla\frac{1}{T} -$$

$$\frac{1}{T}\left[\boldsymbol{P} - \left(p_{\text{fl}} - \varepsilon + \varrho\frac{\partial\varepsilon}{\partial\varrho} + \frac{\varrho^2}{2}\nabla\cdot\left(\frac{1}{\varrho}\frac{\partial\varepsilon}{\partial\left(\nabla\varrho\right)}\right)\right)\boldsymbol{I} +\right.$$

$$\left.\frac{\varrho^2}{2}\nabla\left(\frac{1}{\varrho}\frac{\partial\varepsilon}{\partial\left(\nabla\varrho\right)}\right) - \frac{\partial\varepsilon}{\partial\left(\nabla\varphi\right)}\otimes\nabla\varphi\right] : \nabla\boldsymbol{v} \geq 0$$

Ideal fluids

Rigorous methods are essential.

$$\frac{1}{T}\left[\frac{\partial \varepsilon}{\partial \varphi} - \nabla\cdot\left(\frac{\partial \varepsilon}{\partial(\nabla\varphi)}\right)\right] f +$$

$$\left[\mathrm{J}_E - \boldsymbol{P}\cdot\boldsymbol{v} - \frac{\varrho}{2}\left(\frac{\partial \varepsilon}{\partial(\nabla\varrho)}(\nabla\cdot\boldsymbol{v}) + \frac{\partial \varepsilon}{\partial(\nabla\varrho)}\cdot\nabla\boldsymbol{v}\right) - \frac{\partial \varepsilon}{\partial(\nabla\varphi)} f\right]\cdot\nabla\frac{1}{T} -$$

$$\frac{1}{T}\left[\boldsymbol{P} - \left(p_{\mathrm{fl}} - \varepsilon + \varrho\frac{\partial \varepsilon}{\partial \varrho} + \frac{\varrho^2}{2}\nabla\cdot\left(\frac{1}{\varrho}\frac{\partial \varepsilon}{\partial(\nabla\varrho)}\right)\right)\boldsymbol{I} + \frac{\varrho^2}{2}\nabla\left(\frac{1}{\varrho}\frac{\partial \varepsilon}{\partial(\nabla\varrho)}\right) - \frac{\partial \varepsilon}{\partial(\nabla\varphi)}\otimes\nabla\varphi\right] : \nabla\boldsymbol{v} \geq 0$$

The ideal field equation is variational:

$$\frac{\partial \varepsilon}{\partial \varphi} - \nabla\cdot\left(\frac{\partial \varepsilon}{\partial(\nabla\varphi)}\right) = \left.\frac{\delta \varepsilon}{\delta \varphi}\right|_{\rho} = 0.$$

Ideal pressure (no mechanical dissipation):

$$\boldsymbol{P}_{perf} = \left(p_{\mathrm{fl}} - \varepsilon + \varrho\frac{\partial \varepsilon}{\partial \varrho} + \frac{\varrho^2}{2}\nabla\cdot\left(\frac{1}{\varrho}\frac{\partial \varepsilon}{\partial(\nabla\varrho)}\right)\right)\boldsymbol{I} - \frac{\varrho^2}{2}\nabla\left(\frac{1}{\varrho}\frac{\partial \varepsilon}{\partial(\nabla\varrho)}\right) + \frac{\partial \varepsilon}{\partial(\nabla\varphi)}\nabla\varphi$$

Classical holography

Surface tractions and bulk forces:

$$\boxed{\nabla \cdot \boldsymbol{P}_{perf} = \rho \nabla \Phi} \quad \longrightarrow \quad \rho \dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P}_{perf} = \rho(\dot{\boldsymbol{v}} + \nabla \Phi) = 0$$

Then:

$$\nabla \cdot \boldsymbol{P}_{perf} = \varrho \nabla \left[\mu + \frac{\partial \varepsilon}{\partial \varrho} - \nabla \cdot \frac{\partial \varepsilon}{\partial (\nabla \varrho)} \right] + \varrho s \nabla T - \left[\frac{\partial \varepsilon}{\partial \varphi} - \nabla \cdot \frac{\partial \varepsilon}{\partial (\nabla \varphi)} \right] \nabla \varphi.$$

Conditions:

- homothermal (or homoentropic),
- ideal scalar field.

$$\nabla \cdot \boldsymbol{P}_{perf} = \varrho \nabla \underbrace{\left[\mu + \left. \frac{\delta \varepsilon}{\delta \varrho} \right|_{\varphi} \right]}_{=\Phi}$$

$$\boxed{\varepsilon_{\mathrm{NG}} = \varrho\phi + \frac{1}{8\pi G}\nabla\phi\cdot\nabla\phi}$$

Perfect fluid pressure, holopotential and field equation :

$$\begin{aligned}
\boldsymbol{P}_{\mathrm{NG}} &= \left(p_{\mathrm{fl}} - \frac{1}{8\pi G}\nabla\phi\cdot\nabla\phi\right)\mathbf{1} + \frac{1}{4\pi G}\nabla\phi\otimes\nabla\phi,\\
\Phi_{\mathrm{NG}} &= \mu_{\mathrm{fl}} + \phi,\\
\Delta\phi &= 4\pi G\varrho.
\end{aligned}$$

Momentum balance + perfect fluid constitutive equation :

$$\varrho\dot{\boldsymbol{v}} + \nabla\cdot\boldsymbol{P}_{\mathrm{NG}} = \varrho\dot{\boldsymbol{v}} + \nabla p_{\mathrm{fl}} - \varrho\mathrm{f}_{\mathrm{vol}},$$

or, equivalently

$$\dot{\boldsymbol{v}} = -\nabla\left(\mu_{\mathrm{fl}} + \phi\right) - s\nabla T,$$

Example 2: Thermodynamical consistency of Korteweg fluids

$$\boxed{\varepsilon_{\mathrm{TcK}} = \frac{b_1(\varrho)}{2}\nabla\varrho\cdot\nabla\varrho}$$

Perfect fluid pressure and holopotential:

$$\boldsymbol{P}_{\mathrm{TcK}} = \left(p_{\mathrm{fl}} - \frac{\varrho b_1}{2}\Delta\varrho\right)\mathbf{1} - \frac{1}{2}\left(\varrho b_1' - b_1\right)\nabla\varrho\otimes\nabla\varrho - \frac{\varrho b_1}{2}\nabla\otimes\nabla\varrho, \quad (1)$$

$$\Phi_{\mathrm{TcK}} = \mu_{\mathrm{fl}} - \frac{1}{2}b_1'\nabla\varrho\cdot\nabla\varrho - b_1\Delta\varrho. \quad (2)$$

Momentum balance + perfect fluid constitutive equation:

$$\varrho\dot{\boldsymbol{v}} + \nabla\cdot\boldsymbol{P}_{\mathrm{Tck}} = \varrho\dot{\boldsymbol{v}} + \nabla p_{\mathrm{fl}} - \varrho\nabla\Phi_{\mathrm{TcK}},$$

If $b_1 = \frac{K}{\varrho}$ then we obtain the Bohm potential (and a family of superfluid equations):

$$\Phi_{\mathrm{sf}} = \mu_{\mathrm{fl}} - K\left(\frac{\Delta\varrho}{\varrho} - \frac{\nabla\varrho\cdot\nabla\varrho}{2\varrho^2}\right) = \mu_{\mathrm{fl}} - 2K\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}$$

Complex field representation

Classical holographic fluids are almost vorticity free:

$$\dot{\rho} + \rho \nabla \cdot \boldsymbol{v} = 0,$$
$$\dot{\boldsymbol{v}} + \nabla \Phi = \boldsymbol{0}.$$

Then the upper convected derivative with the vorticity $\boldsymbol{\omega} := \nabla \times \boldsymbol{v}$ is obtained:

$$\dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\, \boldsymbol{\omega} = \overset{\nabla}{\boldsymbol{\omega}} + (\nabla \cdot \boldsymbol{v})\, \boldsymbol{\omega} = \varrho \left(\frac{\boldsymbol{\omega}}{\varrho} \right)^{\nabla} = 0$$

Complex field construction, with a velocity potential $\boldsymbol{v} = \nabla S$:

$$2R\mathrm{f}_1 \left(\frac{\partial R}{\partial t} + \nabla R \cdot \nabla S + \frac{1}{2} R \Delta S \right) + \mathrm{f}_2 \left(\frac{\partial S}{\partial t} + \frac{1}{2} \nabla S \cdot \nabla S + \Phi \right) = 0.$$

Choosing a suitable f_1 and f_2, one obtains for $\Psi := R e^{\mathrm{i} \frac{S}{S_0}}$:

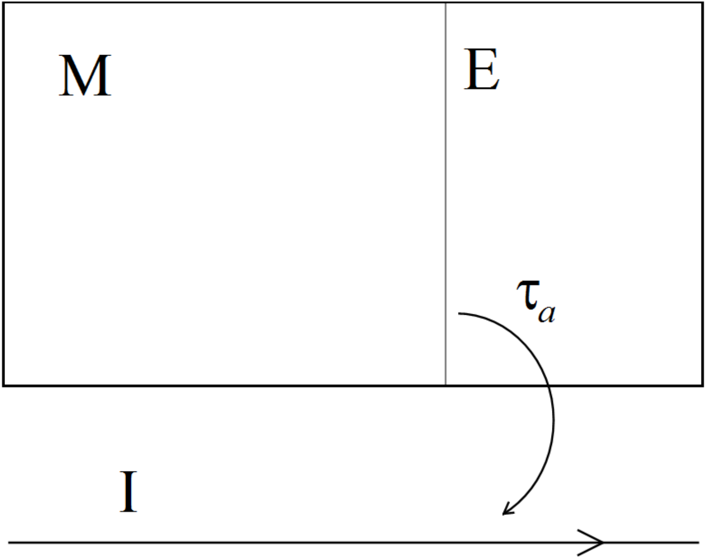
$$\mathrm{i} S_0 \frac{\partial \Psi}{\partial t} = \left(-\frac{S_0^2}{2} \Delta + \left(\Phi + \frac{S_0^2}{2} \frac{\Delta |\Psi|}{|\Psi|} \right) \right) \Psi.$$

If $\Phi = -\frac{S_0^2}{2} \frac{\Delta |\Psi|}{|\Psi|}$ one obtains a linear equation.

Material frame indifference: nonrelativistic covariance

Matolcsi-Ván (PLA, 2006)
Fülöp-Ván (MMAS, 2012)
Ván (CMaT, 2017)

The four dimensions of Galilean relativistic space-time

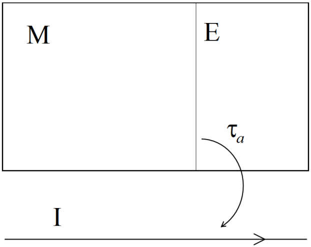


Mathematical structure of Galilean relativistic space-time

1. The *space-time* $\mathbb{M}$ is an oriented four dimensional vector space of the $x^a \in \mathbb{M}$ *world points or events*. There are no Euclidean or pseudoeuclidean structures on $\mathbb{M}$: the length of a space-time vector does not exist.
2. The *time* $\mathbb{I}$ is a one dimensional oriented vector space of $t \in \mathbb{I}$ *instants*.
3. $\tau_a : \mathbb{M} \to \mathbb{I}$ is the *timing* or *time evaluation*, a linear surjection.
4. $\delta_{\bar{a}\bar{b}} : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \otimes \mathbb{R}$ Euclidean structure is a symmetric bilinear mapping, where $\mathbb{E} := Ker(\tau) \subset \mathbb{M}$ is the three dimensional vector space of *space vectors*.

- Simplification: space-time and time are affine spaces
- Simplification: measure lines.
- Abstract indexes: $a, b, c, ...$ for $\mathbb{M}$, $\bar{a}, \bar{b}, \bar{c}, ...$ for S

Vectors an covectors are different



$$\boxed{A'^{\alpha} B'_{\beta} = A^{\alpha} B_{\beta} = AB + A^i B_i}$$

$$\begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ x^i + v^i t \end{pmatrix}$$

Vector transformations (extensives):

$$\begin{pmatrix} A' \\ A'^i \end{pmatrix} = \begin{pmatrix} A \\ A^i + v^i A \end{pmatrix}$$

Covector transformations (derivatives):

$$\begin{pmatrix} B' & B'_i \end{pmatrix} = \begin{pmatrix} B - B_k v^k & B_i \end{pmatrix}$$

Balances: absolute, local and substantial

$$\boxed{\partial_a A^a = 0} \qquad \longrightarrow \qquad \begin{aligned} u^a : \quad D_u A + \partial_i A^i &= d_t A + \partial_i A^i = 0, \\ u'^a : \quad D_{u'} A + \partial_i A'^i &= \partial_t A + \partial_i A'^i = 0. \end{aligned}$$

$(a, b, c \in \{0,1,2,3\})$

Transformed: $(d_t - v^i \partial_i) A + \partial_i (A^i + A v^i) = d_t A + A \partial_i v^i + \partial_i A^i = 0$

Mass, energy and momentum

What kind of quantity is the energy?

- Kinetic energy, square of the relative velocity: 2nd order tensor
- Kinetic theory: trace of a contravariant second order tensor.
- Energy density and flux: additional order

Basic field:

$$\boxed{Z^{abc} = z^{bc} u^a + z^{\bar{a}bc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$a, b, c \in \{0,1,2,3\}, \quad \bar{a}, \bar{b}, \bar{c} \in \{1,2,3\}$

$$z^{bc} \to \begin{pmatrix} \rho & p^{\bar{b}} \\ p^{\bar{c}} & e^{\bar{b}\bar{c}} \end{pmatrix}, \qquad z^{\bar{a}bc} \to \begin{pmatrix} j^{\bar{a}} & P^{\bar{a}\bar{b}} \\ P^{\bar{a}\bar{c}} & q^{\bar{a}\bar{b}\bar{c}} \end{pmatrix}, \qquad \varrho_e = \frac{e^{\bar{b}}{}_{\bar{b}}}{2}$$

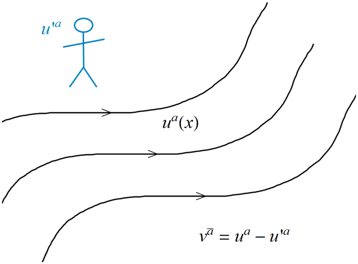
Galilean transformation

$$\boxed{Z'^{\alpha\beta\gamma} = G^{\alpha}_{\mu} G^{\beta}_{\nu} G^{\gamma}_{\kappa} Z^{\mu\nu\kappa}}$$

$$Z^{\alpha\beta\gamma} = \left(\begin{pmatrix} \rho & p^i \\ p^j & e^{ji} \end{pmatrix} \quad \begin{pmatrix} j^k & P^{ki} \\ P^{kj} & q^{kij} \end{pmatrix} \right), \quad G^{\alpha}_{\nu} = \begin{pmatrix} 1 & 0^i \\ v^j & \delta^{ji} \end{pmatrix}, \quad \varrho_e = \frac{e^i{}_i}{2}$$

Transformation rules follow:

$$\begin{aligned} \rho' &= \rho, \\ p'^i &= p^i + \rho v^i, \\ \varrho'_e &= \varrho_e + p^i v_i + \rho \frac{v^2}{2}, \\ j'^i &= j^i + \rho v^i, \end{aligned}$$



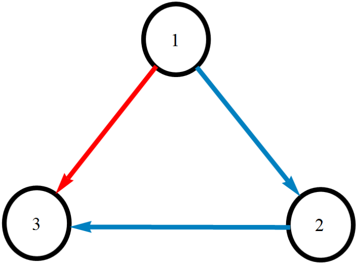
$$\begin{aligned} P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\ q'^i &= q^i + \varrho_e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i)\frac{v^2}{2}. \end{aligned}$$

Galiean transformation of energy

Transitivity :

$$\left.\begin{aligned} \varrho_{e2} &= \varrho_{e1} + p_1 v_{12} + \rho\frac{v_{12}^2}{2} \\ \varrho_{e3} &= \varrho_{e2} + p_2 v_{23} + \rho\frac{v_{23}^2}{2} \end{aligned}\right\} \to \varrho_{e3} = \varrho_{e1} + p_1 v_{13} + \rho\frac{v_{13}^2}{2}$$

$$p_2 = p_1 + \rho v_{12}, \qquad v_{13} = v_{12} + v_{23}$$



Balance transformations

Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{\bar{a}bc} = 0$$

Rest frame

$$\begin{aligned}
\dot{\rho} + \partial_i j^i &= 0, \\
\dot{p}^i + \partial_k P^{ik} &= 0^i, \\
\dot{\varrho}_e + \partial_i q^i &= 0.
\end{aligned}$$

Inertial reference frame

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i, \\
\dot{\varrho}_e + \varrho_e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

Extensivity

Extensivity : body and continuum

$X_A = (X_1, X_2, ..., X_n)$ extensive physical quantities
Gibbs relation :

$$dS = Y_1 dX_1 + Y_2 dX_2 + ... + Y_n dX_n.$$

Extensivity 3 = Euler homogeneity

- $S(\lambda X_A) = \lambda S(X_A)\ \forall \lambda \in \mathbb{R}^+$, the entropy is a first order Euler homogeneous function ;
- $\exists s$, density ($X_1 = V$), specific ($X_1 = M$) :

$$S(X_A) = X_1 s\left(\frac{X_B}{X_1}\right), \qquad B = 2, ..., n;$$

- Gibbs-Duhem : $S = Y_1 X_1 + Y_2 X_2 + ... + Y_n X_n$.

Extensivity : a connection between continuum and homogeneous bodies.
Gibbs relation : measures, Radon-Nykodim derivatives.
Thermodynamic limit : $\lim_{N\to\infty} \frac{S(N)}{N} < \infty$

Thermodynamics of small systems : T. Hill (1962)

$$S(X_1, X_2), \quad dS = Y_1 dX_1 + Y_2 dX_2, \qquad \Lambda S(X_1, X_2) \neq S(\Lambda X_1, \Lambda X_2).$$

Extensive is constructed from nonextensive

$$\hat{S} := NS, \quad \hat{X}_1 := NX_1, \quad \hat{X}_2 := NX_2 \quad \Rightarrow$$

$$d\hat{S} = \mathrm{Y}_1 d\hat{X}_1 + \mathrm{Y}_2 d\hat{X}_2 - \phi dN, \quad \boxed{\mu = S - Y_1 X_1 + Y_2 X_2}$$

$$\hat{S}(\hat{X}_1, \hat{X}_2, N) = NS\left(\frac{\hat{X}_1}{N}, \frac{\hat{X}_2}{N}\right)$$

- Surface and volume. Aerosols, nano systems, etc..
- The origin of chemical potential.
- The original idea is due to Gibbs.

Challenge: new thermodynamic state variables?

Early example: internal variables for acoustic damping:
📖?Mandelstam-Leontovich, JETP 1937, 📖?Landau-Lifshitz, Statistical Physics

Extensivity restored

Example: elasticity $s(e, \epsilon^{ij})$ for specific quantities ($s = S/M$) vs densities:

$$de = Tds - \frac{P_{ij}}{\rho} d\epsilon^{ij} \quad \Leftrightarrow \quad d\rho_e = Td\rho_s - P_{ij}d\epsilon^{ij} + (e - Ts)d\rho.$$

Meaning: partial derivatives. P^{ij} is the Cauchy pressure. Compatibility with fluids.
Thermodynamic equilibrium becomes shape dependent, because elasticity is a weakly nonlocal theory: $\epsilon^{ij} = \partial^{(i}u^{j)}$.
Thermodynamic equilibrium is inhomogeneous and boundary condition dependent. Continuum mechanics: homogeneous thermodynamics is neglected.

Thermodynamics is LOCAL. Better upscaling instead of downscaling.

Summary

- Fields and gradients can be thermodynamic state variables iff extensivity and spacetime are properly represented.
- Classical holography of perfect fluids is a general consequence of the Second Law.
- Euler-Lagrange forms are obtained from thermodynamics, without variational principles.
- Newtonian gravity, quantum fluids and their various modifications emerge.

Universality? local equilibrium? effective theory? stability?

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The Second Law of Thermodynamics is a first principle.

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Universality? local equilibrium? effective theory? stability?

Is the Second Law of Thermodynamics a first principle?

Thank you for your attention !

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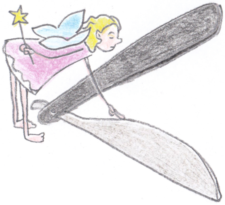


II.

Thank you for your attention!

Probabilistic Korteweg fluids – additivity

Zeroth Law of thermodynamics: separability of independent physical systems.
Multicomponent normal fluids. Notation: $\rho_1 = \rho_1(\boldsymbol{x}_1)$.

$$\mathrm{u}(\rho_1 + \rho_2) = \mathrm{u}(\rho_1) + \mathrm{u}(\rho_2).$$

Multicomponent probabilistic fluids:

$$\mathrm{u}(\rho_1 \rho_2) = \mathrm{u}(\rho_1) + \mathrm{u}(\rho_2).$$

Functional condition, $\rho_{tot} = \rho_1 \rho_1$:

$$\mathrm{u}(\rho_{tot}, (\nabla \rho_{tot})^2) = \mathrm{u}\left(\rho_1 \rho_2, (\rho_2 \nabla_1 \rho_1)^2 + (\rho_1 \nabla_2 \rho_2)^2\right) = \mathrm{u}(\rho_1, (\nabla_1 \rho_1)^2) + \mathrm{u}(\rho_2, (\nabla_2 \rho_2)^2).$$

Unique solution:

$$\boxed{\mathrm{u}(\rho, (\nabla \rho)^2) = k \ln \rho + \frac{\kappa}{2} \frac{(\nabla \rho)^2}{\rho^2}}$$

Independent Schrödinger equations for independent particles/components.

QFT, GR can be fluids: 📖Jackiw et al. (JP A, 2004), 📖Biró-VP(FP, 2015), ...

More Newtonian gravity

📖VP-Abe (Physica A, 2022)
📖Abe-VP (Symmetry, 2022)
📖Pszota-VP (Physics of the Dark Universe, 2024)

Thermodynamic gravity

Constitutive state variables: $(e, \nabla e, \rho, \nabla\rho, (\boldsymbol{v}), \nabla\boldsymbol{v}, \varphi, \nabla\varphi, \nabla^2\varphi)$

$\rightarrow$ thermodynamic state variables: $(e, \rho, \varphi, \nabla\varphi)$

Ideal energy: $\varepsilon_{gr} = \varrho\varphi + \frac{\nabla\varphi\cdot\nabla\varphi}{8\pi G}$ (not the usual one)

$$\rho\dot{s} + \nabla\cdot\boldsymbol{J} = \left(\boldsymbol{q} + \frac{\dot{\varphi}}{4\pi G}\nabla\varphi\right)\cdot\nabla\left(\frac{1}{T}\right) + \boxed{\frac{f}{4\pi G T}\left(\Delta\varphi - 4\pi G\rho\right)} - \left[\boldsymbol{P} - p\boldsymbol{I} - \frac{1}{4\pi G}\left(\nabla\varphi\nabla\varphi - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{I}\right)\right] : \frac{\nabla\boldsymbol{v}}{T} \geq 0$$

- Perfect self-gravitating (isothermal) fluids are holographic:

$$\nabla\cdot\left(p\boldsymbol{I} + \frac{1}{4\pi G}\left(\nabla\varphi\nabla\varphi - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{I}\right)\right) = \rho\nabla(\mu + \varphi)$$

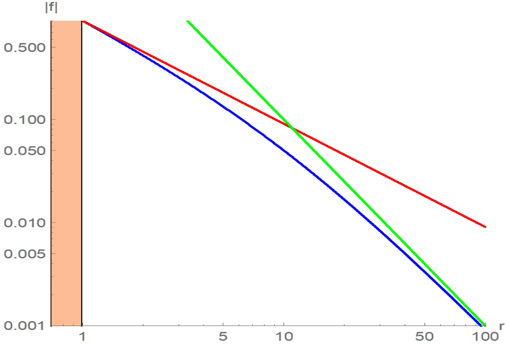
Nonlinear extension, static, nondissipative field

Stationary nondissipative field equation:

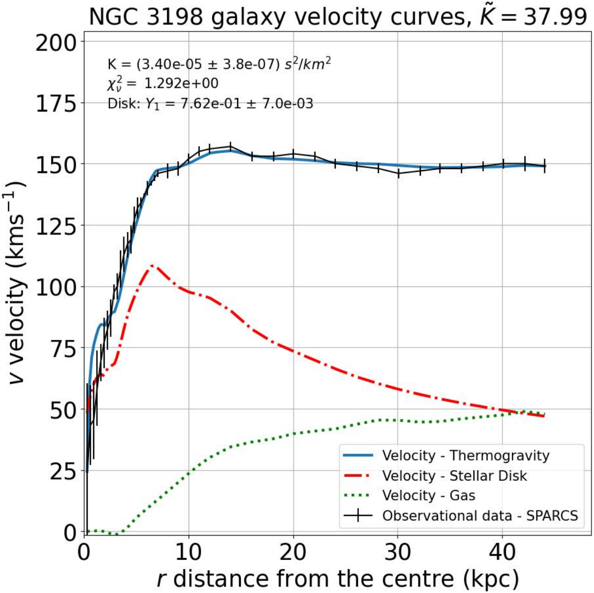
$$\boxed{0 = \Delta\varphi - 4\pi G\rho - K\nabla\varphi\cdot\nabla\varphi.}$$

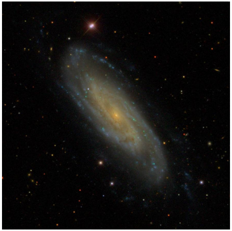
Spherical symmetric force field. Crossover. Apparent and real masses:

$$f(r) = -\frac{r_1}{Kr(r+r_1)} = -\frac{GM_{aa}}{r(r+r_1)}$$



Thermodynamic gravity, MOND and Dark Matter





NGC 3198	
M_{DM+BM}	M_{aa}
205.0	227.5

Unit: $10^9 M_\odot$

Thanks to M. Pszota