April 22^{nd} 2025

Thermodynamic inequalities for relativistic matter

Lorenzo Gavassino

Main collaborators:

- Jorge Noronha;
- Marcelo Disconzi;
- Masoud Shokri;
- Gabriel Soares-Rocha;
- Nicki Mullins;
- Brynmor Haskell;
- Marco Antonelli;

VANDERBILT UNIVERSITY

Funding:

- Polish National Science Centre grant OPUS 2129/33/B/ST9/00942
- Vanderbilt Seeding Success Grant

Nicolaus Copernicus Astronomical Center of the Polish Academy of Sciences

Main references

- GL *Can We Make Sense of Dissipation without Causality?* **Physical Review X, 12, 041001** (Oct 2022)
- GL *Bounds on transport from hydrodynamic stability* **Physics Letters B, Volume 840, 10 May 2023, 137854** (May 2023)
- GL; Disconzi, M. M.; Noronha, J. *Universality Classes of Relativistic Fluid Dynamics: Foundations* **Physical Review Letters 132, 222302** (May 2024)
- GL.; Shokri, M. *Stability of multi-component Israel-Stewart-Maxwell theory for charge diffusion* **Physical Review D 108, 096010** (Nov 2023)
- GL; Antonelli, M.; Haskell, B. *Symmetric-hyperbolic quasi-hydrodynamics* **Physical Review D 106, 056010** (Sep 2022)
- GL; Antonelli, M.; Haskell, B. *Thermodynamic Stability Implies Causality* **Physical Review Letters 128, 010606** (Jan 2022)
- GL *Applying the Gibbs stability criterion to relativistic hydrodynamics* **Classical and Quantum Gravity 38 21LT02** (Oct 2021)
- Chiarini; Sammet; Shokri, *Semi-Classical Spin Hydrodynamics in Flat and Curved Spacetime: Covariance, Linear Waves, and Bjorken Background (https://arxiv.org/abs/2412.19854)*

Before we start

My conventions/assumptions:

- Signature: $(-, +, +, +)$;
- Units: $\boldsymbol{c = \hbar = k_B = 1}$;
- The metric $\boldsymbol{g_{\mu\nu}}$ is a fixed background;

What is this talk about?

As you surely know…

There are some universal thermodynamic inequalities.

- $c_p \geq c_v > 0$;
- $c_s^2 > 0$;
- …

(see also talk by Andrei Starinets)

They are usually derived for homogeneous fluids in a box, with scalar conserved charges (e.g. baryon number).

Landau-Lifshitz, Volume 5 (Statistical Physics: Part 1), section 21

… but

What if:

a) The conserved charges are not scalars (e.g. spin);

b) The fluid is rotating or accelerating;

c) The spacetime is curved;

d) The fluid is inhomogeneous due to, e.g., electromagnetic fields;

e) The substance of interest is an elastic medium;

f) The phase of matter is "exotic" (e.g. superfluid, supersolid,…);

Then, we expect many more inequalities to emerge.

Are we able to find them all?

Statement of the problem

We need a universal procedure for finding inequalities such that:

1) No inequality is missed;
2) No inequality is exaggerated;
3) It is clear where each inequality comes from (i.e. what happens if we break it);
4) Calculations can be carried out in one afternoon;
5) It is fun!

Here, we provide such a procedure.

General Procedure

The rules of the game

We are given a macroscopic description of matter. This amounts to 2 things:

1. A list of macroscopic fields: $\boldsymbol{\Psi} = \{\boldsymbol{T}, \boldsymbol{u}^{\alpha}, \boldsymbol{A}^{\alpha} \ldots\}$;
2. Some expressions for the macroscopic fluxes:

$\boldsymbol{T}^{\mu\nu} = \boldsymbol{T}^{\mu\nu}[\boldsymbol{\Psi}]$ (stress energy tensor, assumed symmetric here)

$\boldsymbol{s}^{\mu} = \boldsymbol{s}^{\mu}[\boldsymbol{\Psi}]$ (entropy current)

$\boldsymbol{J}^{\mu}_{\boldsymbol{A}} = \boldsymbol{J}^{\mu}_{\boldsymbol{A}}[\boldsymbol{\Psi}]$ (possibly other currents)

Main rule of the game: Local first and second laws of thermodynamics are obeyed, namely

$\boldsymbol{\nabla}_{\mu}\boldsymbol{T}^{\mu\nu} = \boldsymbol{0}$ (first law)

$\boldsymbol{\nabla}_{\mu}\boldsymbol{s}^{\mu} \geq \boldsymbol{0}$ (second law)

Goal of the game: Determine thermodynamic inequalities for $\{\boldsymbol{T}^{\mu\nu}[\boldsymbol{\Psi}], \boldsymbol{s}^{\mu}[\boldsymbol{\Psi}], \boldsymbol{J}^{\mu}_{\boldsymbol{A}}[\boldsymbol{\Psi}]\}$

Global first law (see Masoud's talk)

Let $\boldsymbol{K^\mu}$ be a Killing vector field.

Then, the vector field $\boldsymbol{J^\mu = -T^{\mu\nu}K_\nu}$ is a conserved current.

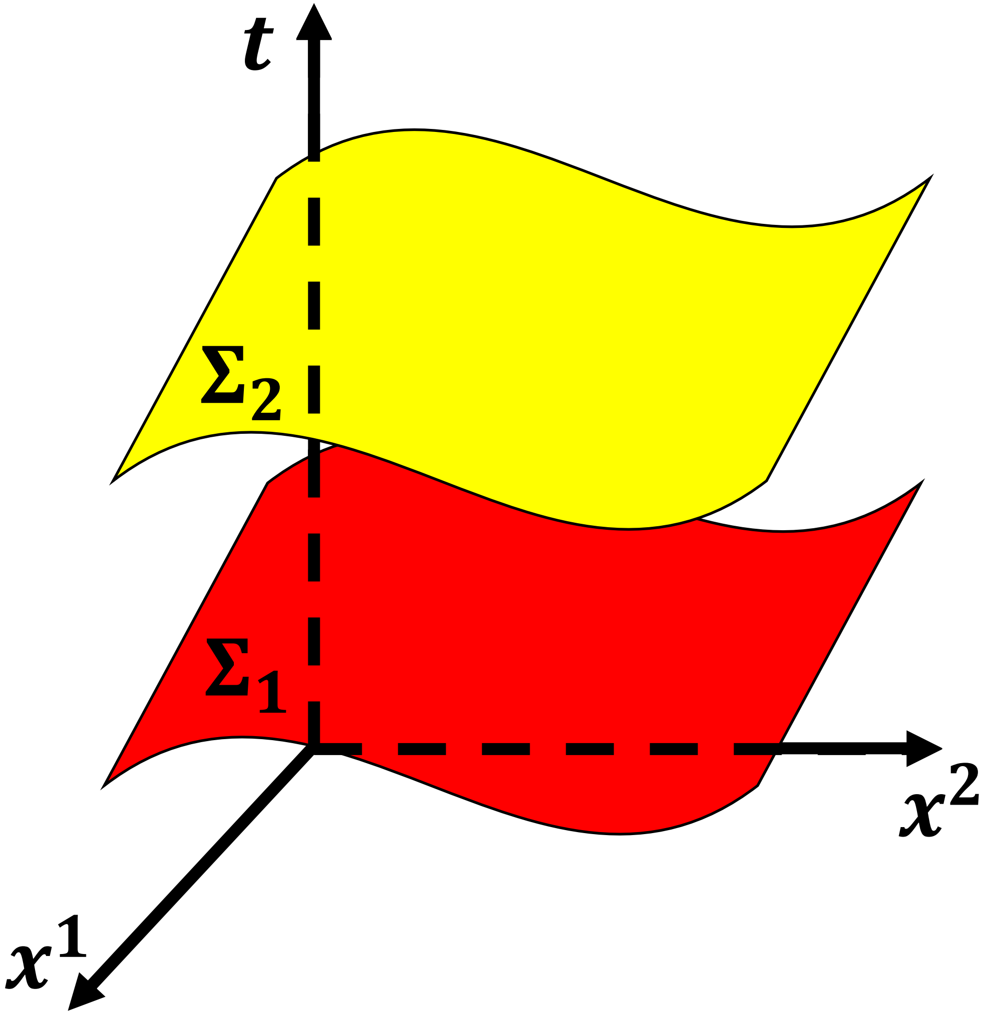
$$\boldsymbol{\nabla_\mu J^\mu = -\nabla_\mu (T^{\mu\nu}K_\nu)}$$
$$\boldsymbol{= -\left(\nabla_\mu T^{\mu\nu}\right)K_\nu - T^{\mu\nu}\nabla_{(\mu}K_{\nu)} = 0}$$

Apply Gauss' Theorem:

$$\boldsymbol{U = \int_{\Sigma_1} J^\mu \, d\Sigma_\mu = \int_{\Sigma_2} J^\mu \, d\Sigma_\mu}$$

$\boldsymbol{U}$ does not depend on the Cauchy surface.

Note that, depending on the choice of $\boldsymbol{K^\mu}$, the Noether charge $\boldsymbol{U}$ could be the energy, the linear momentum, the angular momentum…



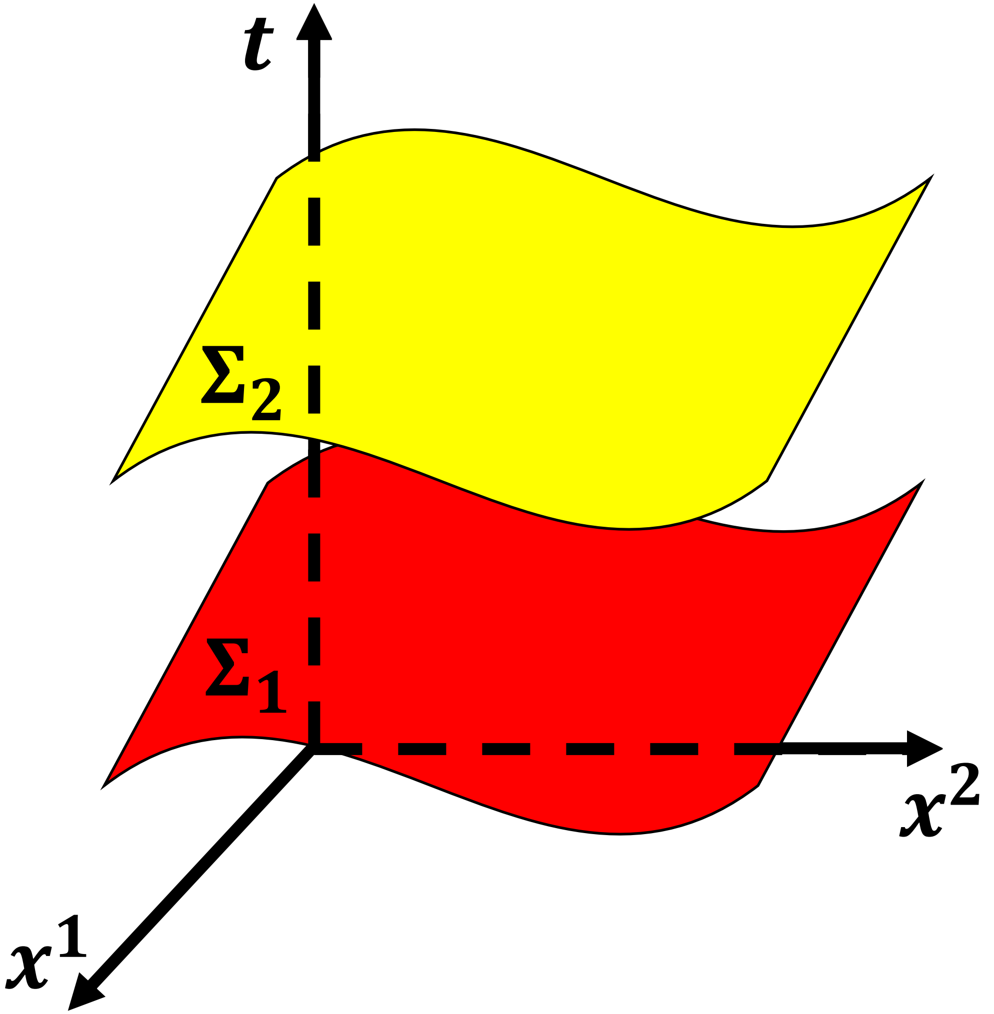
Global second law

The local second law reads $\boldsymbol{\nabla_\mu s^\mu \geq 0}$

Apply Gauss' Theorem assuming that $\boldsymbol{\Sigma_2}$ is the causal future of $\boldsymbol{\Sigma_1}$. Then

$$\boldsymbol{S_1 = \int_{\Sigma_1} s^\mu \, d\Sigma_\mu \leq \int_{\Sigma_2} s^\mu \, d\Sigma_\mu = S_2}$$

$\boldsymbol{S}$ increases in (causal) time.



Maximum Entropy Principle

Fix an isolated finite system.

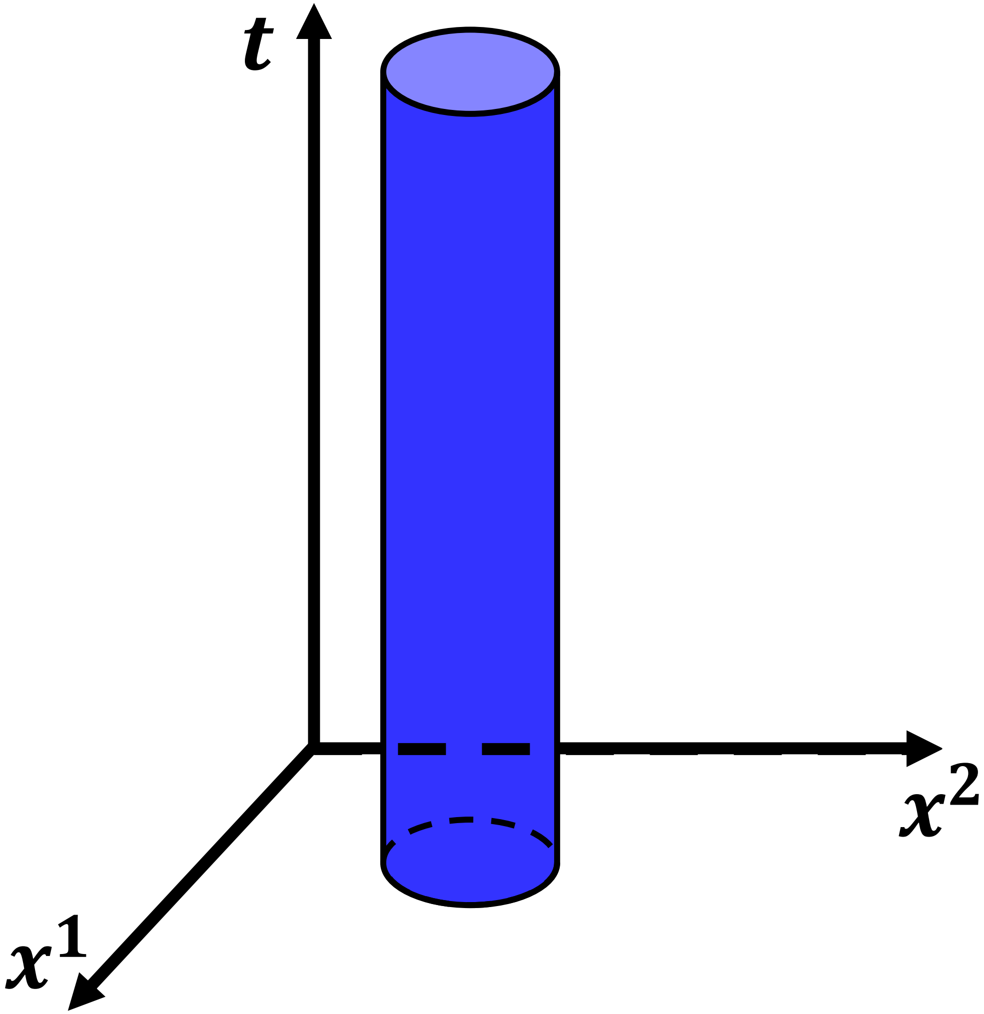
Choose a timelike coordinate t.

This defines a foliation $\Sigma(t)$. Then

$\dot{S} \geq 0$ (ordinary second law)

$\dot{Q}_I = 0$ (conservation of all Noether charges)

Max. Entropy Principle: An isolated system evolves towards a late-time state that maximizes S for fixed values of all charges Q_I. Such state is Lyapunov-stable, and should be identified with the state of global thermodynamic equilibrium of the system.



Minimum Grand-Potential Principle

Bring the system in contact with a bath.

Ideal bath: $S^B\left(Q_I^B\right) \approx const - \alpha_\star^I Q_I^B$

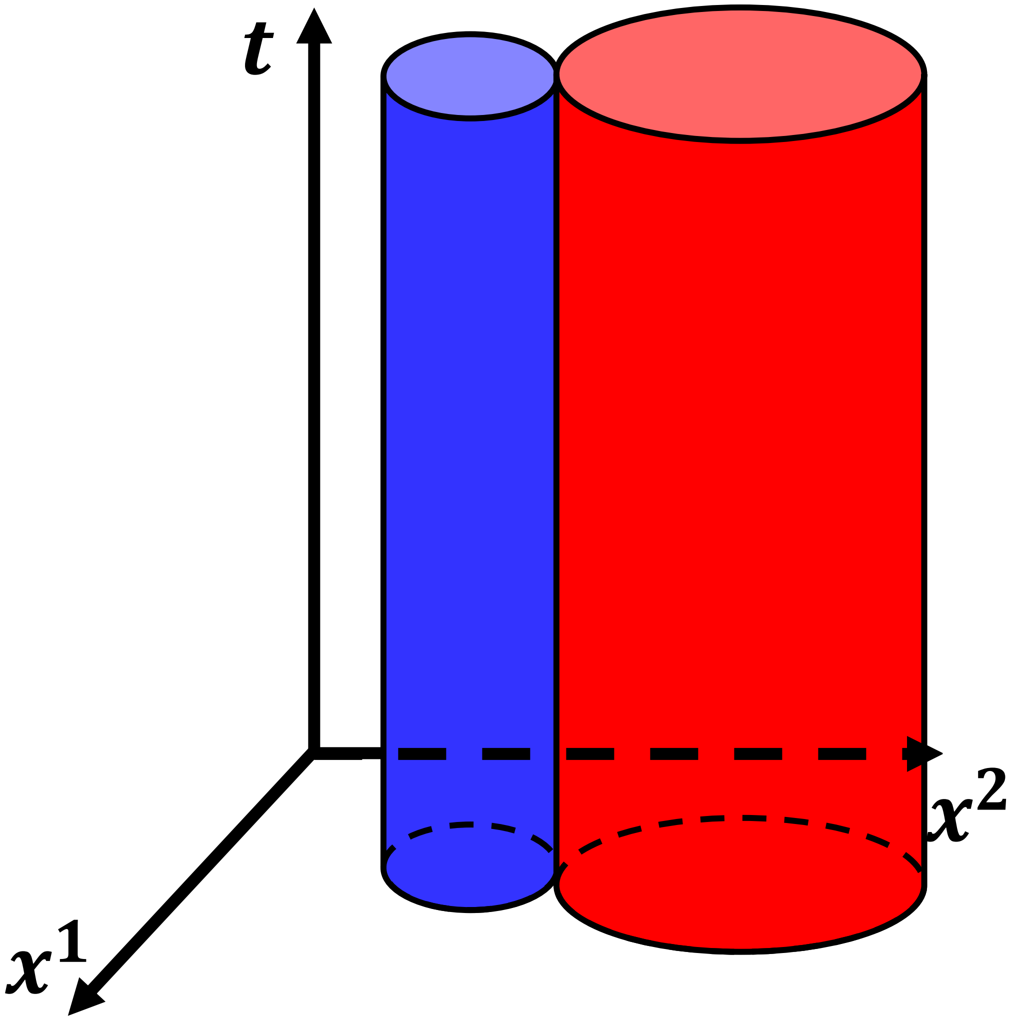
with $\alpha_\star^I = const.$

Then:

$$\dot{Q}_I = -\dot{Q}_I^B$$

$$0 \leq \dot{S} + \dot{S}^B = \dot{S} - \alpha_\star^I \dot{Q}_I^B = \dot{S} + \alpha_\star^I \dot{Q}_I$$

Min. GP Principle: A system in contact with an ideal bath evolves towards a late-time state that maximizes $\Phi = S + \alpha_\star^I Q_I$ for arbitrary variations. Such state is Lyapunov-stable, and should be identified with the state of global thermodynamic equilibrium of the system.



The "grand-potential functional"

$$\Phi[\Psi;\alpha_\star^I] = S[\Psi] + \alpha_\star^I Q_I[\Psi] = \int_\Sigma (s^\mu + \alpha_\star^I J_I^\mu) d\Sigma_\mu$$

Note that

$$\alpha_\star^I J_I^\mu = \alpha_\star^A J_A^\mu + \alpha_\star^a(-K_a^\nu T_\nu^\mu) = \alpha_\star^A J_A^\mu + \beta_\star^\nu T_\nu^\mu$$

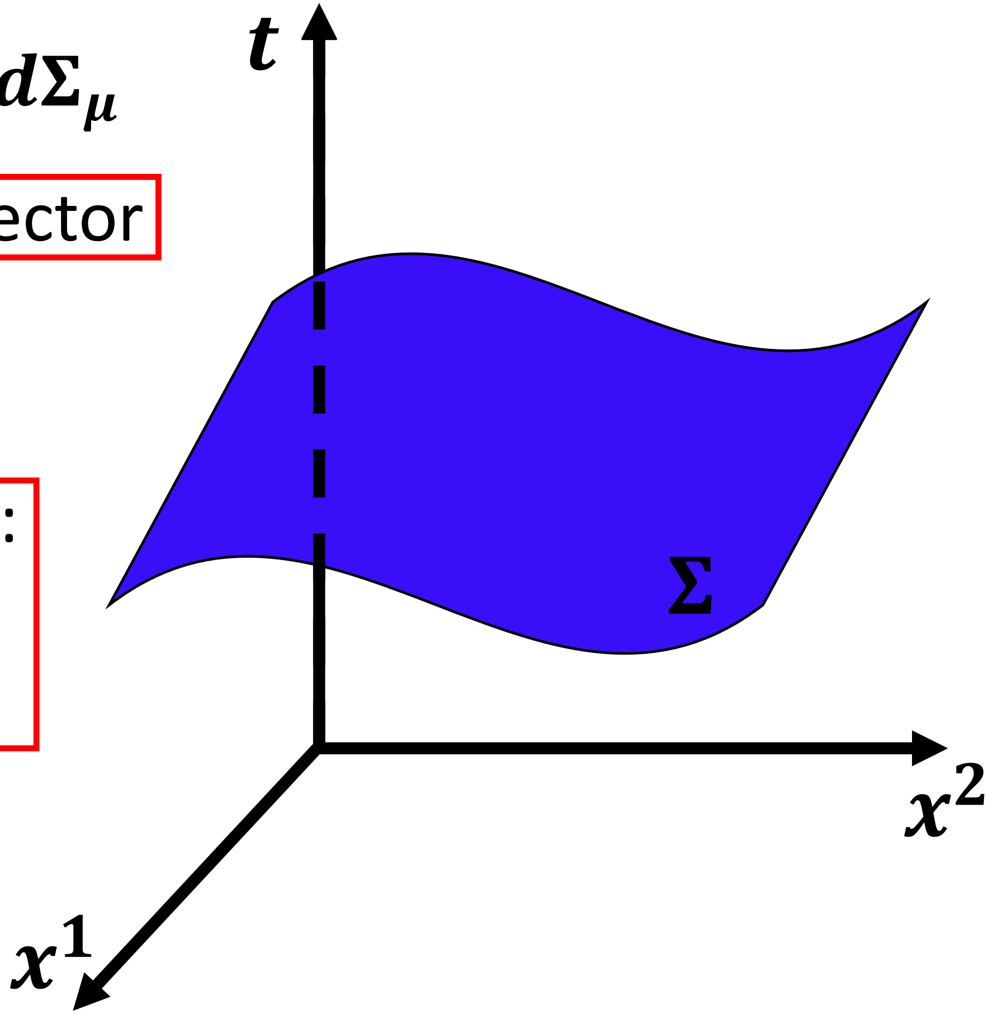
Killing vector

Quantum numbers:
Baryon number,
electric charge…

Spacetime symmetries:
Energy, momentum,
angular momentum

$$\Phi[\Psi;\alpha_\star^A,\beta_\star^\nu] = \int_\Sigma (s^\mu + \alpha_\star^A J_A^\mu + \beta_\star^\nu T_\nu^\mu) d\Sigma_\mu$$

We need to maximize it!



The procedure (have fun with it!)

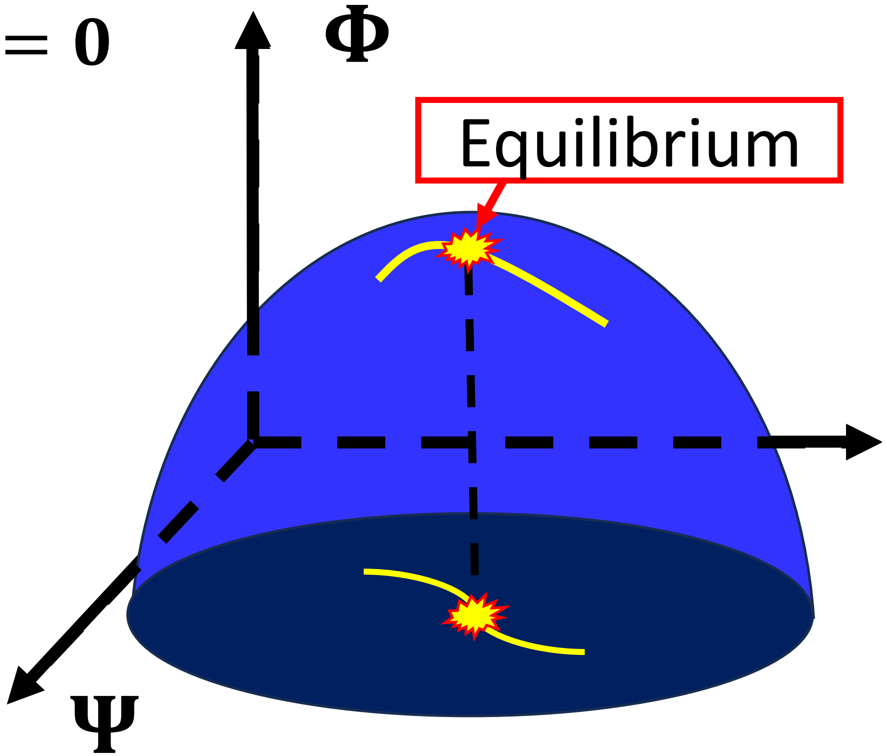
Consider a one-parameter family of states $\boldsymbol{\Psi}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = \mathbf{0}$ is the (still unknown) equilibrium state.

Write the function $\boldsymbol{\Phi}(\boldsymbol{\lambda}) = \boldsymbol{\Phi}\left[\boldsymbol{\Psi}(\boldsymbol{\lambda}); \boldsymbol{\alpha}^A_\star, \boldsymbol{\beta}^\nu_\star\right]$

We must impose (for all choices of $\boldsymbol{\Sigma}$)

- Stationarity: $\dot{\boldsymbol{\Phi}}(\mathbf{0}) = \mathbf{0}$ (identifies the equilibrium)
- Maximum: $\ddot{\boldsymbol{\Phi}}(\mathbf{0}) \leq \mathbf{0}$ (produces the inequalities)

Then, the equilibrium state is Lyapunov-stable!



A useful result: Under quite general assumptions,

$$-\frac{\mathbf{1}}{\mathbf{2}}\ddot{\boldsymbol{\Phi}}(\mathbf{0}) =: \int_{\boldsymbol{\Sigma}} \boldsymbol{E}^{\boldsymbol{\mu}} \boldsymbol{d\Sigma}_{\boldsymbol{\mu}} \geq \mathbf{0}$$

holds for all $\boldsymbol{\Psi}(\boldsymbol{\epsilon})$ and for all $\boldsymbol{\Sigma}$ if and only if $\boldsymbol{E}^{\boldsymbol{\mu}}$ is timelike future-directed for all $\left\{\boldsymbol{\Psi}(\mathbf{0}), \dot{\boldsymbol{\Psi}}(\mathbf{0})\right\}$.

"Information current"

Concrete examples

Example 1: Kinetic theory

Recall what we need: $\boldsymbol{\Phi}\left[\boldsymbol{\Psi}; \boldsymbol{\alpha}_\star^A, \boldsymbol{\beta}_\star^\nu\right] = \int_\Sigma \left(\boldsymbol{s}^\mu + \boldsymbol{\alpha}_\star^A \boldsymbol{J}_A^\mu + \boldsymbol{\beta}_\star^\nu \boldsymbol{T}_\nu^\mu\right) \boldsymbol{d\Sigma}_\mu$

Take $\boldsymbol{\Psi} = \boldsymbol{f}(\boldsymbol{x}^\mu, \boldsymbol{p}^\nu)$, and

$$\boldsymbol{s}^\mu = \int_p \boldsymbol{p}^\mu[-\boldsymbol{f}\ \mathbf{ln}(\boldsymbol{f}) - (\mathbf{1} - \boldsymbol{f})\ \mathbf{ln}(\mathbf{1} - \boldsymbol{f})]$$

$$\boldsymbol{J}^\mu = \int_p \boldsymbol{p}^\mu \boldsymbol{f}$$

$$\boldsymbol{T}^{\mu\nu} = \int_p \boldsymbol{p}^\mu \boldsymbol{p}^\nu \boldsymbol{f}$$

$$\boldsymbol{\Phi}[\boldsymbol{f}; \boldsymbol{\alpha}_\star, \boldsymbol{\beta}_\star^\nu] = \int_{\Sigma \times p} [-\boldsymbol{f}\ \mathbf{ln}(\boldsymbol{f}) - (\mathbf{1} - \boldsymbol{f})\ \mathbf{ln}(\mathbf{1} - \boldsymbol{f}) + \boldsymbol{\alpha}_\star \boldsymbol{f} + \boldsymbol{\beta}_\star^\nu \boldsymbol{p}_\nu \boldsymbol{f}]\ \boldsymbol{p}^\mu \boldsymbol{d\Sigma}_\mu$$

Example 1: Kinetic theory

Consider a one-parameter family of states $\boldsymbol{f(\lambda)}$ (with $\boldsymbol{\lambda = 0}$ equilibrium state).

$$\boldsymbol{\Phi(\lambda) = \int_{\Sigma\times p} [-f\ln(f) - (1-f)\ln(1-f) + \alpha_\star f + \beta_\star^\nu p_\nu f]\, p^\mu d\Sigma_\mu}$$

$$\boldsymbol{\dot{\Phi}(\lambda) = \int_{\Sigma\times p} \left[\ln\left(f^{-1} - 1\right) + \alpha_\star + \beta_\star^\nu p_\nu\right] \dot{f}\, p^\mu d\Sigma_\mu}$$

$$\boldsymbol{\ddot{\Phi}(\lambda) = \int_{\Sigma\times p} \left[\ln\left(f^{-1} - 1\right) + \alpha_\star + \beta_\star^\nu p_\nu\right] \ddot{f}\, p^\mu d\Sigma_\mu - \int_{\Sigma\times p} \frac{\dot{f}^2}{f(1-f)} p^\mu d\Sigma_\mu}$$

Example 1: Kinetic theory

Recall what we need to do: $\dot{\boldsymbol{\Phi}}(\mathbf{0}) = \mathbf{0}$ (find equilibrium), $\ddot{\boldsymbol{\Phi}}(\mathbf{0}) \leq \mathbf{0}$ (stability)

$$\dot{\boldsymbol{\Phi}}(\mathbf{0}) = \int_{\Sigma \times p} \left[\ln\left(f^{-1} - 1\right) + \alpha_\star + \beta_\star^\nu p_\nu\right] \dot{f}\, p^\mu d\Sigma_\mu = 0$$

Gives Fermi-Dirac:

$$f = \frac{1}{e^{-(\alpha_\star + \beta_\star^\nu p_\nu)} + 1}$$

And we learn that

Tolman law

Klein law

$$\frac{u^\mu}{T} = \beta_\star^\mu = \text{Killing vector}, \qquad \frac{\mu}{T} = \alpha_\star = \text{constant}$$

Example 1: Kinetic theory

$$\ddot{\Phi}(0) = \int_{\Sigma\times p} \left[\ln\left(f^{-1} - 1\right) + \alpha_\star + \beta_\star^\nu p_\nu\right] \dot{f}\, p^\mu d\Sigma_\mu - \int_{\Sigma\times p} \frac{\dot{f}^2}{f(1-f)} p^\mu d\Sigma_\mu$$

Recall what we need to extract:

$$-\frac{1}{2}\ddot{\Phi}(0) =: \int_\Sigma E^\mu d\Sigma_\mu \quad \longrightarrow \quad E^\mu = \frac{1}{2}\int_p \frac{\dot{f}^2 p^\mu}{f(1-f)}$$

As expected, the information current is timelike future directed.

The Fermi-Dirac distribution is Lyapunov stable.

Example 2: Ideal fluids

Recall what we need: $\boldsymbol{\Phi}\left[\boldsymbol{\Psi}; \alpha_\star^A, \beta_\star^\nu\right] = \int_\Sigma \left(s^\mu + \alpha_\star^A J_A^\mu + \beta_\star^\nu T_\nu^\mu\right) d\Sigma_\mu$

Take $\boldsymbol{\Psi} = \{s, n, u^\mu\}$, and

$$s^\mu = s u^\mu$$
$$J^\mu = n u^\mu$$
$$T^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu}$$

With $\varepsilon = \varepsilon(s, n)$, $d\varepsilon = T ds + \mu dn$, $\varepsilon + P = Ts + \mu n$

$$\boldsymbol{\Phi}[s, n, u^\mu; \alpha_\star, \beta_\star^\nu] = \int_\Sigma \left[(1 + T\beta_\star^\nu u_\nu) s u^\mu + (\alpha_\star + \mu\beta_\star^\nu u_\nu) n u^\mu + P\beta_\star^\mu\right] d\Sigma_\mu$$
$$= \int_\Sigma \phi^\mu d\Sigma_\mu$$

Example 2: Ideal fluids

Consider a one-parameter family of states $\boldsymbol{\Psi}(\boldsymbol{\lambda})$ (with $\boldsymbol{\lambda} = \mathbf{0}$ equilibrium state).

$$\phi^\mu(\lambda) = (1 + T\beta_\star^\nu u_\nu)s^\mu + (\alpha_\star + \mu\beta_\star^\nu u_\nu)J^\mu + P\beta_\star^\mu$$

$$\dot\phi^\mu(\lambda) = (1 + T\beta_\star^\nu u_\nu)\dot s^\mu + (\alpha_\star + \mu\beta_\star^\nu u_\nu)\dot J^\mu + \dot P\beta_\star^\mu$$
$$+ \left(\dot T\beta_\star^\nu u_\nu + T\beta_\star^\nu \dot u_\nu\right)s^\mu + (\dot\mu\beta_\star^\nu u_\nu + \mu\beta_\star^\nu \dot u_\nu)J^\mu$$

$$\ddot\phi^\mu(\lambda) = (1 + T\beta_\star^\nu u_\nu)\ddot s^\mu + (\alpha_\star + \mu\beta_\star^\nu u_\nu)\ddot J^\mu + \ddot P\beta_\star^\mu$$
$$+ 2\left(\dot T\beta_\star^\nu u_\nu + T\beta_\star^\nu \dot u_\nu\right)\dot s^\mu + 2(\dot\mu\beta_\star^\nu u_\nu + \mu\beta_\star^\nu \dot u_\nu)\dot J^\mu$$
$$+ \left(\ddot T\beta_\star^\nu u_\nu + 2\dot T\beta_\star^\nu \dot u_\nu + T\beta_\star^\nu \ddot u_\nu\right)s^\mu + (\ddot\mu\beta_\star^\nu u_\nu + 2\dot\mu\beta_\star^\nu \dot u_\nu + \mu\beta_\star^\nu \ddot u_\nu)J^\mu$$

Example 2: Ideal fluids

Let's identify the equilibrium state using $\dot{\boldsymbol{\Phi}}(\mathbf{0}) = \mathbf{0}$.

$$\dot{\phi}^{\mu}(0) = (1 + T\beta_{\star}^{\nu}u_{\nu})\dot{s}^{\mu} + (\alpha_{\star} + \mu\beta_{\star}^{\nu}u_{\nu})\dot{j}^{\mu} + \dot{P}\beta_{\star}^{\mu}$$
$$+ \left(\dot{T}\beta_{\star}^{\nu}u_{\nu} + T\beta_{\star}^{\nu}\dot{u}_{\nu}\right)su^{\mu} + (\dot{\mu}\beta_{\star}^{\nu}u_{\nu} + \mu\beta_{\star}^{\nu}\dot{u}_{\nu})nu^{\mu} = 0$$

Again, we find that

$$\frac{u^{\mu}}{T} = \beta_{\star}^{\mu} = \text{Killing vector}, \qquad \frac{\mu}{T} = \alpha_{\star} = \text{constant}$$

(keep in mind that $u^{\mu}\dot{u}_{\mu} = 0$, which is the derivative of $u^{\mu}u_{\mu} = -1$)

Example 2: Ideal fluids

The information current is $\boldsymbol{E}^{\mu} = -\ddot{\boldsymbol{\phi}}^{\mu}(\mathbf{0})/\mathbf{2}$.

$$\ddot{\phi}^{\mu}(0) = (1 + T\beta^{\nu}_{\star}u_{\nu})\ddot{s}^{\mu} + (\alpha_{\star} + \mu\beta^{\nu}_{\star}u_{\nu})\ddot{J}^{\mu} + \ddot{P}\beta^{\mu}_{\star}$$
$$+ 2\left(\dot{T}\beta^{\nu}_{\star}u_{\nu} + T\beta^{\nu}_{\star}\dot{u}_{\nu}\right)\dot{s}^{\mu} + 2(\dot{\mu}\beta^{\nu}_{\star}u_{\nu} + \mu\beta^{\nu}_{\star}\dot{u}_{\nu})\dot{J}^{\mu}$$
$$+ \left(\ddot{T}\beta^{\nu}_{\star}u_{\nu} + 2\dot{T}\beta^{\nu}_{\star}\dot{u}_{\nu} + T\beta^{\nu}_{\star}\ddot{u}_{\nu}\right)s^{\mu} + (\ddot{\mu}\beta^{\nu}_{\star}u_{\nu} + 2\dot{\mu}\beta^{\nu}_{\star}\dot{u}_{\nu} + \mu\beta^{\nu}_{\star}\ddot{u}_{\nu})J^{\mu}$$

Example 2: Ideal fluids

The information current is $E^\mu = -\ddot{\phi}^\mu(0)/2$.

$$\ddot{\phi}^\mu(0) = \cancel{(1 + T\beta_\star^\nu u_\nu)\ddot{s}^\mu + (\alpha_\star + \mu\beta_\star^\nu u_\nu)\ddot{J}^\mu} + \ddot{P}\beta_\star^\mu$$
$$+ 2\left(\dot{T}\beta_\star^\nu u_\nu + \cancel{T\beta_\star^\nu \dot{u}_\nu}\right)\dot{s}^\mu + 2(\dot{\mu}\beta_\star^\nu u_\nu + \cancel{\mu\beta_\star^\nu \dot{u}_\nu})\dot{J}^\mu$$
$$+ \left(\ddot{T}\beta_\star^\nu u_\nu + \cancel{2\dot{T}\beta_\star^\nu \dot{u}_\nu} + T\beta_\star^\nu \ddot{u}_\nu\right)s^\mu + (\ddot{\mu}\beta_\star^\nu u_\nu + \cancel{2\dot{\mu}\beta_\star^\nu \dot{u}_\nu} + \mu\beta_\star^\nu \ddot{u}_\nu)J^\mu$$

$$T\ddot{\phi}^\mu(0) = -2\dot{T}\dot{s}^\mu - 2\dot{\mu}\dot{J}^\mu - \left(s\ddot{T} + Ts\dot{u}^\nu\dot{u}_\nu + n\ddot{\mu} + \mu n\dot{u}^\nu\dot{u}_\nu - \ddot{P}\right)u^\mu$$

(I used the identity $u^\nu\ddot{u}_\nu = -\dot{u}^\nu\dot{u}_\nu$, which is the derivative of $u^\mu\dot{u}_\mu = 0$)

Example 2: Ideal fluids

After some cleaning, we obtain

$$TE^{\mu} = \left[\dot{T}\dot{s} + \dot{\mu}\dot{n} + (\varepsilon + P)\dot{u}^{\nu}\dot{u}_{\nu}\right]\frac{u^{\mu}}{2} + \dot{P}\dot{u}^{\mu}$$

Requiring E^{μ} to be timelike future-directed for all $\dot{\Psi}$ produces all inequalities.

Ex 1: Set $\dot{n} = \dot{u}^{\nu} = 0$ and $\dot{T} = 1$. This gives

$$TE^{\mu} = \left(\frac{\partial s}{\partial T}\right)_n \frac{u^{\mu}}{2} = \frac{nc_v}{2T}u^{\mu} \longrightarrow nc_v \geq 0$$ (stability to isochoric heating)

Ex 2: Set $\dot{n} = \dot{T} = 0$ and $\dot{u}^{\nu}\dot{u}_{\nu} = 1$. This gives

$$TE^{\mu} = (\varepsilon + P)\frac{u^{\mu}}{2} \longrightarrow \varepsilon + P \geq 0$$ (stability to accelerations)

The null energy condition is a universal thermodynamic inequality for fluids in equilibrium. If you break it, shear waves grow (recall the dispersion relation $\omega_{\text{shear}} = -\frac{i\eta k^2}{\varepsilon+P}$).

Example 2: Ideal fluids

To get necessary and sufficient condition, we change variables (define $\mathfrak{s} = s/n$)

$$TE^\mu = \left[\frac{nT}{c_p}\dot{\mathfrak{s}}^2 + \frac{\dot{P}^2}{(\varepsilon+P)c_s^2} + (\varepsilon+P)\dot{u}^\nu\dot{u}_\nu\right]\frac{u^\mu}{2} + \dot{P}\dot{u}^\mu$$

And we work in a local reference frame such that $u^\mu = (1,0,0,0)$ and $\dot{u}^\mu = (0,\dot{u},0,0)$.
Then, I only need to set $\mathcal{E} = 2T\left(E^0 - E^1\right) \geq 0$, which gives

$$\mathcal{E} = (\dot{\mathfrak{s}}, \dot{P}, \dot{u})\begin{bmatrix} nT/c_p & 0 & 0 \\ 0 & (\varepsilon+P)^{-1}c_s^{-2} & -1 \\ 0 & -1 & \varepsilon+P \end{bmatrix}\begin{pmatrix} \dot{\mathfrak{s}} \\ \dot{P} \\ \dot{u} \end{pmatrix}$$

We get $nc_p \geq 0$, $\varepsilon + P \geq 0$, and $c_s^2 \geq 0$ (stability to compression), plus…

Example 2: Ideal fluids

To get necessary and sufficient condition, we change variables (define $\mathfrak{s} = s/n$)

$$TE^{\mu} = \left[\frac{nT}{c_p}\dot{\mathfrak{s}}^2 + \frac{\dot{P}^2}{(\varepsilon + P)c_s^2} + (\varepsilon + P)\dot{u}^{\nu}\dot{u}_{\nu}\right]\frac{u^{\mu}}{2} + \dot{P}\dot{u}^{\mu}$$

And we work in a local reference frame such that $u^{\mu} = (1,0,0,0)$ and $\dot{u}^{\mu} = (0,\dot{u},0,0)$.
Then, I only need to set $\mathcal{E} = 2T\left(E^0 - E^1\right) \geq 0$, which gives

$$\mathcal{E} = (\dot{\mathfrak{s}}, \dot{P}, \dot{u})\begin{bmatrix} nT/c_p & 0 & 0 \\ 0 & (\varepsilon + P)^{-1}c_s^{-2} & -1 \\ 0 & -1 & \varepsilon + P \end{bmatrix}\begin{pmatrix} \dot{\mathfrak{s}} \\ \dot{P} \\ \dot{u} \end{pmatrix}$$

We get $nc_p \geq 0$, $\varepsilon + P \geq 0$, and $c_s^2 \geq 0$ (stability to compression), plus

$$\det\begin{bmatrix} (\varepsilon + P)^{-1}c_s^{-2} & -1 \\ -1 & \varepsilon + P \end{bmatrix} = c_s^{-2} - 1 \geq 0 \quad \Longrightarrow \quad \boxed{c_s^2 \leq 1}$$

Example 2: Ideal fluids

The condition $c_s^2 \leq 1$ is often identified with "causality", but they are not the same.

Counter-example by Bludman and Ruderman (1968):

$$\left(\partial_t^2 - \partial_x^2\right)^2 P + \left(\partial_t^2 - c_s^2 \partial_x^2\right) P = 0 \qquad (\text{with } c_s^2 > 1)$$

Perfectly causal phase of matter, yet unstable:

$$2\omega^2 = 1 + 2k^2 \pm \sqrt{1 + 4k^2(1 - c_s^2)}$$

Example 3: Viscous fluids / Elastic media

Recall what we need: $\boldsymbol{\Phi}\left[\boldsymbol{\Psi};\boldsymbol{\alpha}_{\star}^{A},\boldsymbol{\beta}_{\star}^{\nu}\right]=\int_{\Sigma}\left(\boldsymbol{s}^{\mu}+\boldsymbol{\alpha}_{\star}^{A}\boldsymbol{J}_{A}^{\mu}+\boldsymbol{\beta}_{\star}^{\nu}\boldsymbol{T}_{\nu}^{\mu}\right)\boldsymbol{d\Sigma}_{\mu}$

Take $\boldsymbol{\Psi}=\{\boldsymbol{s},\boldsymbol{n},\boldsymbol{u}^{\mu}\}$, and

$$\boldsymbol{s}^{\mu}=\left(\boldsymbol{s}-\frac{\mathbf{1}}{\mathbf{4}\boldsymbol{G}\boldsymbol{T}}\boldsymbol{\Pi}^{\alpha\beta}\boldsymbol{\Pi}_{\alpha\beta}\right)\boldsymbol{u}^{\mu}$$
$$\boldsymbol{J}^{\mu}=\boldsymbol{n}\boldsymbol{u}^{\mu}$$
$$\boldsymbol{T}^{\mu\nu}=(\boldsymbol{\varepsilon}+\boldsymbol{P})\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}+\boldsymbol{P}\boldsymbol{g}^{\mu\nu}+\boldsymbol{\Pi}^{\mu\nu}$$

With $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\boldsymbol{s},\boldsymbol{n}),\ \boldsymbol{d\varepsilon}=\boldsymbol{T}\boldsymbol{ds}+\boldsymbol{\mu}\boldsymbol{dn},\ \boldsymbol{\varepsilon}+\boldsymbol{P}=\boldsymbol{T}\boldsymbol{s}+\boldsymbol{\mu}\boldsymbol{n},\ \boldsymbol{G}=\boldsymbol{G}(\boldsymbol{s},\boldsymbol{n})$ (shear modulus)

$$\boldsymbol{\Phi}[\boldsymbol{\Psi};\boldsymbol{\alpha}_{\star},\boldsymbol{\beta}_{\star}^{\nu}]=\int_{\Sigma}\left[(\mathbf{1}+\boldsymbol{T}\boldsymbol{\beta}_{\star}^{\nu}\boldsymbol{u}_{\nu})\boldsymbol{s}\boldsymbol{u}^{\mu}+(\boldsymbol{\alpha}_{\star}+\boldsymbol{\mu}\boldsymbol{\beta}_{\star}^{\nu}\boldsymbol{u}_{\nu})\boldsymbol{n}\boldsymbol{u}^{\mu}+\boldsymbol{P}\boldsymbol{\beta}_{\star}^{\mu}-\frac{\boldsymbol{\Pi}^{\alpha\beta}\boldsymbol{\Pi}_{\alpha\beta}}{\mathbf{4}\boldsymbol{G}\boldsymbol{T}}+\boldsymbol{\Pi}_{\nu}^{\mu}\boldsymbol{\beta}_{\star}^{\nu}\right]\boldsymbol{d\Sigma}_{\mu}$$
$$=\int_{\Sigma}\boldsymbol{\phi}^{\mu}\boldsymbol{d\Sigma}_{\mu}$$

Landau-Lifshitz, Volume 7 (Theory of Elasticity), section 4

Example 3: Viscous fluids / Elastic media

$$\Phi = \int_\Sigma \left[(1 + T\beta_\star^\nu u_\nu)\mathfrak{s} n u^\mu + (\alpha_\star + \mu\beta_\star^\nu u_\nu) n u^\mu - \frac{\Pi^{\alpha\beta}\Pi_{\alpha\beta}}{4GT} + P\beta_\star^\mu + \Pi^\mu_\nu \beta_\star^\nu \right] d\Sigma_\mu$$

Stationarity ($\dot{\Phi}(0) = 0$):

Equilibrium is non-deformed

$$\frac{u^\mu}{T} = \beta_\star^\mu, \qquad \frac{\mu}{T} = \alpha_\star, \qquad \Pi^{\mu\nu} = 0$$

Information current:

$$TE^\mu = \left[\frac{nT}{c_p}\dot{\mathfrak{s}}^2 + \frac{\dot{P}^2}{(\varepsilon + P)c_s^2} + (\varepsilon + P)\dot{u}^\nu \dot{u}_\nu + \frac{1}{2G}\dot{\Pi}^{\alpha\beta}\dot{\Pi}_{\alpha\beta} \right] \frac{u^\mu}{2} + \dot{P}\dot{u}^\mu + \dot{\Pi}^{\mu\nu}\dot{u}_\nu$$

Example 3: Viscous fluids / Elastic media

Recall: $\boldsymbol{E^\mu}$ must be timelike future-directed for all $\dot{\boldsymbol{\Psi}} = \{\dot{P}, \dot{s}, \dot{u}^\alpha, \dot{\Pi}^{\alpha\beta}\}$

$$TE^\mu = \left[\frac{nT}{c_p}\dot{s}^2 + \frac{\dot{P}^2}{(\varepsilon + P)c_s^2} + (\varepsilon + P)\dot{u}^\nu \dot{u}_\nu + \frac{1}{2G}\dot{\Pi}^{\alpha\beta}\dot{\Pi}_{\alpha\beta}\right]\frac{u^\mu}{2} + \dot{P}\dot{u}^\mu + \dot{\Pi}^{\mu\nu}\dot{u}_\nu$$

We get the following inequalities (assuming $n, T > 0$)

$c_p > 0,\ 0 < c_s^2 \leq 1, \varepsilon + P > 0$ (same as before)

$G > 0$ (stability to isochoric deformations)

$c_{tr}^2 = \frac{G}{\varepsilon+P} \leq 1$ (causality of transversal elastic waves)

$c_{lg}^2 = c_s^2 + \frac{4G}{3(\varepsilon+P)} \leq 1$ (causality of longitudinal elastic waves)

Cfr. Landau-Lifshitz, Volume 7 (Theory of Elasticity), section 22

Application to fluctuating hydrodynamics

A quick calculation

The grand-canonical density operator is

$$\hat{\rho} = \frac{e^{\alpha_\star^I \hat{Q}_I}}{Z}$$

Let $\hat{P}[\mathbf{\Psi}]$ be the projector onto the space of microscopic states that realize the macroscopic state $\mathbf{\Psi}$. Then, the grandcanonical probability of being in $\mathbf{\Psi}$ is

$$\wp[\Psi] = \mathrm{Tr}\left(\hat{P}[\Psi]\hat{\rho}\right) = \frac{\mathrm{Tr}\left(\hat{P}[\Psi]e^{\alpha_\star^I \hat{Q}_I}\right)}{Z}$$

$$\approx \frac{e^{\alpha_\star^I Q_I[\Psi]}}{Z}\mathrm{Tr}\left(\hat{P}[\Psi]\right) = \frac{e^{S[\Psi]+\alpha_\star^I Q_I[\Psi]}}{Z} = \frac{e^{\Phi[\Psi]}}{Z}$$

A quick calculation

Let $\boldsymbol{\Psi}$ be the equilibrium macrostate and $\boldsymbol{\Psi} + \boldsymbol{\lambda\delta\Psi}$ a non-equilibrium macrostate (with $\boldsymbol{\lambda}$ small). Then

$$\frac{\wp[\boldsymbol{\Psi} + \boldsymbol{\lambda\delta\Psi}]}{\wp[\boldsymbol{\Psi}]} = e^{\overbrace{\boldsymbol{\Phi}[\boldsymbol{\Psi}+\boldsymbol{\lambda\delta\Psi}]-\boldsymbol{\Phi}[\boldsymbol{\Psi}]}^{\boldsymbol{\Phi}(\lambda)-\boldsymbol{\Phi}(0)}} \approx e^{\frac{1}{2}\ddot{\boldsymbol{\Phi}}(0)\lambda^2} = e^{-\int_\Sigma E^\mu[\lambda\delta\Psi]d\Sigma_\mu}$$

In conclusion, the probability distribution for fluctuations is

$$\wp \propto e^{-\int_\Sigma E^\mu d\Sigma_\mu}$$

Appendix

The first stability-causality theorem

Theor: Thermodynamic stability implies linear causality.

Proof: See the picture, and recall that

$\boldsymbol{\Phi(\Sigma_1) \leq \Phi(\Sigma_2)}$.

Expand to order $\boldsymbol{\epsilon^2}$, and recall that $\boldsymbol{E^\mu}$ is timelike future-directed:

$$0 = \int_{\Sigma_1} \boldsymbol{E^\mu d\Sigma_\mu} \geq \int_{\Sigma_2} \boldsymbol{E^\mu d\Sigma_\mu} \geq \mathbf{0}.$$

Thus, $\boldsymbol{E^\mu} = \mathbf{0}$ on $\boldsymbol{\Sigma_2}$, and so is $\boldsymbol{\dot{\Psi}}$.

The propagation of linear signals is causal.