Spectra and transport in the RTA (and beyond)

Based on:
arXiv:2403.17769 with Matej Bajec and Sašo Grozdanov
arXiv:2501.00099 with Sašo Grozdanov

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Overview

Motivation

RTA kinetic theory

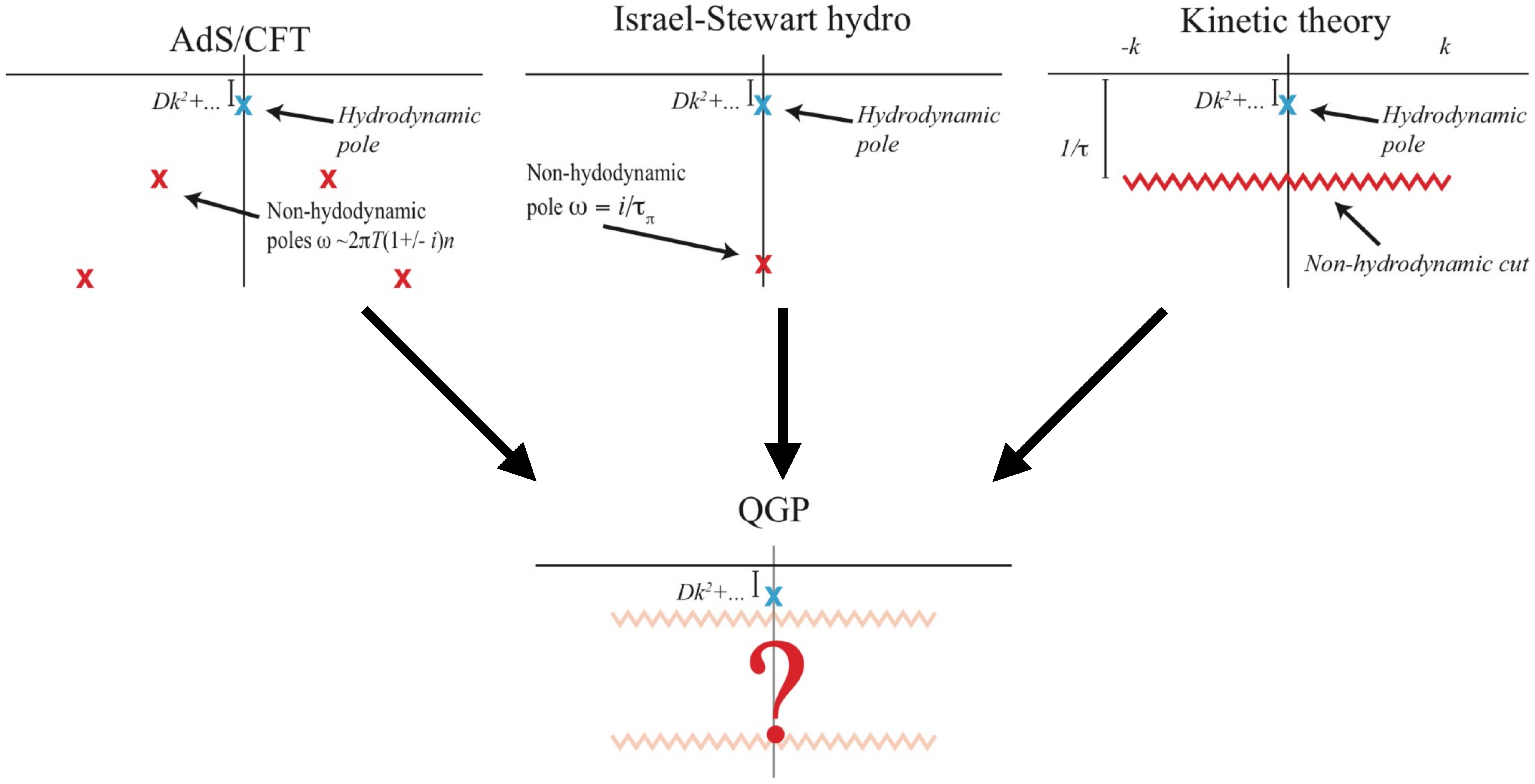
Analytic correlators and their transport properties

Beyond the RTA

Motivation

Thermal correlators of conserved operators carry important transport information

Strong coupling correlators extensively computed: Reissner-Nördstrom black holes Son, Starinets 2006, Myers, Starinets, Thomson 2007, Davison, Kaplis 2011…, Einstein-Maxwell-dilaton Davison, Goutéraux 2014, finite coupling: Kaplis, Grozdanov, Starinets 2016, …



Adapted from Kurkela, Wiedemann, Wu 2019

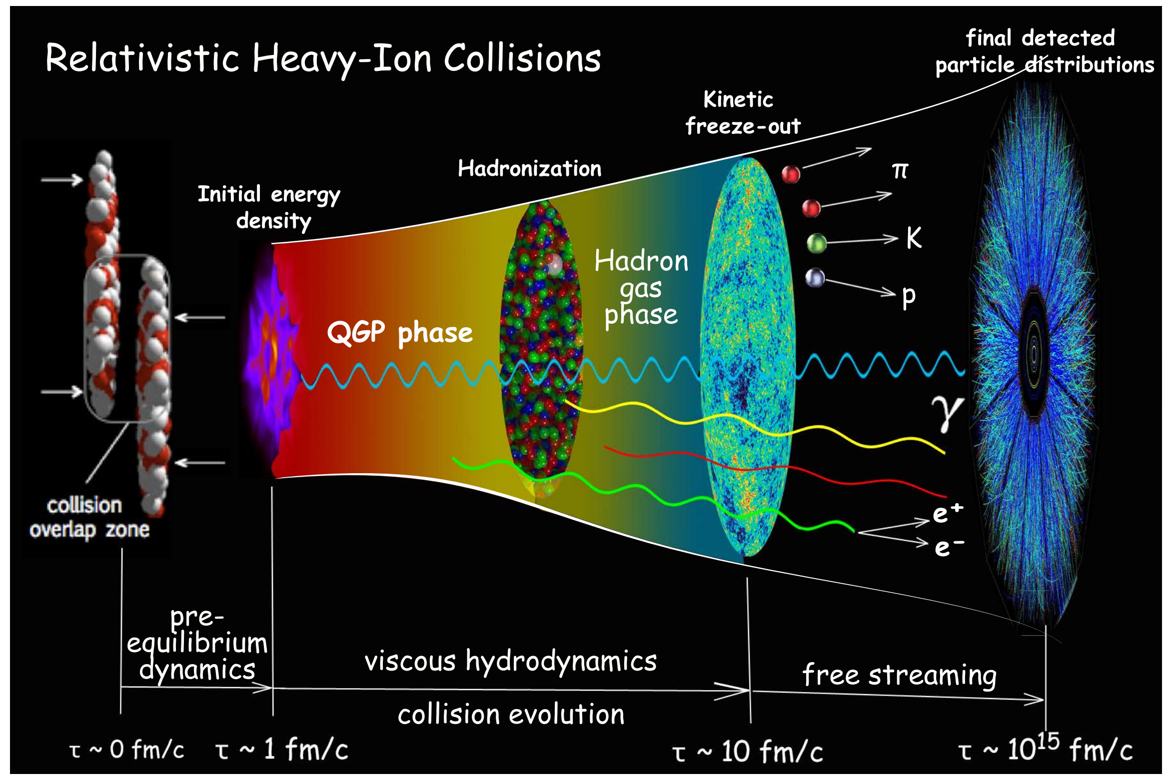
How does weak coupling compare? Romatschke 2015, Kurkela, Wiedemann 2017

Core questions

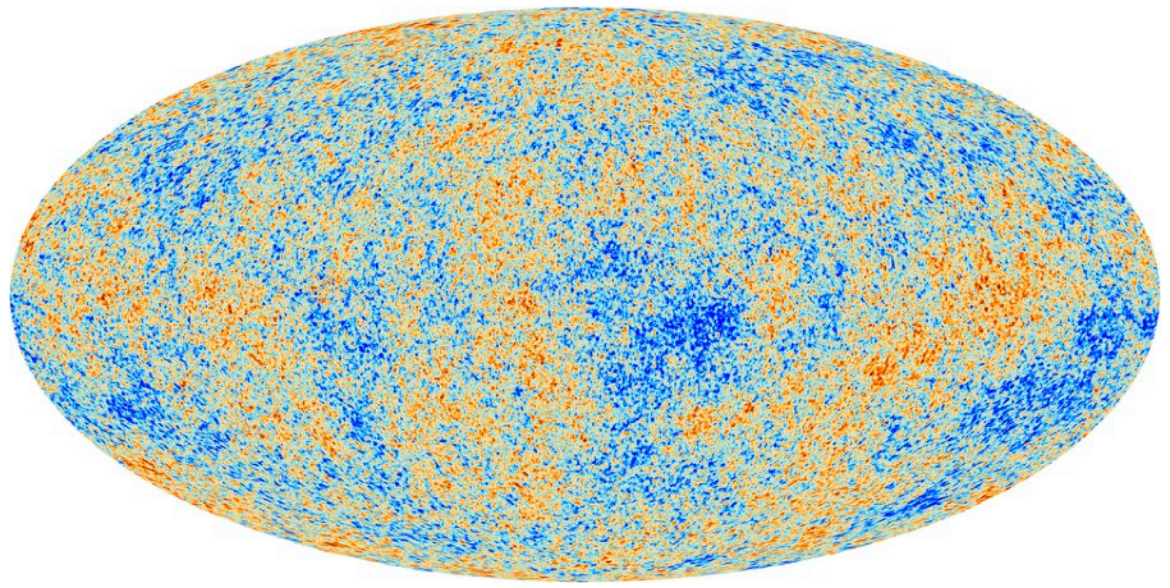
- What is the analytic structure of correlators from the Bolzmann Equation due to external electric field/ temperature gradient:
- … in 3+1 dimensions? QGP, cosmology
- … in 2+1 dimensions? Condensed matter
- … in the presence of impurities?

Method: Relaxation Time Approximation (RTA)

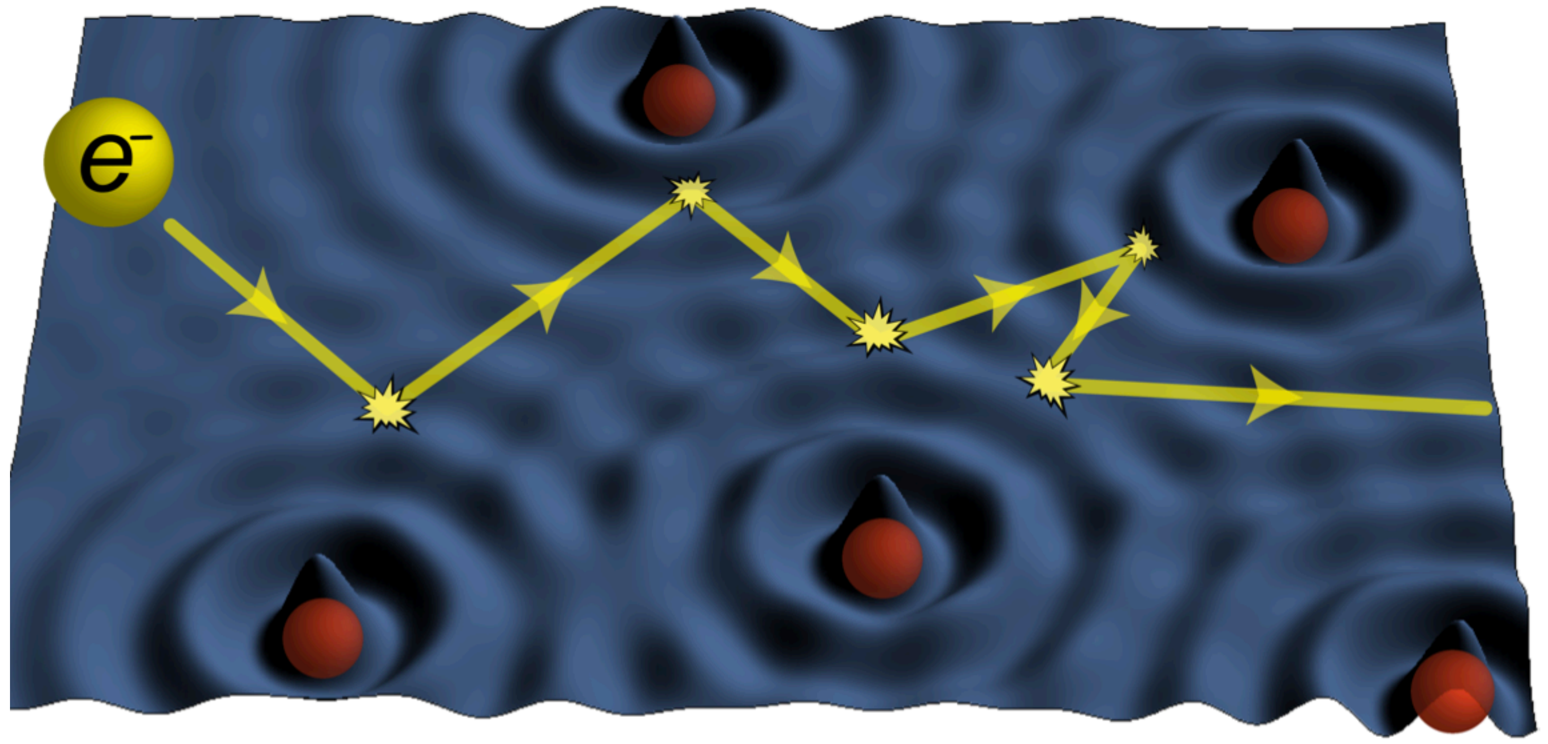
Results: Analytic expression for correlators, thermoelectric transport coefficients



Heinz, Shen 2015



ESA and Planck collaboration



Xie, Chou, Foster 2015

Kinetic Theory at a glance

- Key object: $f(t, \mathrm{x}, \mathrm{p})$, one particle distribution function
- Evolution is given by Boltzmann equation:

$$\underbrace{\left[p^\mu \partial_\mu + F^\mu \nabla_{p^\mu}\right] f}_{\text{free streaming}} = \underbrace{C[f]}_{\text{collisions}}$$

- In practice, very complicated: $C[f] = C_{1\to 2} + C_{2\to 2} + \dots$

RTA kinetic theory

- Replace collision kernel with deviation from equilibrium

$$\left[p^\mu \partial_\mu + F^\mu \partial_{p^\mu}\right] f = \frac{p^\mu u_\mu}{\tau_R} \left(f - f^{\rm eq}\right)$$

Bhatnagar, Gross, Krook 1954
Anderson, Witting 1974

Electric field $F^{\mu\nu} p_\nu$

Gravity $-\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta$

Relaxation time

Maxwell-Boltzmann $e^{\frac{p \cdot u(t,x) + \mu(t,x)}{T(t,x)}}$

- Work with a massless gas $p^2 = 0$

Moments of f

- Integrating over momenta → macroscopic quantities
- Number density

$$n(t,x) = \int \frac{d^d p}{(2\pi)^d} f(t,x,p)$$

- Current and EMT are higher moments:

$$J^\mu = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{p^0} f \qquad T^{\mu\nu} = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^0} f$$

Conservation equations in the RTA

$$p^\mu \partial_\mu f = \frac{p^\mu u_\mu}{\tau_R}\left(f - f^{\text{eq}}\right)$$

$$J^\mu = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{p^0} f$$

- Integrate over momenta

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^0} p^\mu \partial_\mu f = \partial_\mu J^\mu = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^0} \frac{p^\mu u_\mu}{\tau_R}\left(f - f^{\text{eq}}\right)$$

- Conservation of currents/emt $\Rightarrow$ RTA matching conditions

$$\int \frac{d^d p}{(2\pi)^d p^0}(f - f^{eq}) = 0 \qquad \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{p^0}(f - f^{eq}) = 0$$

Linear response in the RTA

- Sources: gauge field, δA_μ, and metric perturbation, $\delta g_{\mu\nu}$
- Sources induce a change in equilibrium values:

$$T(t, x^i) = T_0 + \delta T(t, x^i)$$

$$\mu(t, x^i) = \mu_0 + \delta\mu(t, x^i)$$

$$u^\mu(t, x^i) = (1{,}0{,}0{,}0) + \delta u^\mu(t, x^i)$$

- Leading to a change in the distribution function:

$$f^{\mathrm{eq}}(t, \mathbf{x}, \mathbf{p}) = f_0(\mathbf{p}) + \delta f^{\mathrm{eq}}(t, \mathbf{x}, \mathbf{p}),$$
$$f(t, \mathbf{x}, \mathbf{p}) = f_0(\mathbf{p}) + \delta f(t, \mathbf{x}, \mathbf{p}).$$

Linear response in the RTA

- The Boltzmann equation reads

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v})\,\delta f - \frac{f_0}{T_0}\left(\mathbf{v}\cdot\mathbf{E} - \Gamma^0_{\alpha\beta}p^0 v^\alpha v^\beta\right) = -\frac{\delta f - \delta f^{\mathrm{eq}}}{\tau_R}$$

- Solution is given by

$$\delta f = \frac{\frac{f_0}{T_0}\tau_R \mathbf{E}\cdot\mathbf{v} - \frac{f_0}{T_0}\tau_R \Gamma^0_{\alpha\beta}p^0 v^\alpha v^\beta + \delta f_{\mathrm{eq}}}{1 + \tau_R(-i\omega + i\mathbf{k}\cdot\mathbf{v})}$$

- Integration over moments gives δJ^μ and $\delta T^{\mu\nu}$

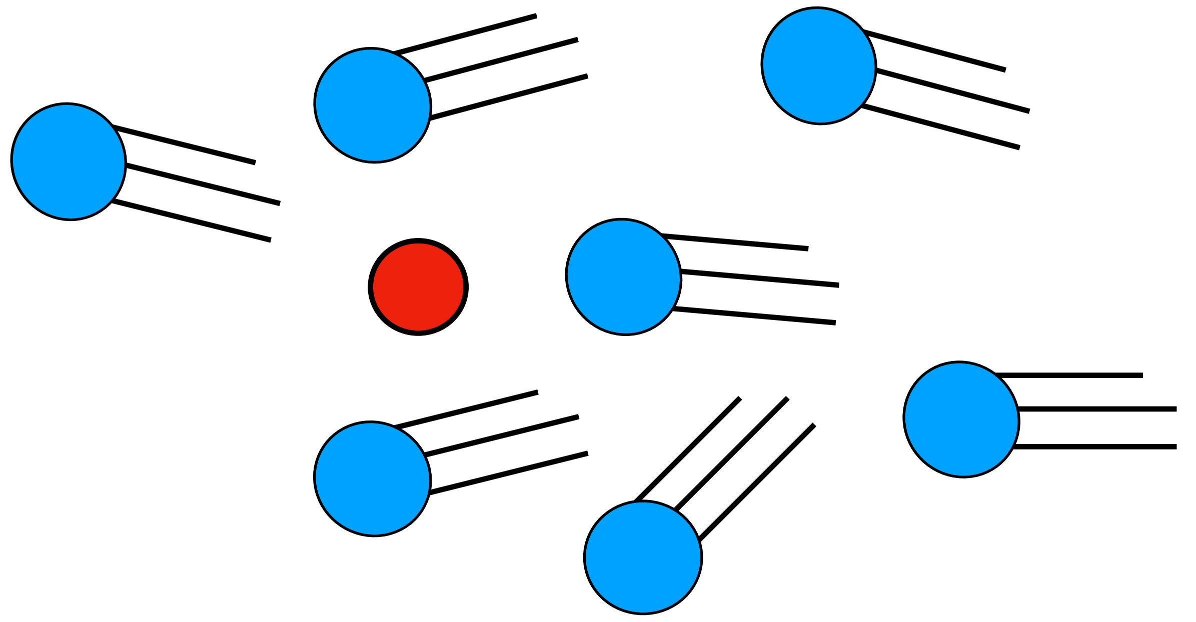
- Use matching conditions to solve self-consistently for $\delta T, \delta\mu, \delta u^\mu$

- Compute correlators via variational approach

$$J^\mu = J_0^\mu - G_{JJ}^{\mu,\nu}\delta A_\nu - \frac{1}{2}G_{JT}^{\mu,\alpha\beta}\delta g_{\alpha\beta} + \ldots,$$

$$T^{\mu\nu} = T_0^{\mu\nu} - \frac{1}{2}G_{TT}^{\mu\nu,\alpha\beta}\delta g_{\alpha\beta} - G_{TJ}^{\mu\nu,\alpha}\delta A_\alpha + \ldots,$$

Momentum relaxation



- Consider a gas of particles with dilute impurities
- Extra scattering in collision term due to impurities

$$C[f] = \frac{p^\mu u_\mu}{\tau_R}\left(f - f_{\mathrm{eq}}\right) - \frac{p^\alpha u_\alpha}{T} u_\mu \Gamma^\mu_\nu p^\nu f$$

where $\Gamma^\mu_\nu = \mathrm{diag}(0, \Gamma_\perp, \Gamma_\parallel)$

- Leads to momentum relaxation in conservation equation

$$\nabla_\mu T^{\mu\nu} = \begin{cases} 0, & \nu \neq i, \\ -\Gamma T^{0i}, & \nu = i, \end{cases}$$

$$\nabla_\mu J^\mu = 0,$$

(Can also have energy/number/etc. relaxation, see Amoretti et al. 2014)

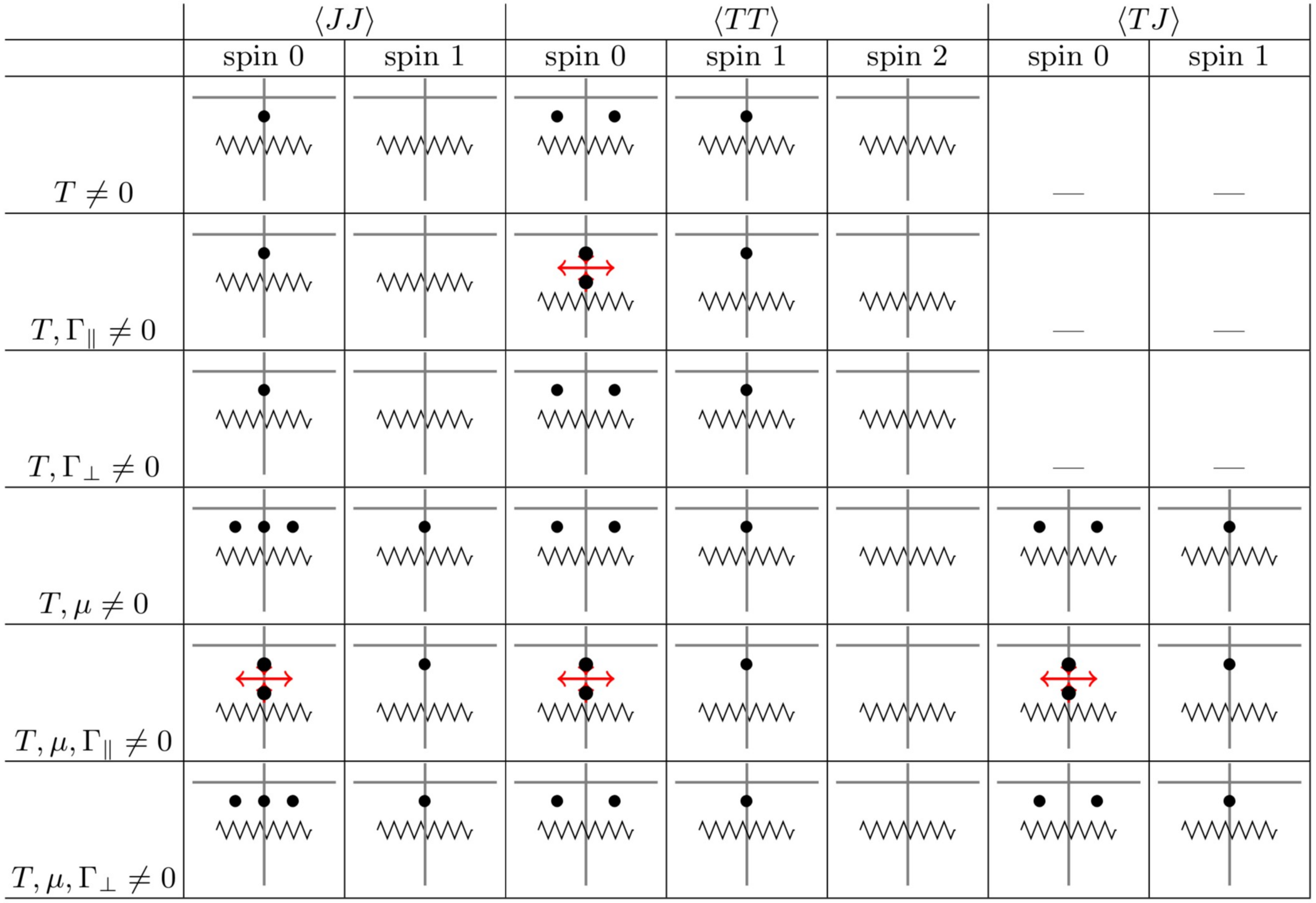
Results: 3+1 d

Analytic structure in 3+1 dimensions

- Black points denote poles
- Squiggle is a logarithmic branch cut

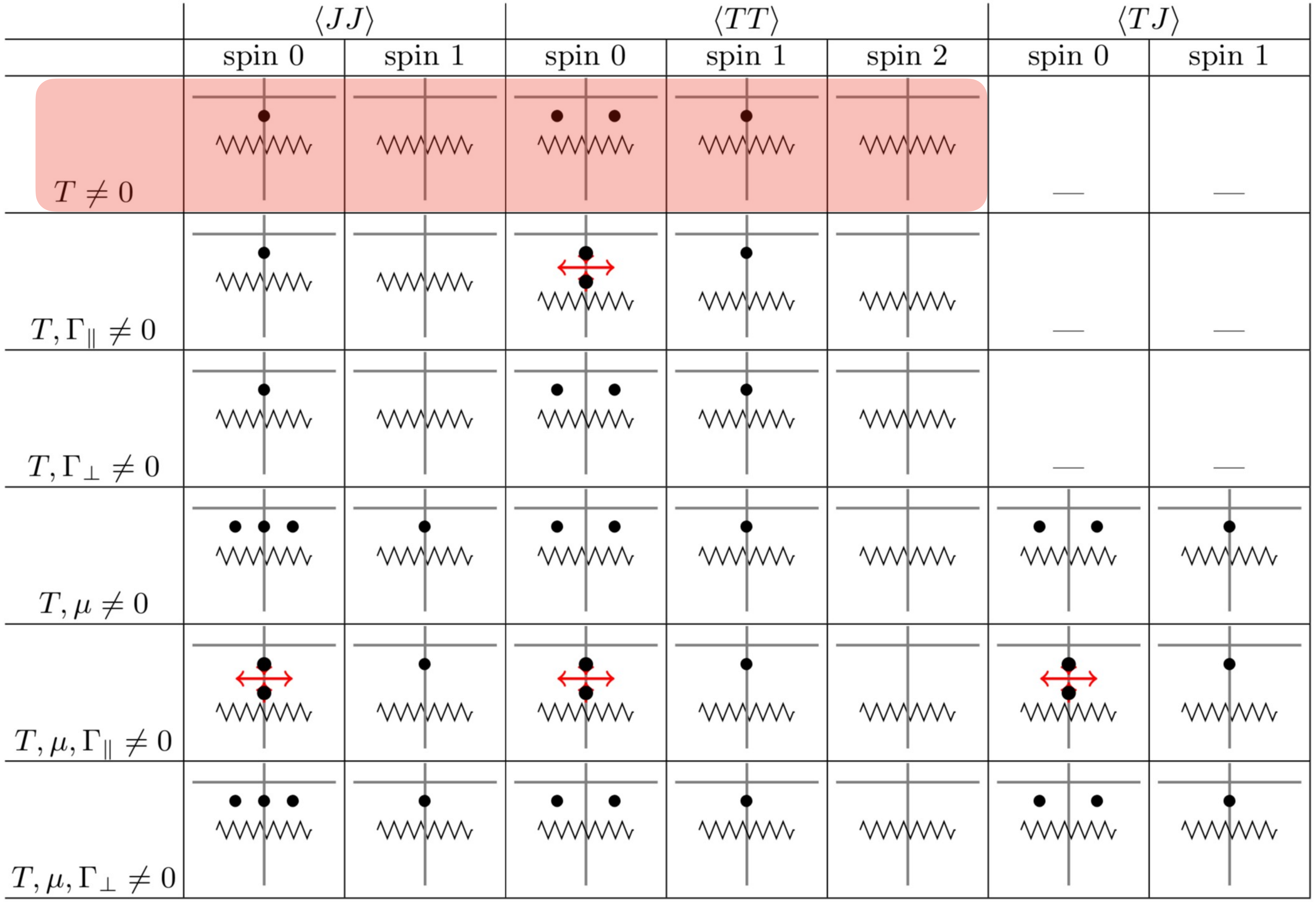
$$\omega = \pm k - \frac{i}{\tau_R}$$

- Red arrows denote quasihydro modes
- Channels arise from $SO(2)$ symmetry around momentum vector



Analytic structure in 3+1 dimensions

First computed by Romatschke 2015 for the massless case



Some massive results from 3 days ago:

Hataei, Heydari, Taghinavaz 2504.14591

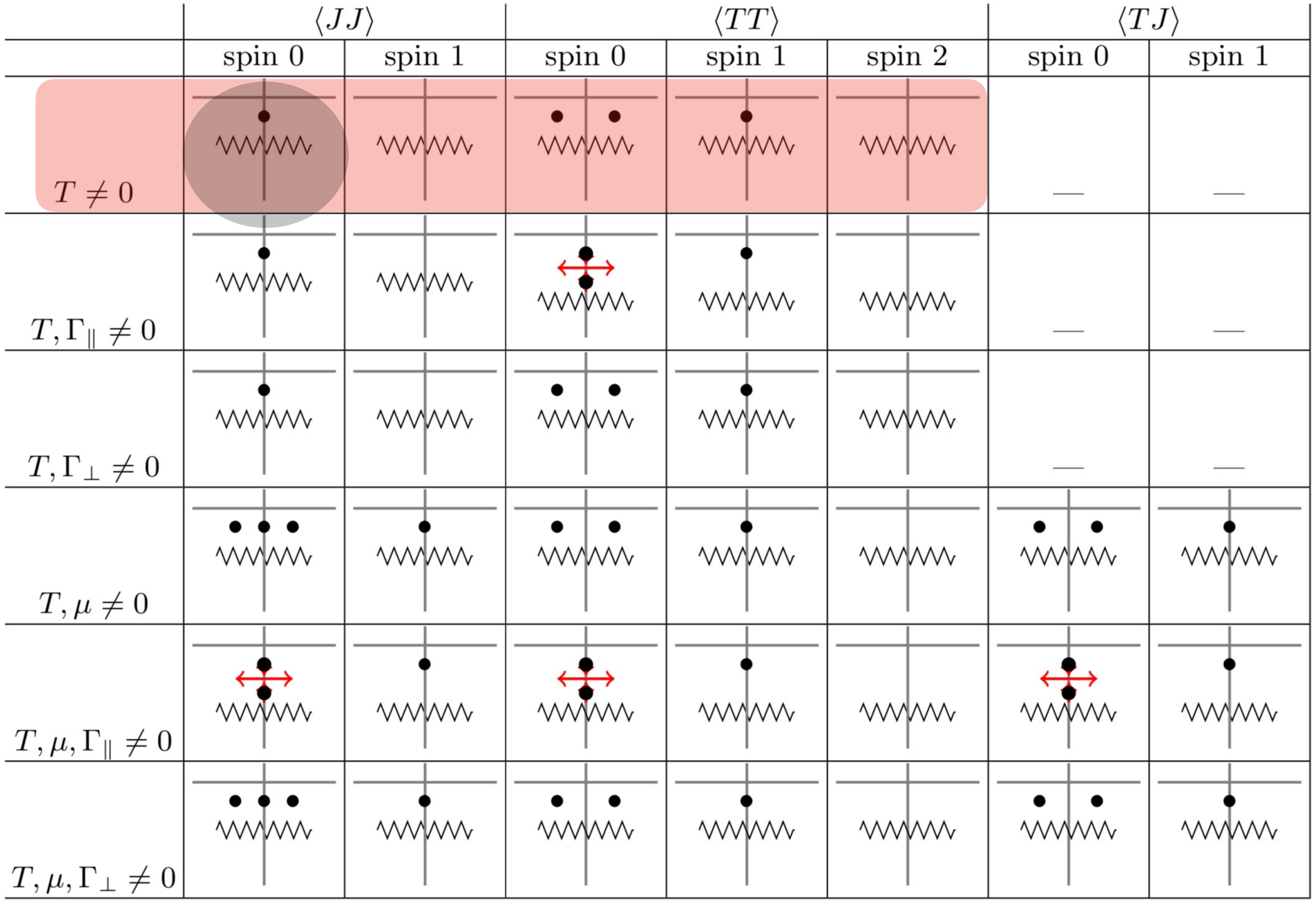
Analytic structure in 3+1 dimensions

$$G_J^{0,0}(\omega,k) = -\chi \frac{1+\left(i\omega - \frac{1}{\tau_D}\right)\frac{1}{2ik}\ln\left(\frac{\omega-k+\frac{i}{\tau_D}}{\omega+k+\frac{i}{\tau_D}}\right)}{1-\frac{1}{2ik\tau_D}\ln\left(\frac{\omega-k+\frac{i}{\tau_D}}{\omega+k+\frac{i}{\tau_D}}\right)}$$

Diffusion pole:

$$\omega = -i\frac{\tau_R}{3}k^2 + \dots$$

Romatschke 2015

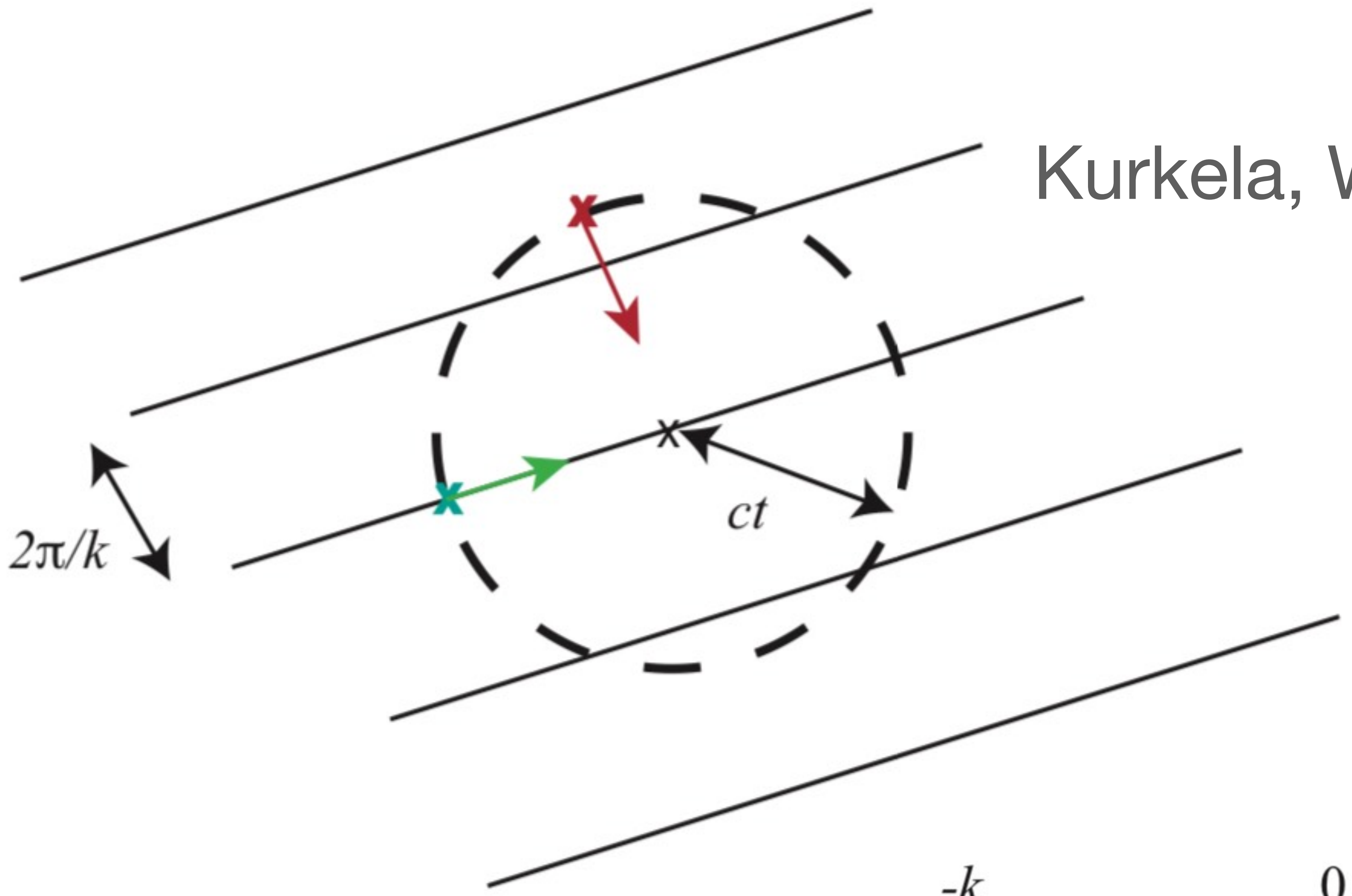


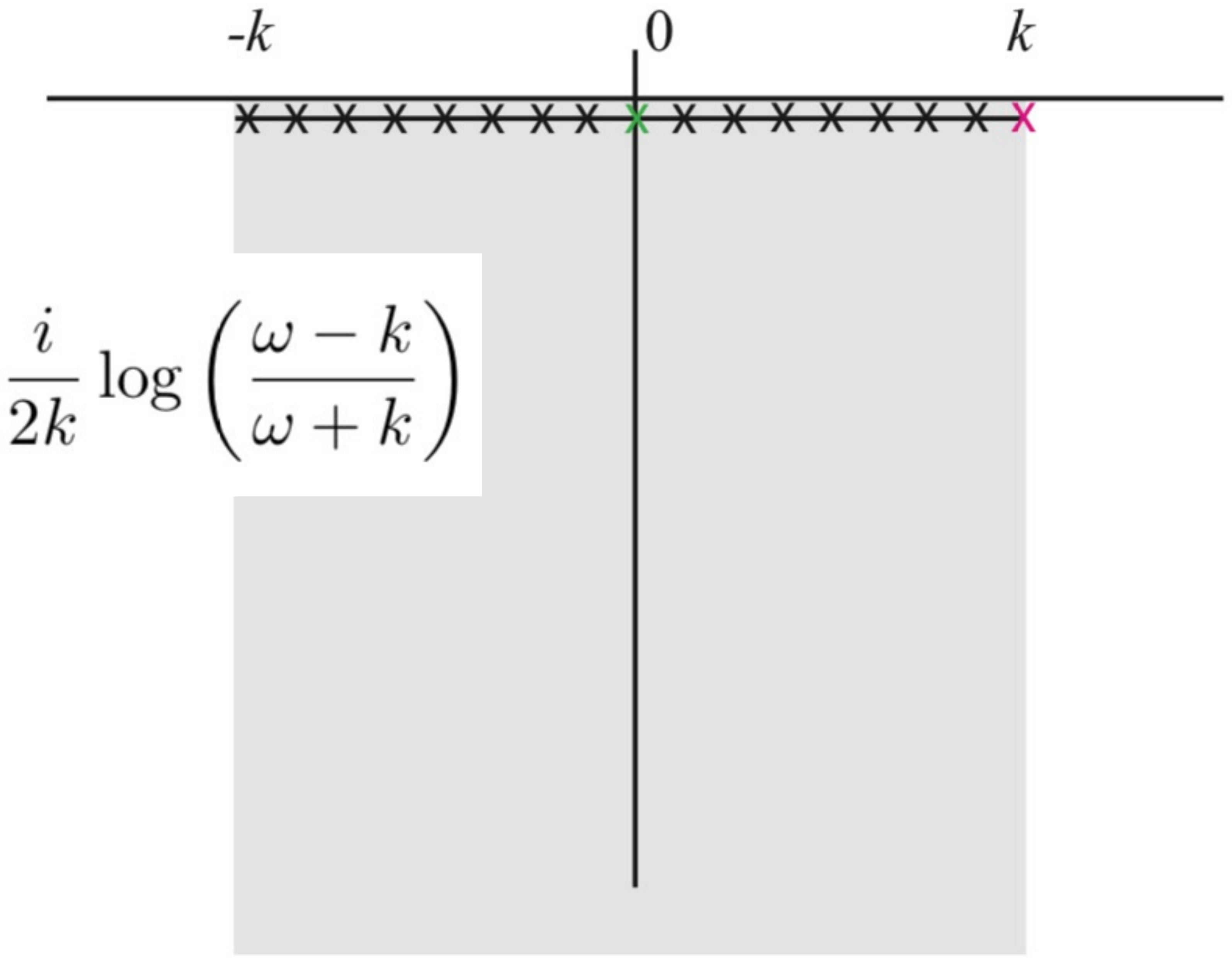
RTA branch cut

Physical meaning

- Picture a massless gas, no interactions
- Perturbations in z-direction induce overdense regions every $2\pi/k$
- Signal from will arrive with frequency $\omega = k\cos\theta$, which corresponds to a pole
- Integrating over all directions, collection of poles assemble to logarithmic branch cut
- Similar structure seen in hard thermal loops, weak coupling QFTs and plasma physics where the branch cut is associated with Landau-damping

Kurkela, Wiedemann 2017





$$\int \frac{d\Omega}{4\pi}\frac{1}{i\omega - i\vec{v}\cdot\vec{k}} = \frac{i}{2k}\log\left(\frac{\omega-k}{\omega+k}\right)$$

Analytic structure in 3+1 dimensions

$$G_T^{00,00}(\omega,k) = -3(\epsilon_0+P_0)\Bigg[1$$

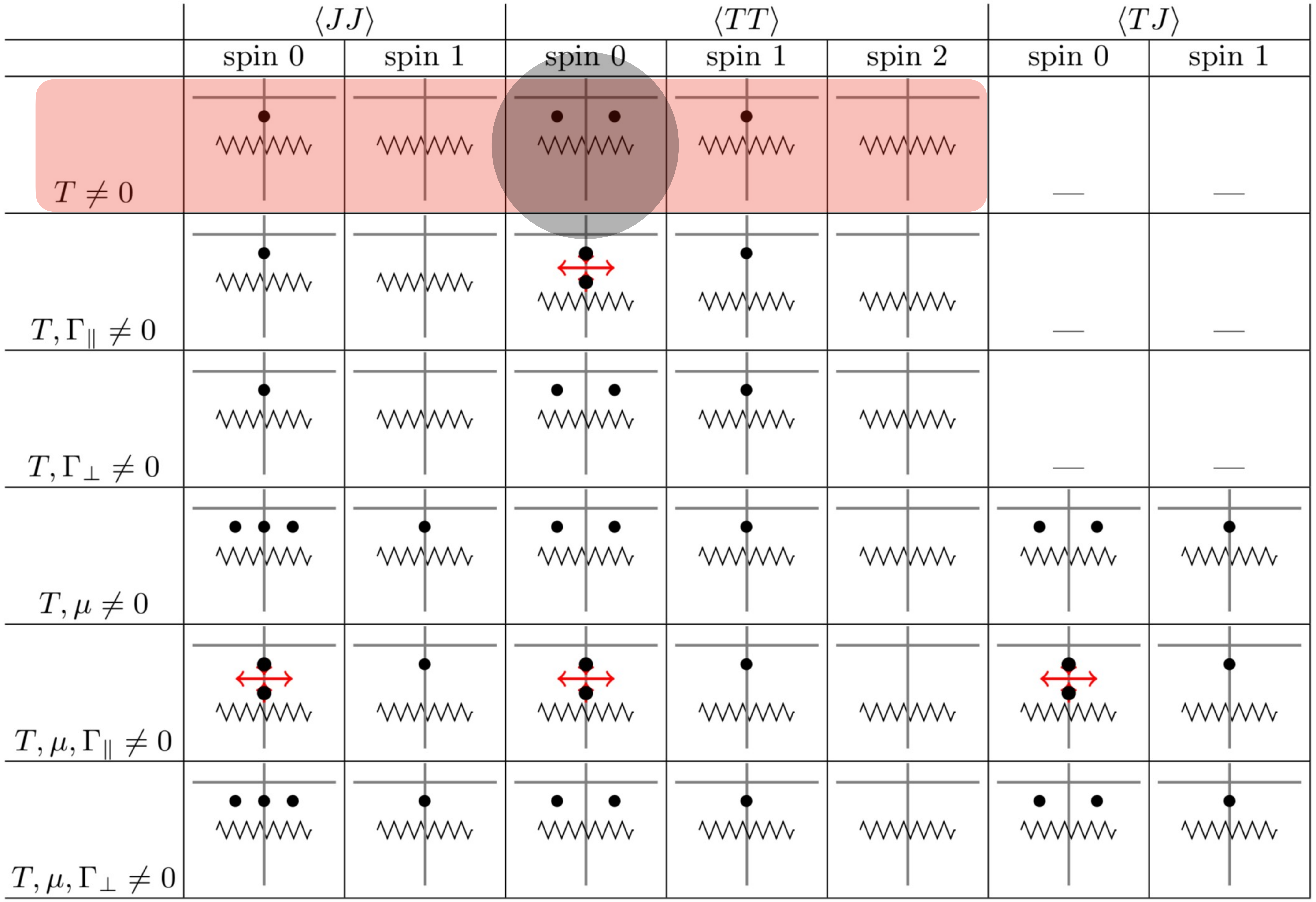
$$+\,k^2\tau_R\,\frac{2k\tau_R + i(1-i\tau_R\omega)L}{2k\tau_R(k^2\tau_R+3i\omega)+i\left(k^2\tau_R+3\omega(i+\tau_R\omega)\right)L}\Bigg]$$

$$L = \ln\left(\frac{\omega - k + \frac{i}{\tau_R}}{\omega + k + \frac{i}{\tau_R}}\right)$$

Sound modes:

$$\omega(k) = \pm\frac{1}{\sqrt{3}}k - \frac{2i}{15}\tau_R\, k^2 + \mathcal{O}(k^3)$$

Romatschke 2015

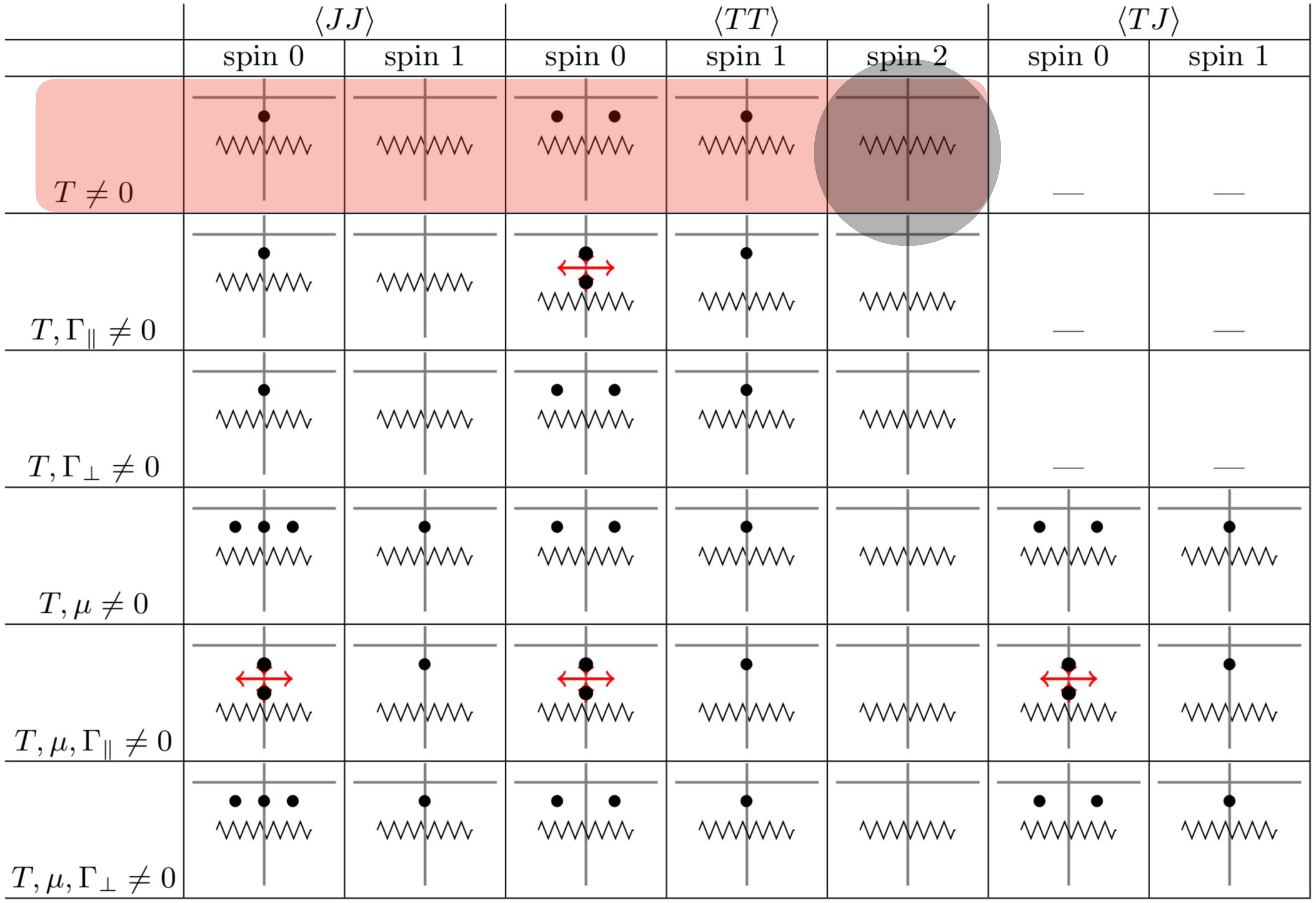


Analytic structure in 3+1 dimensions

$$G_T^{12,12}(\omega,k) = \frac{3i\omega\tau_R(\epsilon_0+P_0)}{16}\left[\frac{10}{3}\frac{1-i\tau_R\omega}{k^2\tau_R^2}\right.$$

$$\left.+\frac{2(1-i\tau_R\omega)^3}{k^4\tau_R^4}+i\frac{((1-i\tau_R\omega)^2+k^2\tau_R^2)^2}{k^5\tau_R^5}L\right]$$

$$L = \ln\left(\frac{\omega-k+\frac{i}{\tau_R}}{\omega+k+\frac{i}{\tau_R}}\right)$$



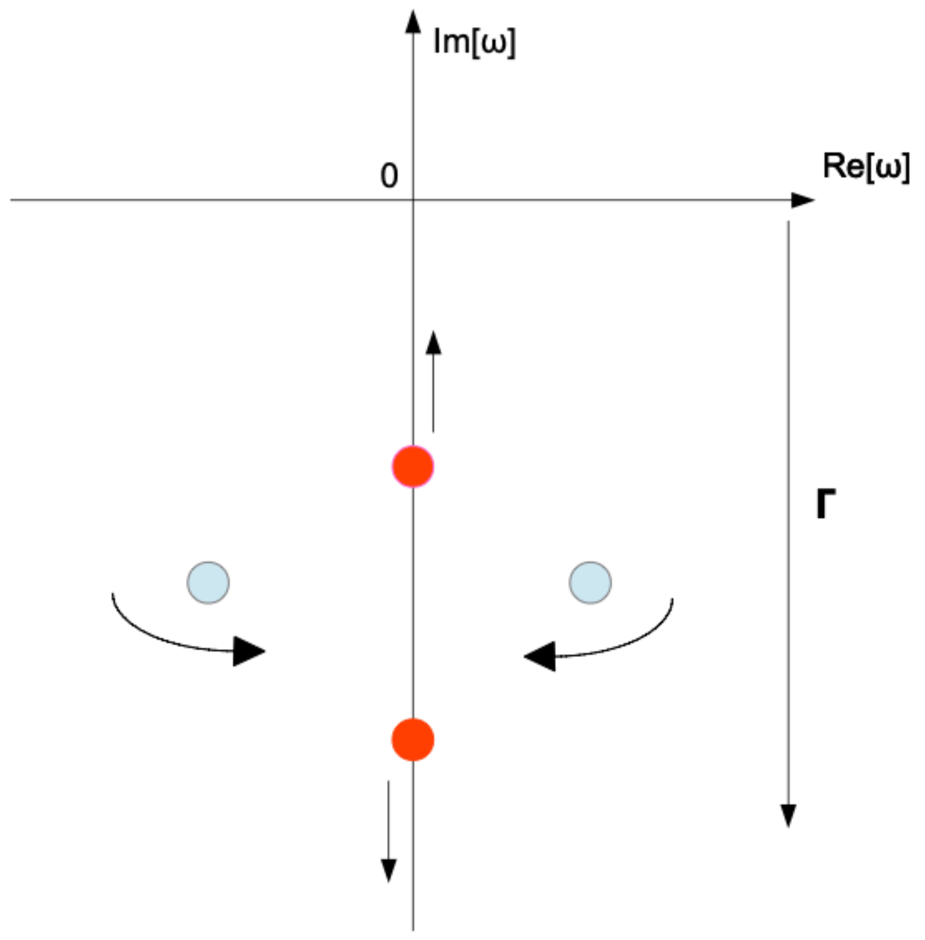
Shear viscosity

$$\frac{\eta}{s} = \frac{1}{s}\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^{12,12} = \frac{\tau_R T_0}{5}$$

Romatschke 2015

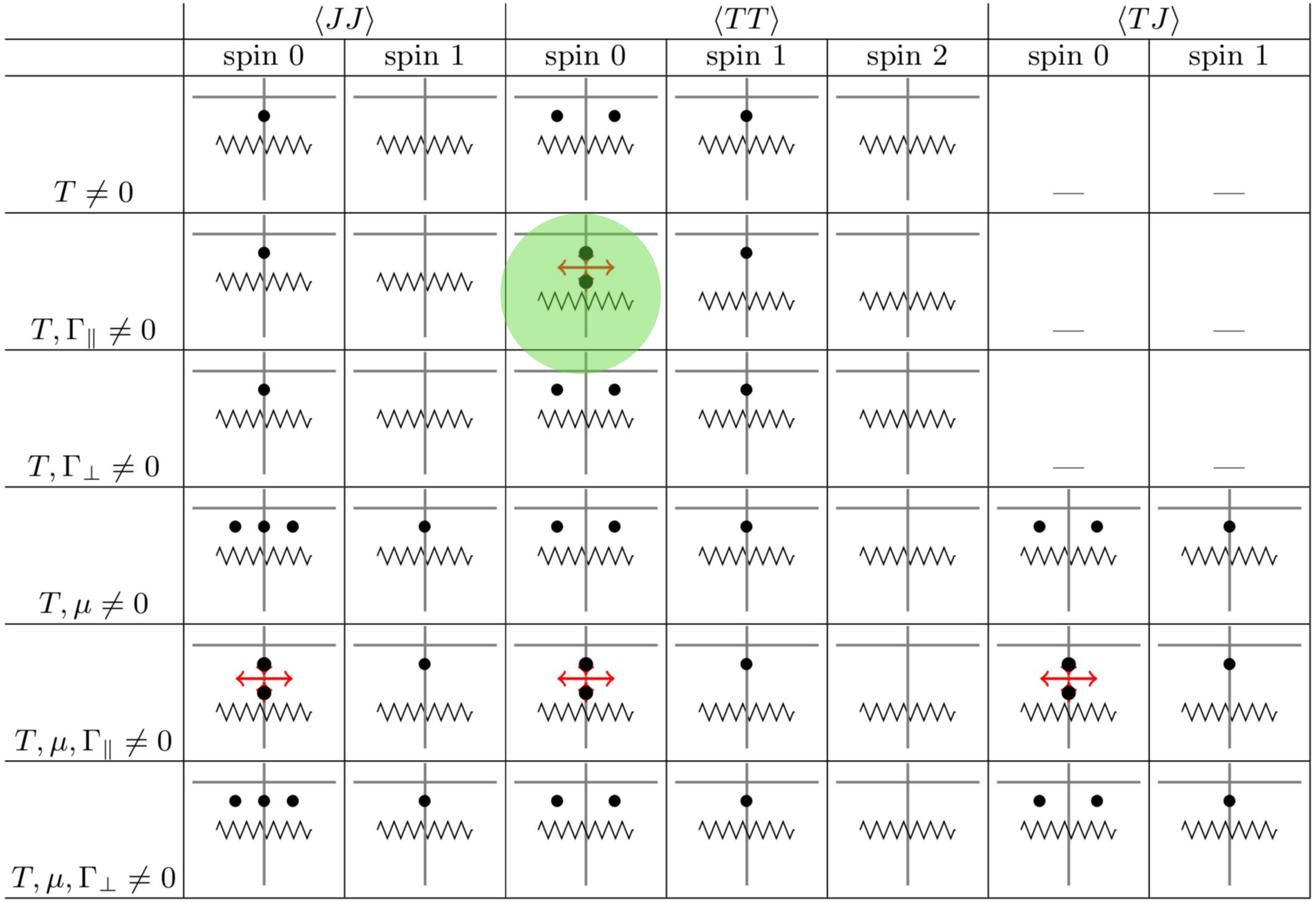
Analytic structure in 3+1 dimensions

Quasihydro crossover, as seen in numerous holographic models

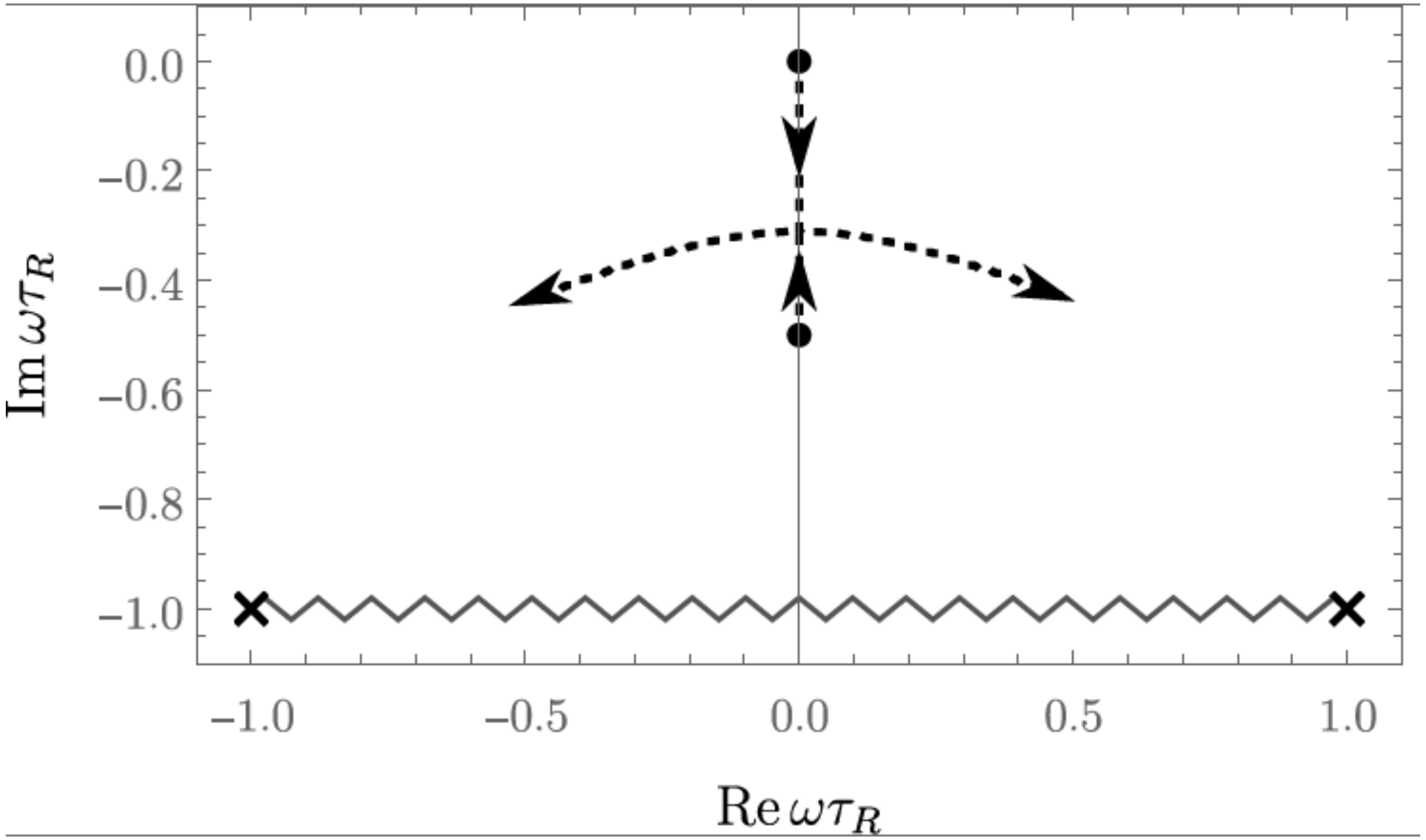


Davison, Gouteraux 2014

Collision occurs at $k_* = \dfrac{\sqrt{3}}{2}\Gamma_{\parallel}$



TT correlator with momentum dissipation



Collision occurs at $k_* = \frac{\sqrt{3}}{2}\Gamma_\parallel$

$$L \equiv \ln\left(\frac{\omega - k + i/\tau_R}{\omega + k + i/\tau_R}\right)$$

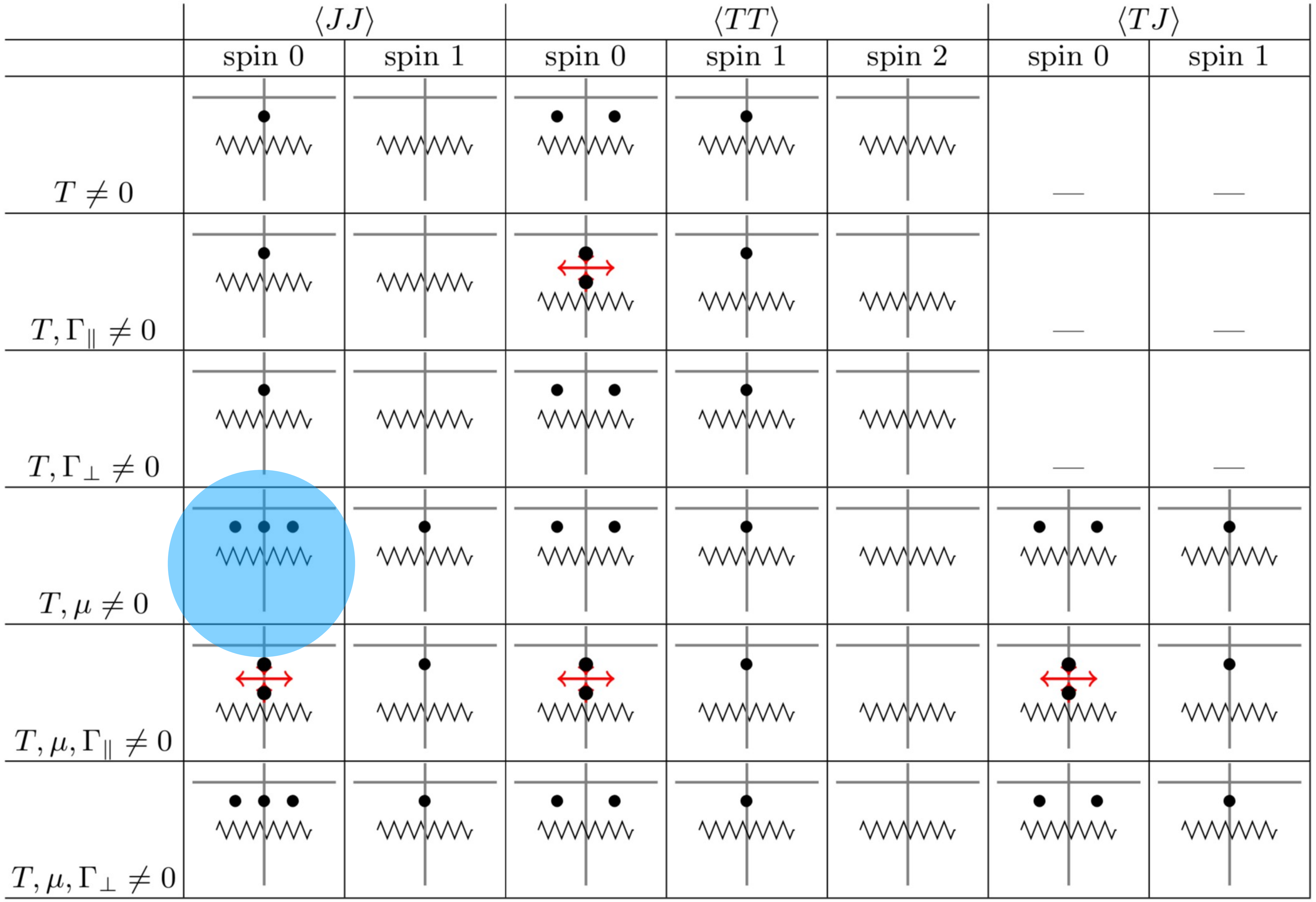
$$G_{TT}^{00,00} = -3(\varepsilon_0 + P_0)\left(1 + \frac{k^2\tau_R(2k\tau_R + L(\tau_R\omega + i))}{-6ik\tau_R\omega(\Gamma_\parallel\tau_R - 1) + 3L\omega(\Gamma_\parallel\tau_R - 1)(1 - i\tau_R\omega) + 2k^3\tau_R^2 + ik^2L\tau_R}\right)$$

Analytic structure in 3+1 dimensions

Finite density and temperature

Factorization of analytic structure!

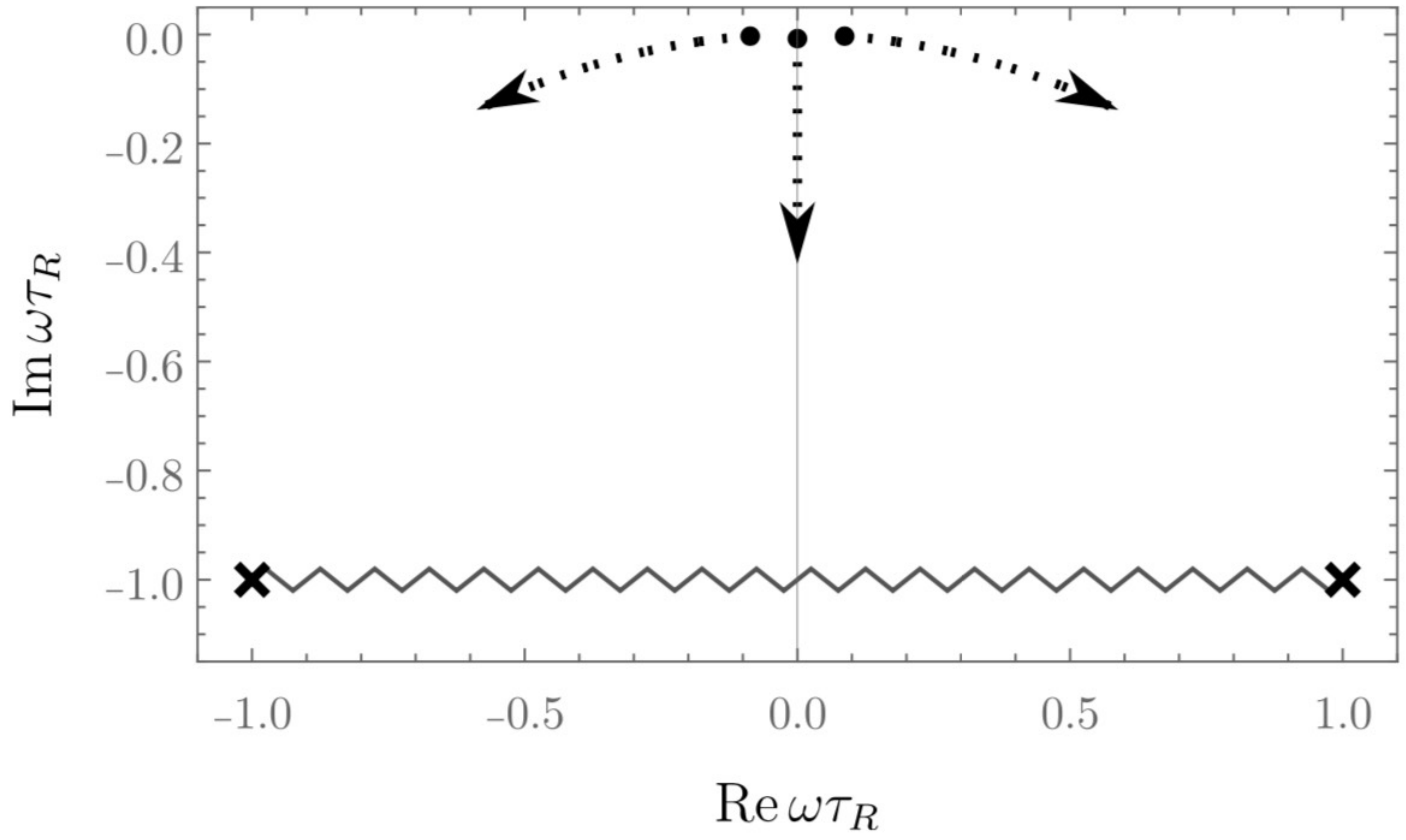
Sound + diffusion



Finite T and μ

$$G_{JJ}^{0,0} = \frac{\chi\left(2k\tau_R + (\omega\tau_R + i)L\right)}{4\left(L - 2ik\tau_R\right)}$$
$$\times\left(i + \frac{3k^2\tau_R^2\left(2ik\tau_R - L\right)}{2k^3\tau_R^3 + ik^2\tau_R^2 L + 6i\omega k\tau_R^2 + 3i\omega\tau_R L\left(\omega\tau - R + i\right)}\right)$$

$$L \equiv \ln\left(\frac{\omega - k + i/\tau_R}{\omega + k + i/\tau_R}\right)$$



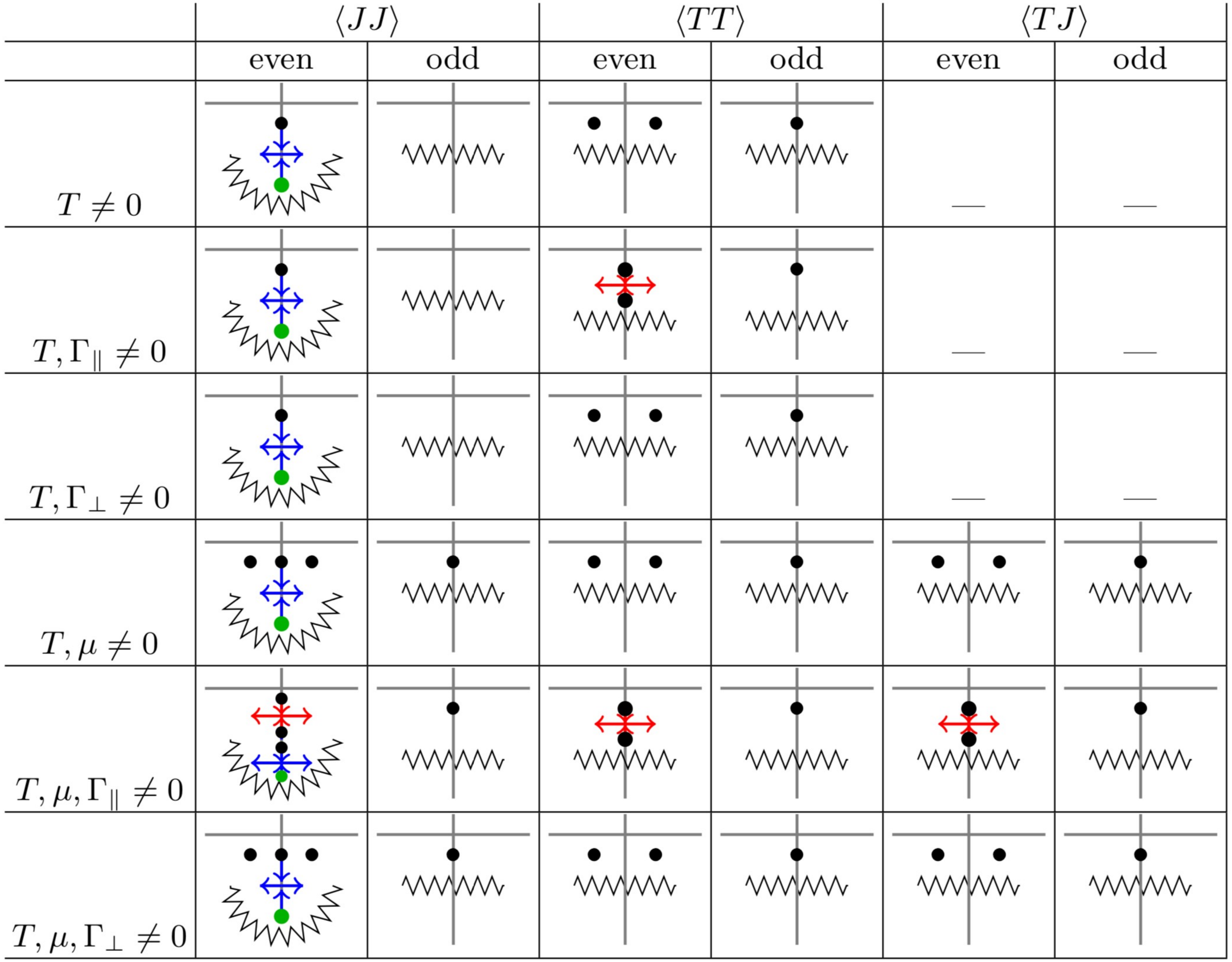
Results: 2+1 d

Analytic structure in 2+1 dimensions

- Black points denote poles
- Squiggle is a square root branch cut

$$\omega = \pm k - \frac{i}{\tau_R}$$

- Two types of collisions!
- Red: collision at $k \sim \Gamma_{\|}$
- Blue: collision at $k = 1/\tau_R$
- Channels due to $\mathbb{Z}_2$ symmetry associated with parity transformations



Analytic structure in 2+1 dimensions

- JJ correlator in the even channel have additional poles

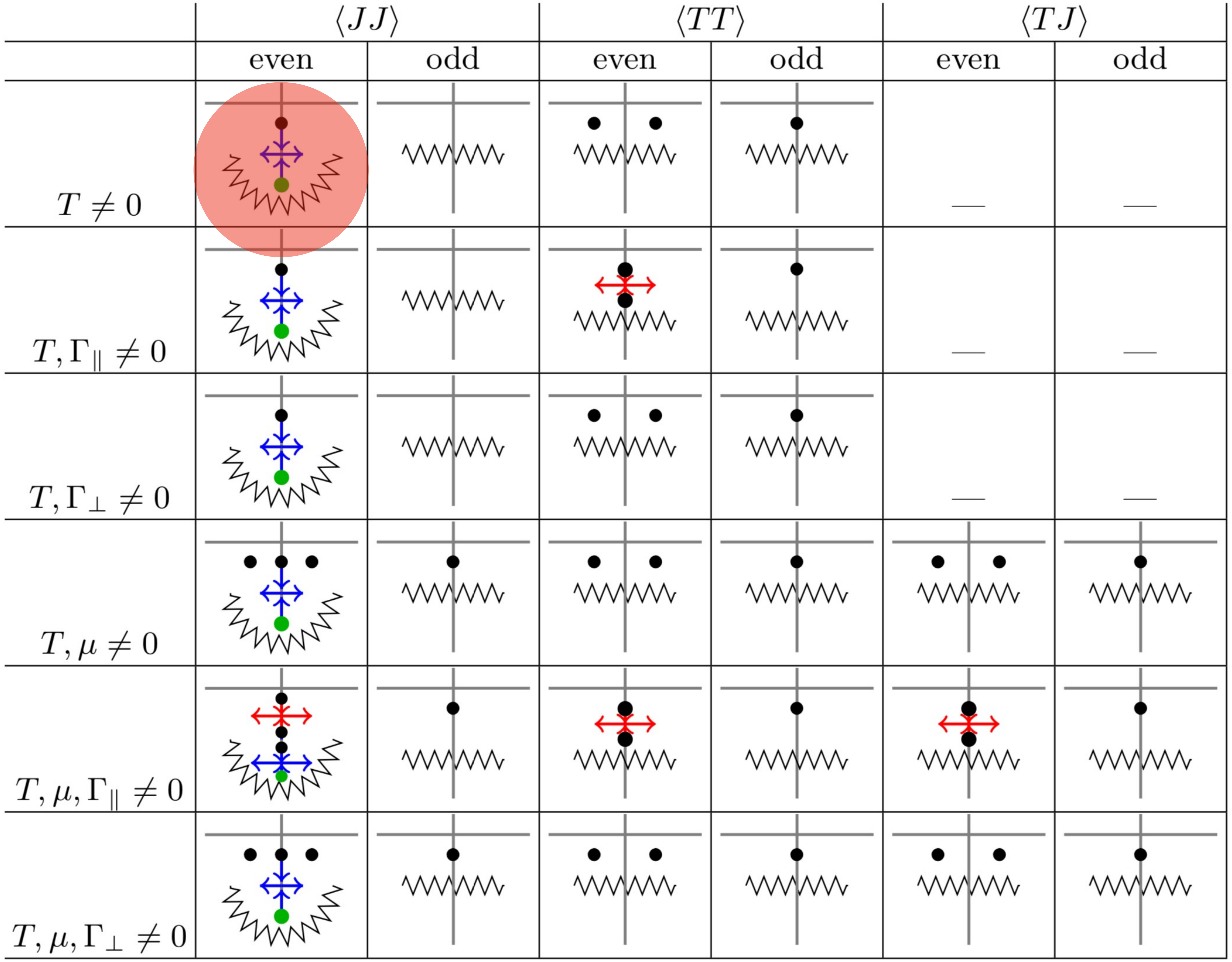
$$\omega = \frac{-i \pm \sqrt{k^2\tau_R^2 - 1}}{\tau_R}$$

- Usual diffusion pole:

$$\omega = -i\frac{\tau_R}{2}k^2 + \dots$$

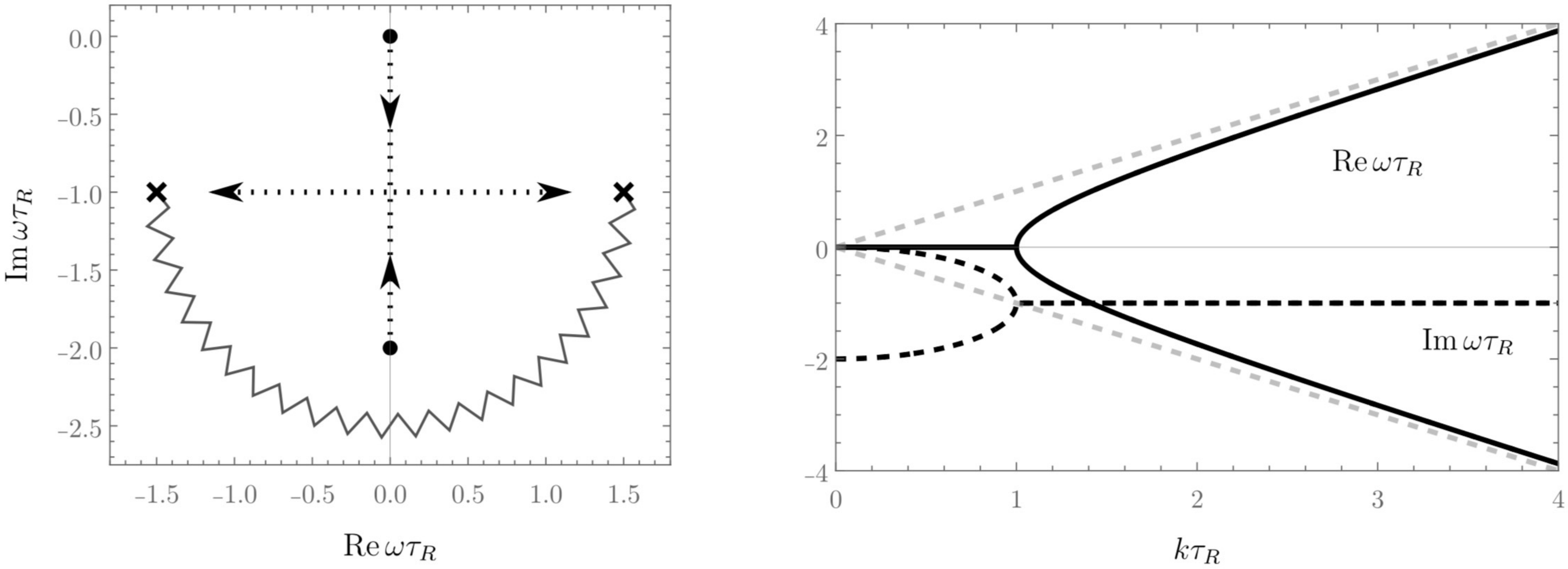
- Gapped mode:

$$\omega = -\frac{2i}{\tau_R} + \frac{i}{2}\tau_R k^2 + \dots$$



JJ correlator in 2+1

$$\omega = \frac{-i \pm \sqrt{k^2\tau_R^2 - 1}}{\tau_R}$$



$$G_{JJ}^{0,0} = \chi \frac{(R + i\omega\tau_R - 1)}{R-1} \frac{\left(-3k^4\tau_R^4 + k^2\tau_R^2(3R + \omega\tau_R(3\omega\tau_R + 4i) - 3) + 2\omega\tau_R(\omega\tau_R + i)(R + i\omega\tau_R - 1)\right)}{3R\left(k^2\tau_R^2(R-1) + 2i(R-1)\omega\tau_R - 2\omega^2\tau_R^2\right)}$$

$$R = \sqrt{k^2\tau_R^2 - (\omega\tau_R + i)^2}$$

JJ correlator in holography

Witczak-Krempa, Sachdev 2013

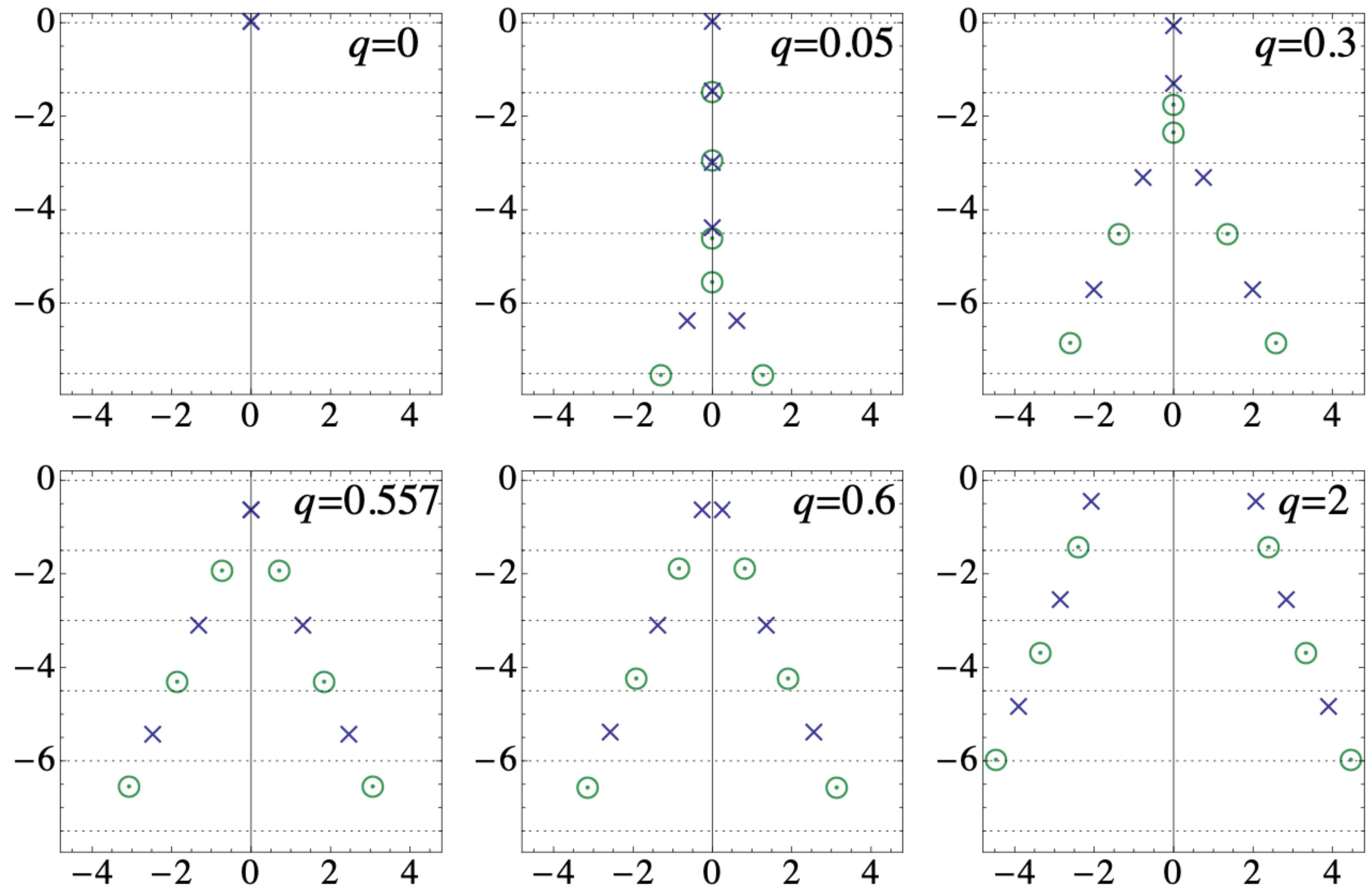
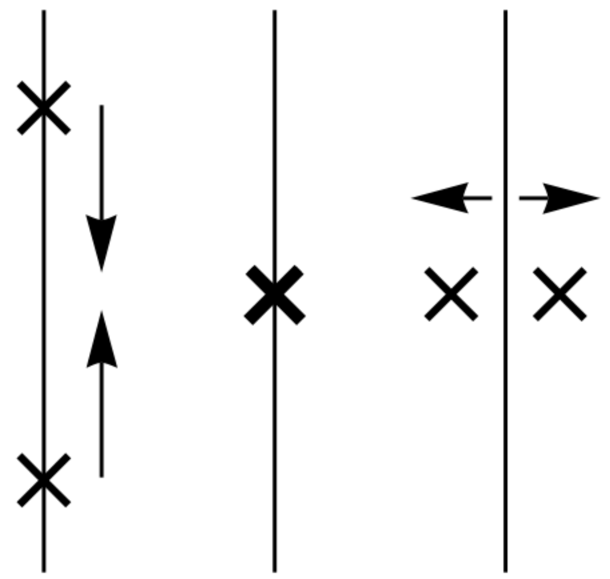


FIG. 3. Poles (crosses) and zeros (circles) in the lower-half complex frequency plane of $C_{tt} = -\chi^2 q^2/\Pi^T(w,q)$, the R-charge density correlation function of the $\mathcal{N} = 8$ supersymmetric Yang-Mills CFT. The positions of the poles and zeros are interchanged for $C_{yy} = \Pi^T$.



Strong coupling - yet similar collision!

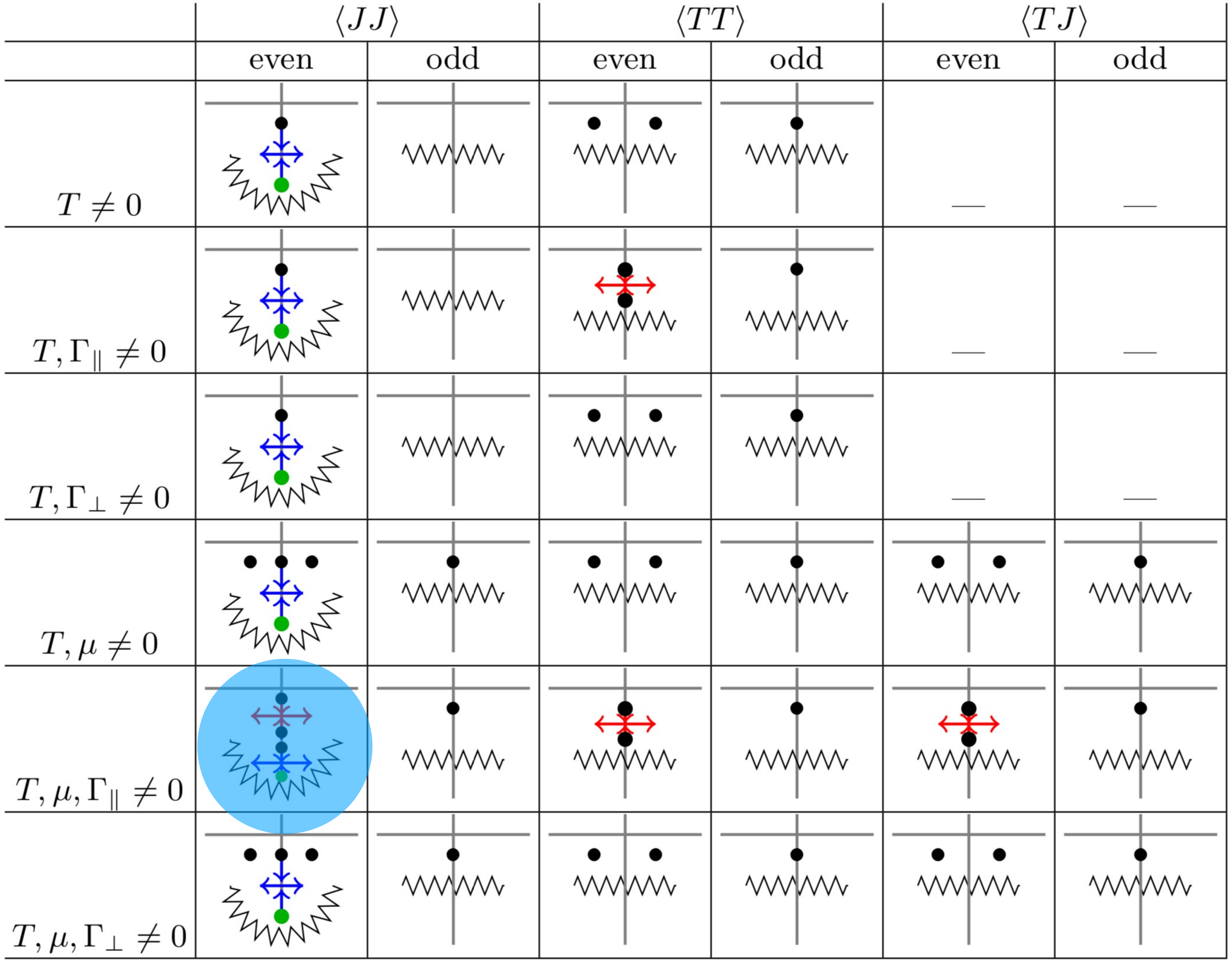
Analytic structure in 2+1 dimensions

Finite temperature, density and momentum relaxation

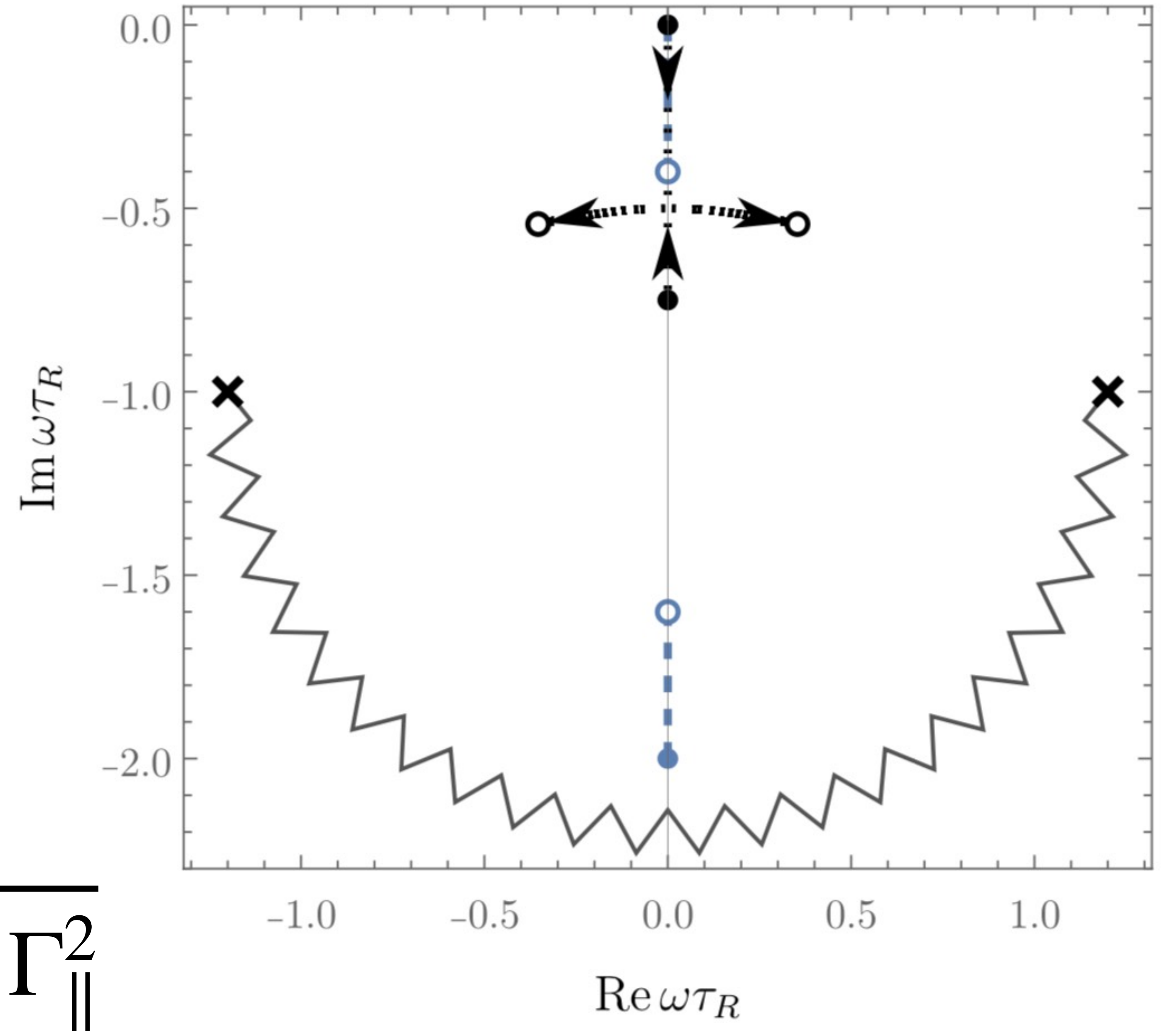
Channel with two collisions!

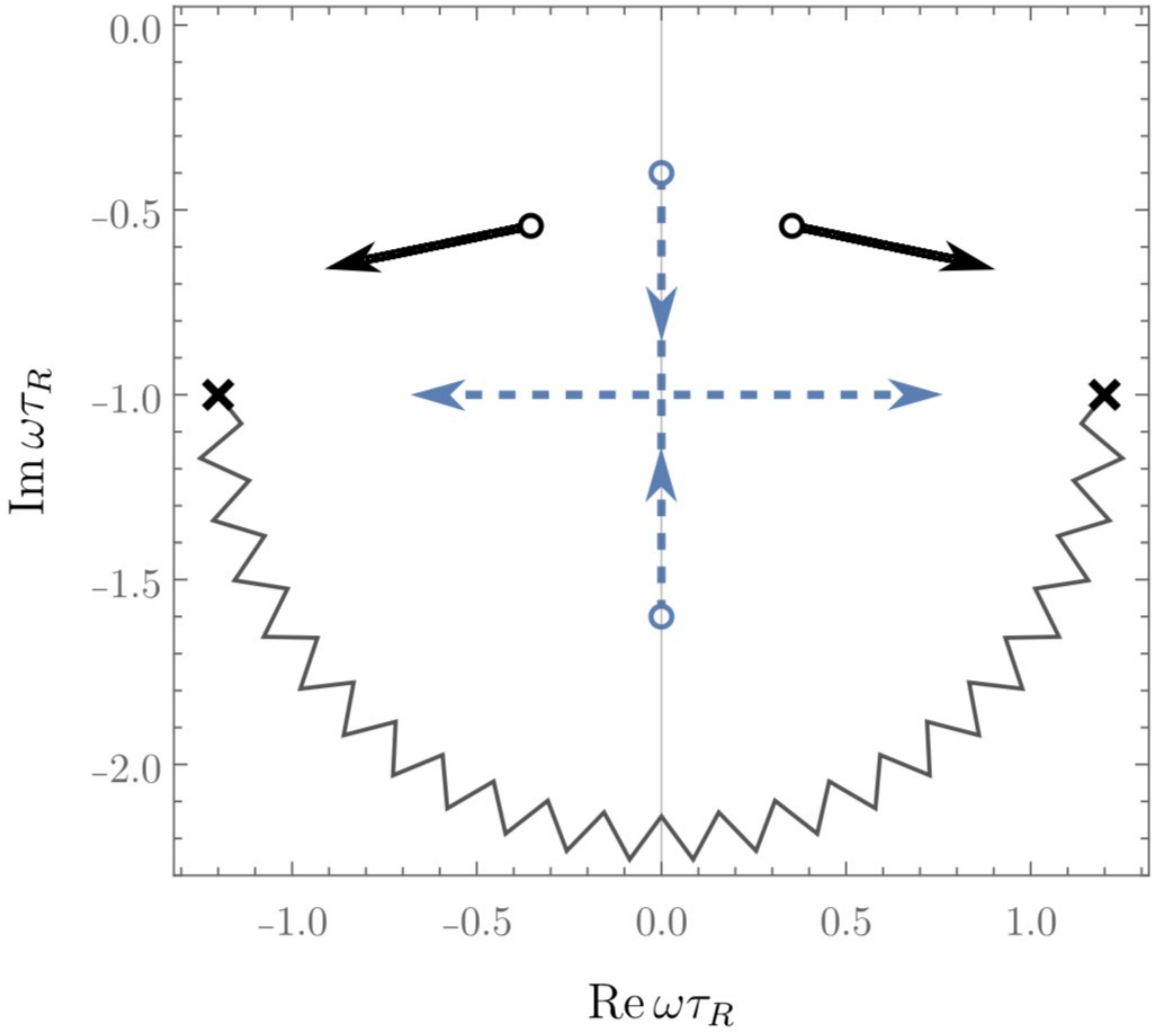
Red at $k_* = \Gamma_\| / \sqrt{2}$

Blue at $k = 1/\tau_R$



Thermoelectric JJ correlator with momentum dissipation





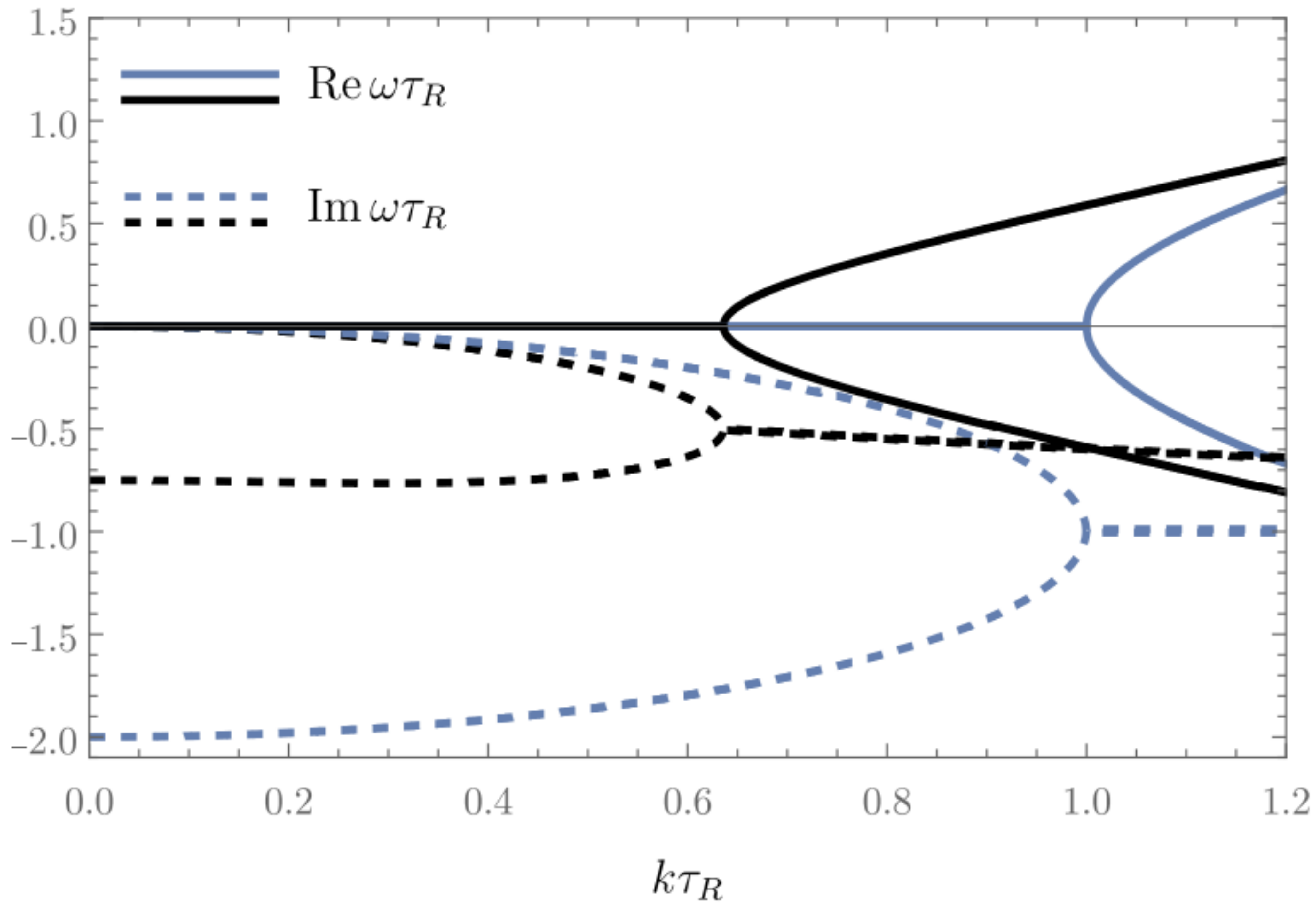
$$\omega = \frac{-i\Gamma_{\|} \pm \sqrt{2k^2 - \Gamma_{\|}^2}}{2} + \ldots$$

$$\omega = \frac{-i \pm \sqrt{k^2\tau_R^2 - 1}}{\tau_R}$$

$$G_{JJ}^{0,0} = \frac{\chi(R + i\tau_R\omega - 1)}{R-1}\frac{1}{3R\left(k^2\tau_R^2(R-1) + 2\omega\tau_R\tilde{\Gamma}_{\|}(-iR + \tau_R\omega + i)\right)}$$
$$\times\Big(k^2\tau_R^2(\tau_R\omega(2i\Gamma_{\|}\tau_R + 3\tau_R\omega + 4i) + 3R - 3)$$
$$- 2\omega\tau_R\tilde{\Gamma}_{\|}(\tau_R\omega + i)(R + i\tau_R\omega - 1) - 3k^4\tau_R^4\Big),$$

$$R = \sqrt{k^2\tau_R^2 - (\omega\tau_R + i)^2}$$

Thermoelectric JJ correlator with momentum dissipation



$$\omega = \frac{-i\Gamma_{\parallel} \pm \sqrt{2k^2 - \Gamma_{\parallel}^2}}{2} + \dots$$

$$\omega = \frac{-i \pm \sqrt{k^2\tau_R^2 - 1}}{\tau_R}$$

$$G_{JJ}^{0,0} = \frac{\chi(R + i\tau_R\omega - 1)}{R-1} \frac{1}{3R\left(k^2\tau_R^2(R-1) + 2\omega\tau_R\tilde{\Gamma}_{\parallel}(-iR + \tau_R\omega + i)\right)}$$
$$\times \left(k^2\tau_R^2(\tau_R\omega(2i\Gamma_{\parallel}\tau_R + 3\tau_R\omega + 4i) + 3R - 3)\right.$$
$$\left. - 2\omega\tau_R\tilde{\Gamma}_{\parallel}(\tau_R\omega + i)(R + i\tau_R\omega - 1) - 3k^4\tau_R^4\right),$$

$$R = \sqrt{k^2\tau_R^2 - (\omega\tau_R + i)^2}$$

Thermoelectric coefficients

Thermoelectric effect

Seebeck effect: temperature gradient ⇒ electric field

Seebeck 1821

Peltier effect: electric gradient ⇒ temperature gradients

Peltier 1834

Thermopile from Leopoldo Nobili

Rooms XV/XVI museo Galileo

⇒Apply computed correlators to determine thermoelectric coefficients

Hartnoll, Kovtun, Müller, Sachdev 2007

Thermoelectric coefficients

Electric and thermal transport coefficients $\begin{pmatrix} \delta J^i \\ \delta Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T_0\alpha^{ij} \\ T_0\tilde{\alpha}^{ij} & T_0\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\partial_j \delta T / T_0 \end{pmatrix}$

Heat current: $\delta Q^i = \delta T^{0i} - \mu_0 \delta J^i$

Compute via Kubo formula, e.g.

$$\sigma^{ij}(\omega) = -\frac{1}{i\omega} \lim_{k\to 0} \left(G^{ij}_{JJ}(\omega, k) - G^{ij}_{JJ}(0,k) \right)$$

Thermoelectric coefficients

Hartnoll, Kovtun, Müller, Sachdev 2007

$$\begin{pmatrix} \delta J^i \\ \delta Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T_0\alpha^{ij} \\ T_0\tilde{\alpha}^{ij} & T_0\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\frac{1}{T_0}\nabla_j \delta T \end{pmatrix}$$

Find $\sigma = \sigma_Q - \frac{1}{i\omega}\frac{n_0}{\varepsilon_0 + P_0}$, where n_0 is the number density and

$$\sigma_Q = \tau_R \frac{e^{\mu_0/T_0}}{12}\frac{T_0^{d-1}}{\pi^{d-1}}$$

s_0 … entropy density

n_0 … number density

- All other coefficients related by the Ward identities

$$(\mu_0\sigma + T_0\alpha)i\omega = -n_0 \qquad (\bar{\kappa} + \mu_0\alpha)i\omega = -s_0$$

- Satisfies the Onsager relations $\alpha = \tilde{\alpha}$

Thermoelectric coefficients

Hartnoll, Kovtun, Müller, Sachdev 2007

$$\begin{pmatrix} \delta J^i \\ \delta Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T_0\alpha^{ij} \\ T_0\tilde{\alpha}^{ij} & T_0\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\frac{1}{T_0}\nabla_j \delta T \end{pmatrix}$$

With momentum breaking

Find $\sigma = \sigma_Q - \frac{1}{i\omega - \Gamma}\frac{n_0}{\varepsilon_0 + P_0}$, where

$$\sigma_Q = \tau_R \frac{e^{\mu_0/T_0}}{12}\frac{T_0^{d-1}}{\pi^{d-1}}$$

s_0 … entropy density

n_0 … number density

- All other coefficients related by the Ward identities

$$(\mu_0\sigma + T_0\alpha)(i\omega - \Gamma) = -n_0 \qquad (\bar{\kappa} + \mu_0\alpha)(i\omega - \Gamma) = -s_0$$

- Satisfies the Onsager relations $\alpha = \tilde{\alpha}$

Outlook: beyond the RTA

Outlook

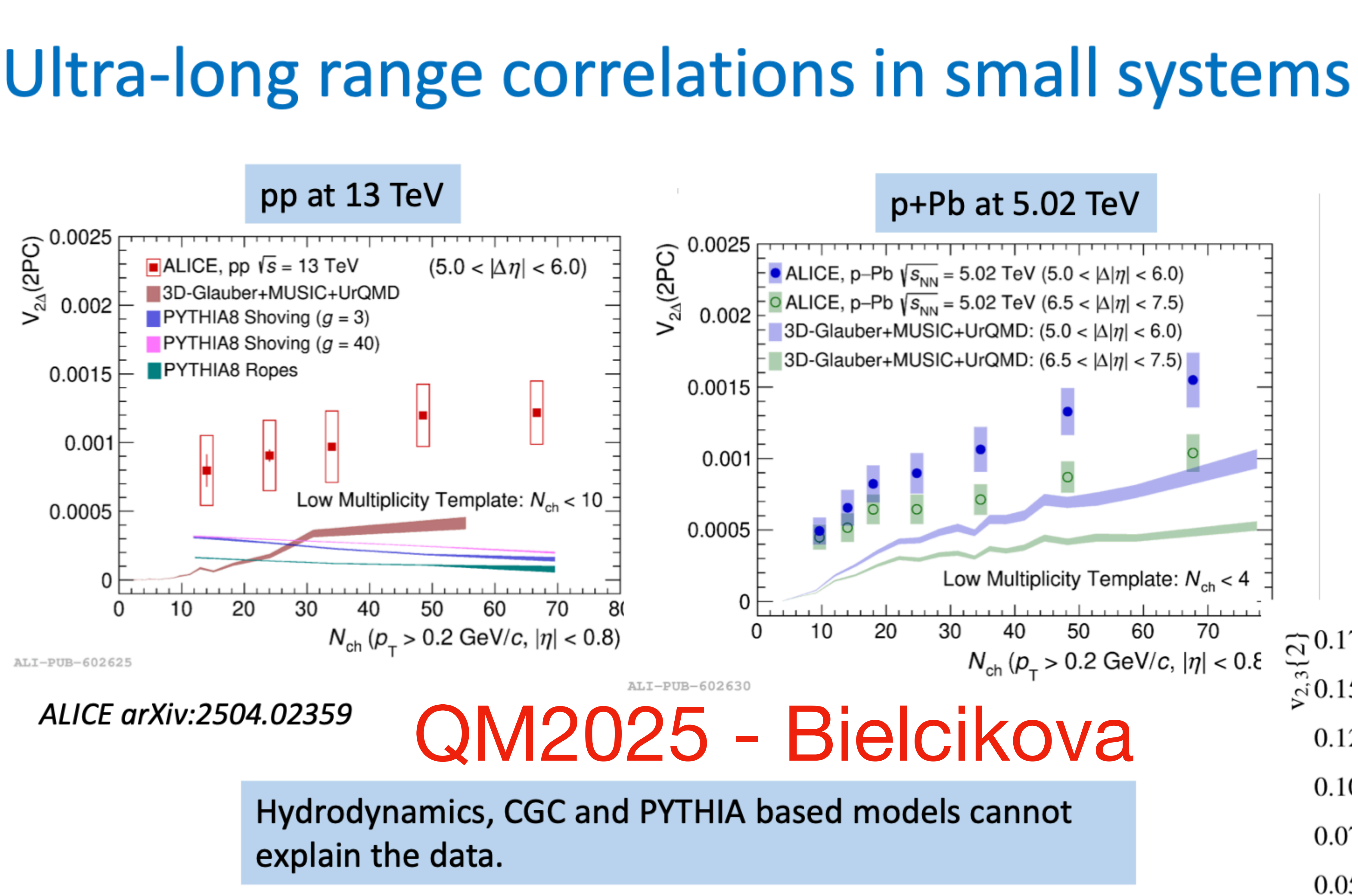
- A century on, kinetic theory is a key workhorse in QGP physics (e.g. KøMPøST, Cooper-Frye formula in freeze-out), plasma physics and other weak coupling settings

- However, starting point is already a truncation from the full Born-Bogoliubov-Kirkwood-Green-Yvon (BBGKY) hierarch

- Of course, this is done for simplifying rea

- However, what if we keep $f_2, f_3, \ldots$?

- Can we maybe say something about long-range correlations?



BBGKY hierarchy

- N-particle distribution function $f_N = f_N(t, x_1, \ldots, x_N, p_1, \ldots, p_N)$
- Evolution is given by Liouville equation

$$\mathscr{L}_N f_N = \partial_t f_N + \{H_N, f\} = 0$$

$$\{A, B\} = \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial x^i}\frac{\partial A}{\partial p_i}$$

- N-particle Hamiltonian is e.g.

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(r_i) + \sum_{i<j} U(r_i - r_j)$$

- Exact, but not practical!

BBGKY hierarchy

- Introducing reduced distribution function

$$f_{n-1} \sim \int d^3x_n d^3p_n f_n$$

- Recast Liouville equation as tower of equations

$$\partial_t f_n - \{f_n, H_n\} = \sum_{i=1}^{n} \int d^3 r_{n+1} d^3 p_{n+1} \frac{\partial U}{\partial \vec{r}} \cdot \frac{\partial f_{n+1}}{\partial \vec{p}}$$

$$\partial_t f_n + \{f_n, H_n\} = C[f_{n+1}]$$

- Still exact! Still unmanageable! Time to truncate…

BBGKY hierarchy truncation

$$\partial_t f_n - \{f_n, H_n\} = C[f_{n+1}]$$

- Typical truncation at first level - stosszahlansatz/molecular chaos

$$C[f_2] \approx C[f_1 f_1]$$

- Key assumption: uncorrelated particles prior to collision

$$C[f_1 f_1] = C_{1\to 2} + C_{2\to 2} + \dots \text{ or the RTA: } C[f_1 f_1] = -\frac{f_1 - f_1^{eq}}{\tau_R}$$

- What is effect of higher levels? Systems with $\tau_R > \tau_C > \dots$

Beyond the RTA

$$\mathscr{L}_1 f_1 = C[f_2], \quad \mathscr{L}_2 f_2 = C[f_3], \ldots$$

- Decompose into uncorrelated and correlated parts

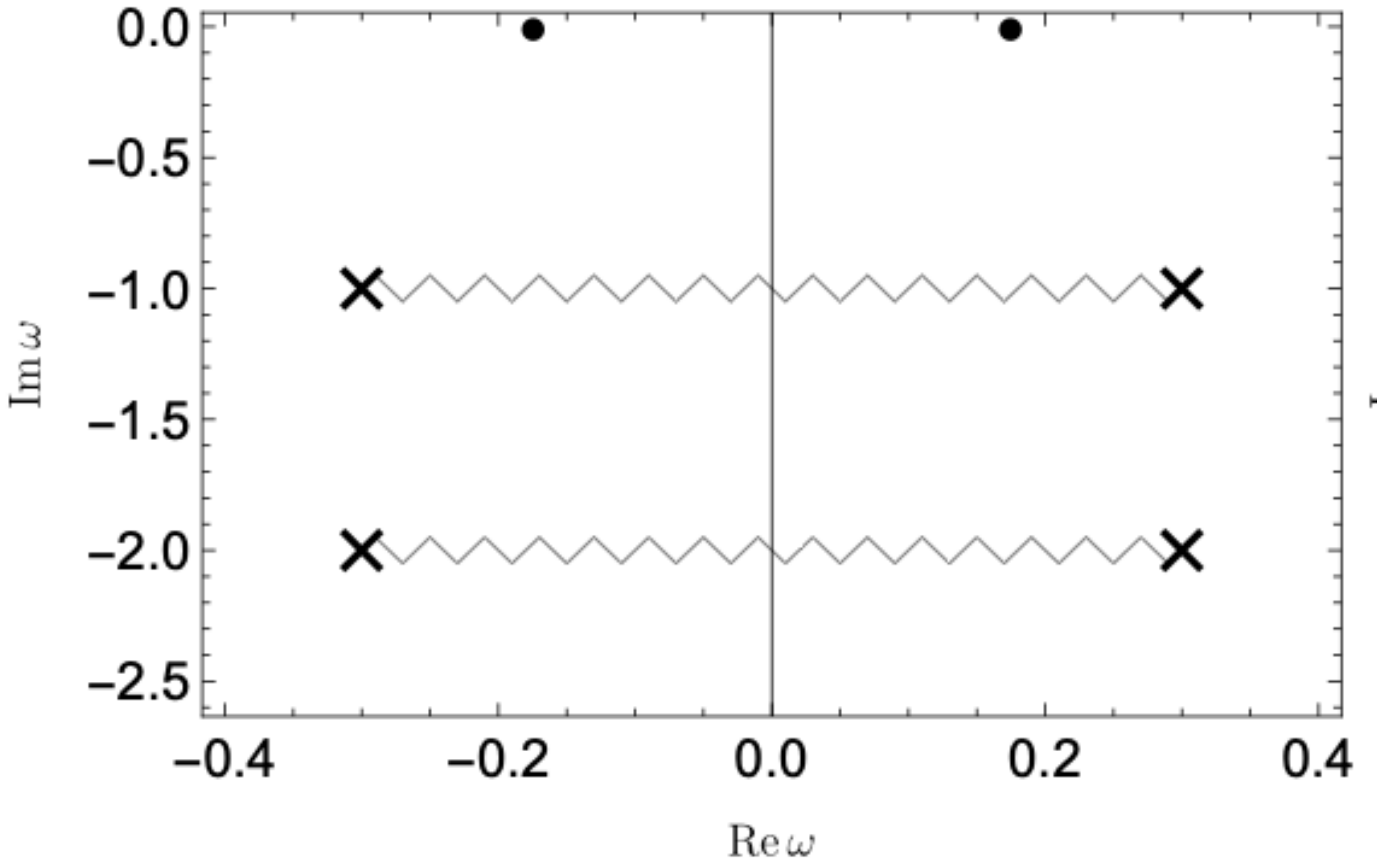
$$f_2 = f_1 f_1 + g_{12}$$

- Approximate with RTA-like term, use long range potential

$$\mathscr{L}_1 f_1 = -\frac{f_1 - f_1^{(eq)}}{\tau_R} + C[g_{12}],$$

$$\mathscr{L}_2 g_{12} = -\frac{g_{12} - g_{12}^{(eq)}}{\tau_C} + C[f_3], \ldots$$

- Analytic access to computed correlators, tower of branch cuts…



Sound channel - $G^{00,00}$

$$\omega = \pm\frac{1}{\sqrt{3}}k - i\left(\frac{2\tau_R}{15} - \hat{\sigma}\frac{\tau_R\tau_C T_0^2}{10\pi^2}\right)k^2 + \mathcal{O}(k^3)$$

Summary

Summary

- Have complete analytical set of correlators for $T, \mu, \Gamma \neq 0$
- Direct access to dispersion relations, transport coefficients
- Future directions: going beyond the RTA, scale dependent $\tau_R(p)$ a la Kurkela-Wiedemann 2017, including magnetic fields, massive particles, multiple species, long time tails …

Backup

Relating sources of response to perturbations

- How do we get from $(\delta g_{\mu\nu}, \delta A_\mu)$ to $(\delta E_\mu, \delta T)$?
- Under diffeos $\delta g'_{\mu\nu} = \delta g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.$ $\quad A'_\mu = A_\mu + A_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu A_\mu.$
- Choose $\xi_\mu = -\frac{1}{i\bar{\omega}} \frac{\delta T}{T_0^3} \delta^{\bar{t}}{}_\mu,$
- Leads to $\delta g'_{tt} = 0, \quad \delta g'_{tj} = -\frac{1}{i\omega} \frac{\partial_j \delta T}{T_0}, \quad \delta A'_j = \frac{E_j}{i\omega} - \mu_0 \frac{1}{i\omega} \frac{\partial_j \delta T}{T_0}.$
- Finally, the thermoelectric matrix reads

$$\begin{pmatrix} \delta J^i \\ \delta Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T_0 \alpha^{ij} \\ T_0 \tilde{\alpha}^{ij} & T_0 \bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\frac{1}{T_0} \nabla_j \delta T \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \delta J^i \\ \delta Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & \alpha^{ij} T_0 \\ \tilde{\alpha}^{ij} T_0 & \bar{\kappa}^{ij} T_0 \end{pmatrix} \begin{pmatrix} i\omega \delta A'_j + i\omega \mu_0 \delta g'_{tj} \\ i\omega \delta g'_{tj} \end{pmatrix}$$

Energy-energy correlator

The stress-energy tensor correlators are given in the longitudinal (sound, or spin-0) channel by

$$\frac{G_T^{00,00}}{3(\varepsilon_0 + P_0)} = \frac{A + B\hat{\sigma}}{C + D\hat{\sigma}}, \tag{A4}$$

where

$$A = 4k^3\tau_R^2 + k^2 L\tau_R(\tau_R\omega + 2i) + 6ik\tau_R\omega + 3iL\omega(\tau_R\omega + i), \tag{A5}$$

$$\begin{aligned}
B = {} & iLL_2(\tau_R - \tau_C)(\tau_C\omega + i)\left(\tau_R^2\omega\left(k^2\tau_C^2 + 3(\tau_C\omega + i)^2\right) + i\tau_R\left(k^2\tau_C^2 + 3\tau_C^2\omega^2 + 6i\tau_C\omega - 6\right) + 3i\tau_C\right) \\
& - 2kL\tau_C\Big(\tau_R^3\omega\left(2ik^2\tau_C^2 - 3i(\tau_C\omega + i)^2\right) + \tau_R^2\left(k^2\tau_C^2(-5 - i\tau_C\omega) + 3i\left(2\tau_C^3\omega^3 + 5i\tau_C^2\omega^2 + \tau_C\omega + 2i\right)\right) \\
& + \tau_R\tau_C\left(4k^2\tau_C^2 + 3\tau_C^2\omega^2 - 18i\tau_C\omega + 9\right) + 3\tau_C^2(-1 + 3i\tau_C\omega)\Big) \\
& + 2kL_2\tau_R(1 - i\tau_C\omega)\left(-2\tau_R^2\left(k^2\tau_C^2 + 3(\tau_C\omega + i)^2\right) + \tau_R\tau_C\left(k^2\tau_C^2 + 3\tau_C^2\omega^2 + 9i\tau_C\omega - 9\right) + 3\tau_C^2\right) \\
& - 4k^2\tau_R\tau_C(\tau_R - \tau_C)\left(2\tau_R\left(2ik^2\tau_C^2 - 3i(\tau_C\omega + i)^2\right) - 3\tau_C(3\tau_C\omega + i)\right),
\end{aligned} \tag{A6}$$

$$C = 4k^3\tau_R^2 + 2ik^2L\tau_R + 12ik\tau_R\omega + 6iL\omega(\tau_R\omega + i), \tag{A7}$$

$$\begin{aligned}
D = {} & 2(\tau_C\omega + i)\Big(2ikL\tau_C\left(\tau_R^2\left(k^2\tau_C^2 + 3\tau_C^2\omega^2 + 3i\tau_C\omega + 3\right) - 6\tau_R\tau_C + 3\tau_C^2\right) \\
& - 2ikL_2\tau_R\left(\tau_R^2\left(k^2\tau_C^2 + 3(\tau_C\omega + i)^2\right) + 3\tau_R\tau_C(2 - i\tau_C\omega) - 3\tau_C^2\right) \\
& + 12k^2\tau_R\tau_C(\tau_R - \tau_C)(-i\tau_R\tau_C\omega + \tau_R - \tau_C) - 3LL_2(\tau_R - \tau_C)^2\Big) \\
& + 4ik\tau_C^3(\tau_R - \tau_C)\left(iL\left(k^2\tau_R + 3\omega(\tau_R\omega + i)\right) + 2k\tau_R\left(k^2\tau_R + 3i\omega\right)\right),
\end{aligned} \tag{A8}$$