On holographic prescriptions for Schwinger-Keldysh effective field theories

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Based on ongoing work w/ **B. Withers** (U. Southampton)

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Foundations and Applications of Relativistic Hydrodynamics
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Motivation

Why SK EFTs?

Limitations of classical hydrodynamics:

- Classical formulation at the level of **equations of motion**: how to write **actions for dissipative systems**?
- Beyond linear response: how to systematically include **interactions** between hydro modes, and with thermal bath? $\Longrightarrow$ long-time tails, stochastic hydrodynamics, ...
- Direct access to G^A, G^S **and higher point TO correlators**?
- **Derivation** of phenomenological restrictions?

Motivation

Overcome by **Schwinger-Keldysh** construction!

- Systematic setup for **non-equilibrium** QFT [Kamenev]
- **Action principle** for hydrodynamics on **SK closed time path contour** [Crossley,Glorioso,Liu; Haehl,Loganayagam,Rangamani; Jensen,Pinzani-Fokeeva,Yarom]
- **Fundamental symmetry principles** $\Longrightarrow$ interactions and fluctuations, implies Onsager relations and 2nd law, locality [Delacrétaz,Goutéraux,VZ], ...
- Beyond linear response, “stochastic” coefficients **invisible to classical constitutive relations** [Jain,Kovtun]

Motivation

Why holography?

Explicit computations in strongly coupled models:

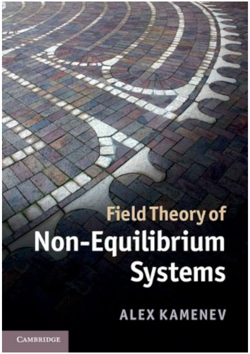
- **Analytic results** (e.g. derivative expansion)
- **Universal bounds** for transport coefficients (eg η/s [Kovtun,Son,Starinets], D [Blake])
- Expect generic behavior $\Rightarrow$ "accidents" help uncover **hidden structures**
- Easier **numerics**

Goal: Understand **gravitational** dual of SK QFTs

- **Complex saddles** in quantum gravity/holography [Witten]

Take scalar field (model A - critical relaxation dynamics [Hohenberg,Halperin]) $\Longrightarrow$ uncover structures relevant for hydrodynamics

Schwinger-Keldysh QFT

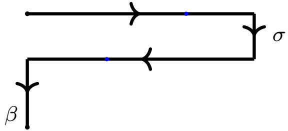


- Time evolution of density matrices

$$\rho(t) = U(t)\rho_0 U^\dagger(t) \quad \Longrightarrow \quad \langle AB\ldots\rangle_{\rho_0} = \mathrm{Tr}(\rho_0 U^\dagger AUB\ldots)$$

$\Longrightarrow$ **Complex time contours**

In - in



Thermal

- General n-point PO correlators from path integrals

$$e^{W_C[s]} = \int_{b.c.} D\varphi e^{i\int_C \mathcal{L}[s]} = \int D\varphi_1 D\varphi_2 e^{iS[\varphi_1,s_1]-iS[\varphi_2,s_2]} \equiv e^{W[s_1,s_2]}$$

- **Various initial states/correlators**: Vacuum-vacuum, in-in, thermal,...

Motivation: Schwinger-Keldysh QFT

Features of SK path integrals:

- **Doubling** of the fields $\Rightarrow$ **physical** $\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2)$, **stochastic** $\varphi_a = \varphi_1 - \varphi_2$
- **Unitarity constraints**
  - $W[s_r, s_a = 0] = 0$
  - $W^*[s_r, s_a] = W[s_r, -s_a]$
  - $\mathrm{Re} W \leq 0$
- For **thermal states**, periodicity in imaginary time $\Rightarrow$ **KMS condition**

$$W[s_1(t), s_2(t)] = W[\Theta s_1(t - i\theta), \Theta s_2(t + i(\beta - \theta))]$$

$W[s_r, s_a] \Longrightarrow$ Direct access to $G^{R,A,S}, \ldots$

$$G^R = G_{ra} \left(= -i \frac{\delta^2 W}{\delta s_a \delta s_r} \right) , \qquad G^A = G_{ar} \,, \qquad G^K = G_{rr} \,, \qquad \ldots$$

Schwinger-Keldysh EFT

Constructing the **effective action** S_{EFT}

$$e^{W[s_r,s_a]} \simeq \int D\phi_r D\phi_a e^{iS_{EFT}[\phi_r,s_r;\phi_a,s_a]}$$

Systematic procedure: [Liu,Glorioso]

- (i) Identify the low-energy dynamical **dofs** (associated to the **symmetries**), and write them down in a **derivative expansion**
  - ϕ_1, ϕ_2 are **coupled**
  - S_{EFT} **complex** $\Leftrightarrow$ **dissipation**
- (ii) Impose **unitarity constraints**
  - $S_{EFT}[\phi_r, s_r; \phi_a = s_a = 0] = 0$
  - $S^*_{EFT}[\phi_r, s_r; \phi_a, s_a] = -S_{EFT}[\phi_r, s_r; -\phi_a, -s_a]$
  - $\mathrm{Im} S_{EFT} \geq 0$

- (iii) Impose **dynamical KMS** condition $S_{EFT}[\phi_r, s_r; \phi_a, s_a] = S_{EFT}[\tilde{\phi}_r, \tilde{s}_r; \tilde{\phi}_a, \tilde{s}_a]$
  [Sieberer,Chiocchetta,Gambassi,Täuber,Diehl; Glorioso,Crossley,Liu]

$$\tilde{s}_1 = \Theta s_1(t - i\theta)\,, \qquad \tilde{s}_2 = \Theta s_2(t + i(\beta - \theta))$$
$$\tilde{\phi}_1 = \Theta \phi_1(t - i\theta)\,, \qquad \tilde{\phi}_2 = \Theta \phi_2(t + i(\beta - \theta))$$

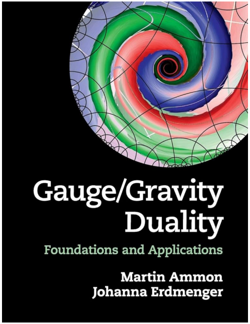
  In the classical limit

$$\tilde{s}_r = \Theta s_r\,, \qquad \tilde{s}_a = \Theta s_a + i\Theta\beta\partial_0 s_r$$
$$\tilde{\phi}_r = \Theta \phi_r\,, \qquad \tilde{\phi}_a = \Theta \phi_a + i\Theta\beta\partial_0 \phi_r$$

Ward identity for this "thermal symmetry" $\Longrightarrow$ **Flucuation-Dissipation relations**

Universal constraints and symmetries of **hydrodynamics**

Holography


Gauge/Gravity Duality
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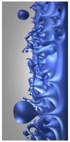
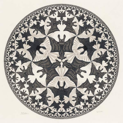
Holography





(Large N) **strongly-coupled QFT**$_d$ $\Longleftrightarrow$ **weakly-coupled GR**$_{(d+1)}$ [Maldacena]

$$e^{W_{QFT}[\varphi_s]} = e^{iS_{on-shell}[\varphi \to \varphi_s]}$$

- AdS spacetime

$$ds^2 = \frac{dr^2}{r^2} + r^2 \left(-dt^2 + \delta_{ij} dx^i dx^j \right)$$

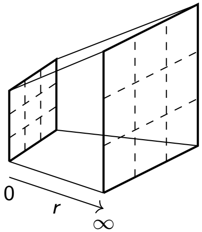
- Near boundary $r \to \infty$ expansion

$$\varphi = r^{\Delta - D - 1} \varphi_s + \cdots + r^{-\Delta} \varphi_V + \cdots$$

corresponds to $\delta S_{QFT} \sim \int \varphi_s \varphi_V$.

Holography

- **AdS-black hole thermodynamics $\Longleftrightarrow$ QFT thermodynamics**

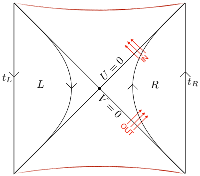


Schwarzschild-AdS$_{d+1}$

$$ds^2 = -r^2 f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 d\Sigma^2$$

$$f(r) = 1 - r_h^d / r^d$$

- Real time prescription for G^R: **infalling** boundary conditions at the horizon [Son,Starinets; Herzog,Son]

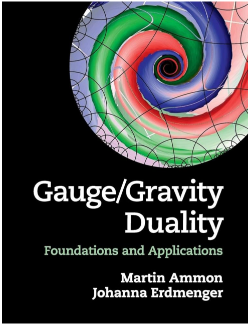


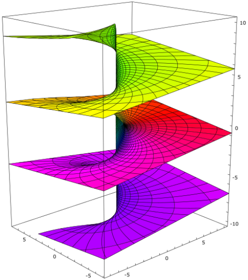
- **Horizon** encodes **dissipation** $\Rightarrow$ Hydro **transport coefficients** from BH horizon (realization of membrane paradigm)! [Iqbal,Liu]

Holographic dictionary

Gravitational Theory	Field Theory
Gauge symmetry	Global symmetry
Bulk field ($g_{\mu\nu}$, A_μ, φ)	Boundary operator (T^{ab}, J^a, $\mathcal{O}$)
• Leading boundary behavior	• Source
• Subleading boundary behavior	• 1-point function
Pure AdS spacetime	Vacuum state
Black hole with temperature T	Thermal state at temperature T
Euclidean on-shell action	Free energy
Boundary/Deep interior	UV/IR
Quasinormal modes with $\omega(k)$	Poles of retarded Green's fn at $\omega(k)$

Glorioso-Crossley-Liu (GCL) prescription





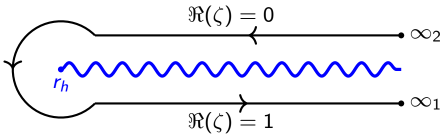
GCL prescription

- Schwarzschild-AdS$_{d+1}$

$$ds^2 = -r^2 f(r) dv^2 + 2 dr dv + r^2 d\Sigma^2 \,, \qquad f(r) = 1 - \frac{r_h^d}{r^d}$$

GCL prescription for SK contour: [Glorioso,Crossley,Liu]

- **Complexify radial coordinate** r and **analytically continue** around the horizon



- Mock tortoise coordinate with branch cut at r_h

$$\frac{d\zeta}{dr} = \frac{2}{i\beta\, r^2 f(r)}$$

GCL prescription

Take a (massless) **probe scalar**

$$S = -\frac{1}{2}\int drdx\sqrt{-g}\,(\partial\phi)^2 + V(\phi)$$

- General solution on **GCL contour** [Jana,Loganayagam,Rangamani; Chakrabarty et al.]

$$\Phi(\zeta,\omega) = c^R(\omega)G^R(\zeta,\omega) + c^A(\omega)G^A(\zeta,\omega)e^{-\beta\omega\zeta}$$

  where G^R, G^A are normalized at boundary and solve

$$\partial_r\left(\sqrt{g_\Sigma}\Pi\right) + \sqrt{g_\Sigma}\left(\partial_r\partial_v - k^2\right)\Phi = 0$$

  with

$$\Pi = [f(r)\partial_r + \partial_v]\,\Phi \equiv \mathbb{D}_+\Phi$$

- $G^R(\omega)$: **ingoing** (regular) at horizon
- $G^A(\omega) = G^R(-\omega)$

Can be found analytically in derivative expansion

GCL prescription

- Boundary sources

$$\Phi|_{\zeta=0,1} = s_{2,1}(\omega)$$

- Find

$$\boxed{\Phi(\zeta,\omega) = -s_F(\omega)\, G^R(\zeta,\omega) + s_P(\omega)\, G^A(\zeta,\omega) e^{\beta\omega(1-\zeta)}}$$

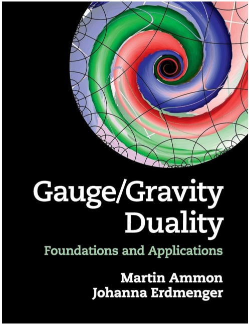
where

$$s_F = -\left[(1+n)s_1 - n\, s_2\right] , \qquad s_P = -n\,(s_1 - s_2)$$

$$n = \frac{1}{e^{\beta\omega} - 1}$$

- Obtain generating functional $W[s_r, s_a]$ by putting **bulk action on-shell**...

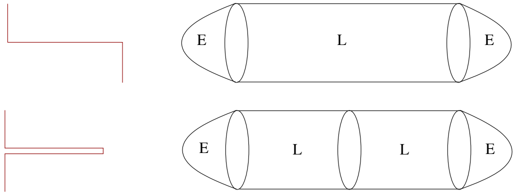
Skenderis - van Rees (SvR) prescription



SvR prescription

(i) **"Fill in"** QFT contour with **complexified/mixed-signature bulk** [Skenderis,van Rees]

- Real time segments $\Longleftrightarrow$ Lorentzian pieces
- Imaginary time segments $\Longleftrightarrow$ Euclidean pieces

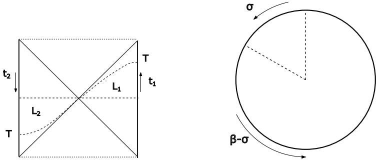


(ii) Impose **matching conditions** at **fixed-time** corners

- **Continuity** of fields ϕ and conjugate momenta $\pi_\phi \sim \partial_t \phi$

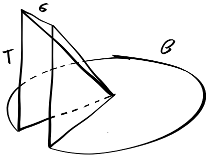
SvR construction

- For thermal state, patch together Schw-AdS [de Boer,Heller,Pinzani-Fokeeva]



$$ds_L^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Sigma^2\,, \qquad ds_E^2 = ds_L^2(t \to -i\tau)$$

- Bulk geometry:



SvR construction

General form of probe scalar

$$\Phi_m = \int \frac{d\omega}{2\pi} e^{-i\omega t} \left(c_m^R(\omega) G^R(\omega, r) + c_m^A(\omega) G^A(\omega, r) \right) \qquad m = 1, \sigma, 2, \beta$$

Matching conditions

$$\begin{aligned} \Phi_1(t = T) &= \Phi_\sigma(t = T) \\ \Phi_\sigma(t = T - i\sigma) &= \Phi_2(t = T - i\sigma) \\ \Phi_2(t = -i\sigma) &= \Phi_\beta(t = -i\sigma) \\ \Phi_\beta(t = -i\beta) &= \Phi_1(t = 0) \end{aligned}$$

Sources on Lorentzian boundaries

$$\begin{aligned} c_m^R(\omega) + c_m^A(\omega) &= s_m(\omega) \qquad && m = 1, 2 \\ c_m^R(\omega) + c_m^A(\omega) &= 0 \qquad && m = \sigma, \beta \end{aligned}$$

SvR construction

Matching at $t = T$ for $s_1 \sim \delta(t_s)$:

$$\int \frac{d\omega}{2\pi} e^{-i\omega T} \left[c_1^R G^R + \left(e^{i\omega t_s} - c_1^R \right) G^A \right] = \int \frac{d\omega}{2\pi} e^{-i\omega T} c_\sigma^R \left(G^R - G^A \right)$$

Closing the contour integral in the LHP, inhomogeneous term drops out $\Rightarrow$

$$c_1^R = c_\sigma^R$$

- Similarly,

$$c_1^R = c_\sigma^R = c_2^R$$
$$c_2^A = c_\beta^A = e^{\beta\omega} c_1^A$$

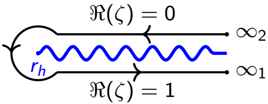
SvR construction

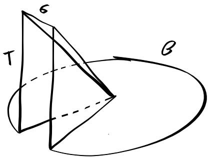
Full solution

$$\boxed{\begin{aligned} \Phi_1 &= -s_F\, G^R + s_P\, G^A \\ \Phi_2 &= e^{-\sigma\omega}\left[-s_F\, G^R + s_P\, G^A e^{\beta\omega}\right] \\ \Phi_\sigma &= -s_F\left[G^R - G^A\right] \\ \Phi_\beta &= -e^{\beta\omega} s_P\left[G^R - G^A\right] \end{aligned}}$$

- Reproduces **GCL on Lorentzian sheets**
  - Similar agreement for (second order) diffusive hydro [de Boer,Heller,Pinzani-Fokeeva]
- For $s_1 = s_2$ recover **ingoing boundary** conditions at horizon [Skenderis,van Rees; van Rees] proposed in [Son,Starinets; Herzog,Son]
- For $\sigma \neq 0$ get Son-Teaney analytic continuation proposal [Son,Teaney] - agrees with **modular flowed** GCL [Sivakumar]

GCL - SvR

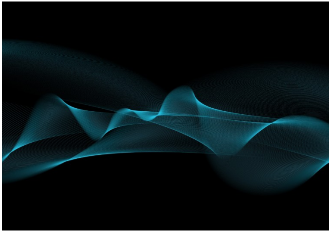




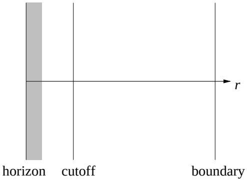
Interpretation: Structure is **fixed by thermal/KMS conditions**, but

- In SvR, QFT contour is periodic **by construction**
- In GCL, KMS is implemented by a **bulk (IR) condition**!

SK membrane constraints



SK membrane constraints



Semi-holography and membrane paradigm [Nickel,Son; Iqbal,Liu; Faulkner,Liu,Rangamani]

- Split IR & UV bulk action:

$$S[\Phi] = S_h[\phi] + S_{UV}[\phi, s]\,, \qquad \phi \equiv \Phi|_h$$

Extremize at the end

$$\frac{\delta S_h}{\delta \phi} + \frac{\delta S_{UV}}{\delta \phi} = 0$$

Horizon fields $\Longleftrightarrow$ low-energy dofs of QFT

- (eg Wilson lines of gauge transformations $\Longleftrightarrow$ Goldstones for $U(1)$ hydro)

SK membrane constraints

Membrane action

$$S_h = -\int_\epsilon dx\sqrt{-\gamma}\Pi(\epsilon,x)\phi(\epsilon,x)\,, \qquad \sqrt{-\gamma}\Pi \equiv \frac{\delta L}{\delta\partial_r\phi}$$

Near horizon solution

$$\phi(\omega) = A(\omega)\epsilon^{\frac{i\beta\omega}{2\pi}} + B(\omega)\,, \qquad \Pi = -i\omega\left(B - A\epsilon^{\frac{i\beta\omega}{2\pi}}\right)$$

- 2 coefficients, fixed by bdy source + horizon condition (e.g. infalling $\Rightarrow A = 0$)

Implement SK EFT constraints on membrane action!

- 2 Lorentzian sheets $\Longleftrightarrow$ **2 copies** of scalar field

SK membrane constraints

Effective Lagrangian

$$-\mathcal{L}_\epsilon = \phi_1(-\omega)\Pi_1(\omega) - \phi_2(-\omega)\Pi_2(\omega) = \phi_r\Pi_a + \phi_a\Pi_r$$

From the near horizon solutions

$$\mathcal{L}_\epsilon = \frac{i\omega\left[-2A_rB_r\phi_a^2 + 2A_aB_a\phi_r^2 - (A_rB_a + A_aB_r)\,\phi_a\phi_r + (A_rB_a + A_aB_r)\,\phi_a\phi_r\right]}{A_aB_r - A_rB_a}$$

Implement **SK constraints**

- (i) $S_{EFT}[\phi_a = 0] = 0$
- (ii) $\mathrm{Im}S_{EFT} \geq 0$

and (iii) **dynamical KMS** [Sieberer,Chiocchetta,Gambassi,Täuber,Diehl]

$$\phi_1(\omega) \to e^{\beta\omega/2}\phi_1(-\omega)$$
$$\phi_2(\omega) \to e^{-\beta\omega/2}\phi_2(-\omega)$$

SK membrane constraints

- (i) $\Rightarrow A_a = 0$ or $B_a = 0$
- (iii) $\Rightarrow B_r = -\frac{1}{2} B_a \coth\left[\frac{\beta\omega}{2}\right]$ or $A_r = -\frac{1}{2} A_a \coth\left[\frac{\beta\omega}{2}\right]$
- (ii) $\Rightarrow$

$$\boxed{B_a = 0\,, \qquad A_r = -\frac{1}{2} A_a \coth\left[\frac{\beta\omega}{2}\right]}$$

Finally

$$\mathcal{L}_\epsilon = i\omega \left[\phi_a(-\omega)\phi_r(\omega) - \phi_r(-\omega)\phi_a(-\omega) + \coth\left[\frac{\beta\omega}{2}\right]\phi_a(-\omega)\phi_a(\omega)\right]$$

Note

- Equivalent to **near-horizon GCL and SvR**: $B_2 = B_1\,,\ A_2 = e^{\beta\omega} A_1$
- Other 2 conditions fixed by **bdy sources** s_1, s_2
- **Classical limit**: $\mathcal{L}_\epsilon \sim \phi_a \dot{\phi}_r - \frac{i}{\beta}\phi_a^2$

SK membrane constraints

To do: move the membrane **towards AdS boundary**

- Connect w/ sources & obtain **non-dissipative** S_{UV}
- Show that SK constraints are **preserved under (holographic) RG flow**

Expect [de Boer,Heller,Pinzani-Fokeeva]

$$\mathcal{L}^{SK} = \phi_a(-\omega) G^R \phi_r(\omega) + \phi_r(-\omega) G^A \phi_a(\omega) + \frac{1}{2}\left(G^R - G^A\right) \coth\left[\frac{\beta\omega}{2}\right] \phi_a(-\omega)\phi_a(\omega)$$

Outlook: Beyond global equilibrium

Outlook: Beyond global equilibrium

GCL is a bulk condition - extend to any **dynamical horizon**! [Glorioso,Crossley,Liu]

$$ds^2 = -f(r,x)dv^2 + 2drdv + \lambda_{ij}(r,x)dx^i dx^j$$

But different notions of horizons ... $\Rightarrow$ IR membrane might capture dissipative effects

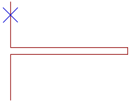
Does GCL define:

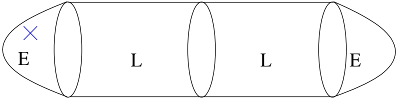
- **"Local equilibrium"** holographic state?
- **"Local dynamical KMS"** with $\beta \to \beta(x)$? [Knysh,Liu,Pinzani-Fokeeva]
- **"Quantum hydrodynamic"** EFTs w/ $\beta\partial \sim 1$? [Blake,Lee,Liu]

- Relation to general **Fluctuation-Dissipation relations** $G^K = G^R \star F - F \star G^A$ and driven systems
- **Fluctuation theorems**

Outlook: Beyond global equilibrium

In-in SvR is well defined at boundary - **excited states** by **euclidean** evolution
[Christodoulou,Skenderis; Pantelidou,Withers]



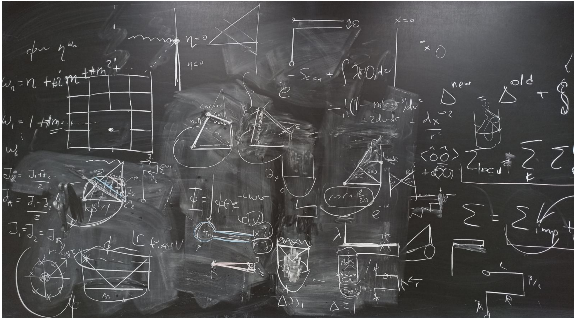


- How are they related to the GCL states?

Further explorations:

- Bulk **gauge field** and **metric** perturbations $\Rightarrow$ hydrodynamics
- Non-linear/higher-point functions [Loganayagam,Rangamani,Virrueta]
- Analytic results in **AdS$_2$/Fluid gravity**
- Understand at the level of the **geometry**

Thank You!



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