QGP, attractors and asymptotics

Michał Spaliński
University of Białystok &
National Centre for Nuclear Research

Galileo Galilei Institute
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Plan of this talk

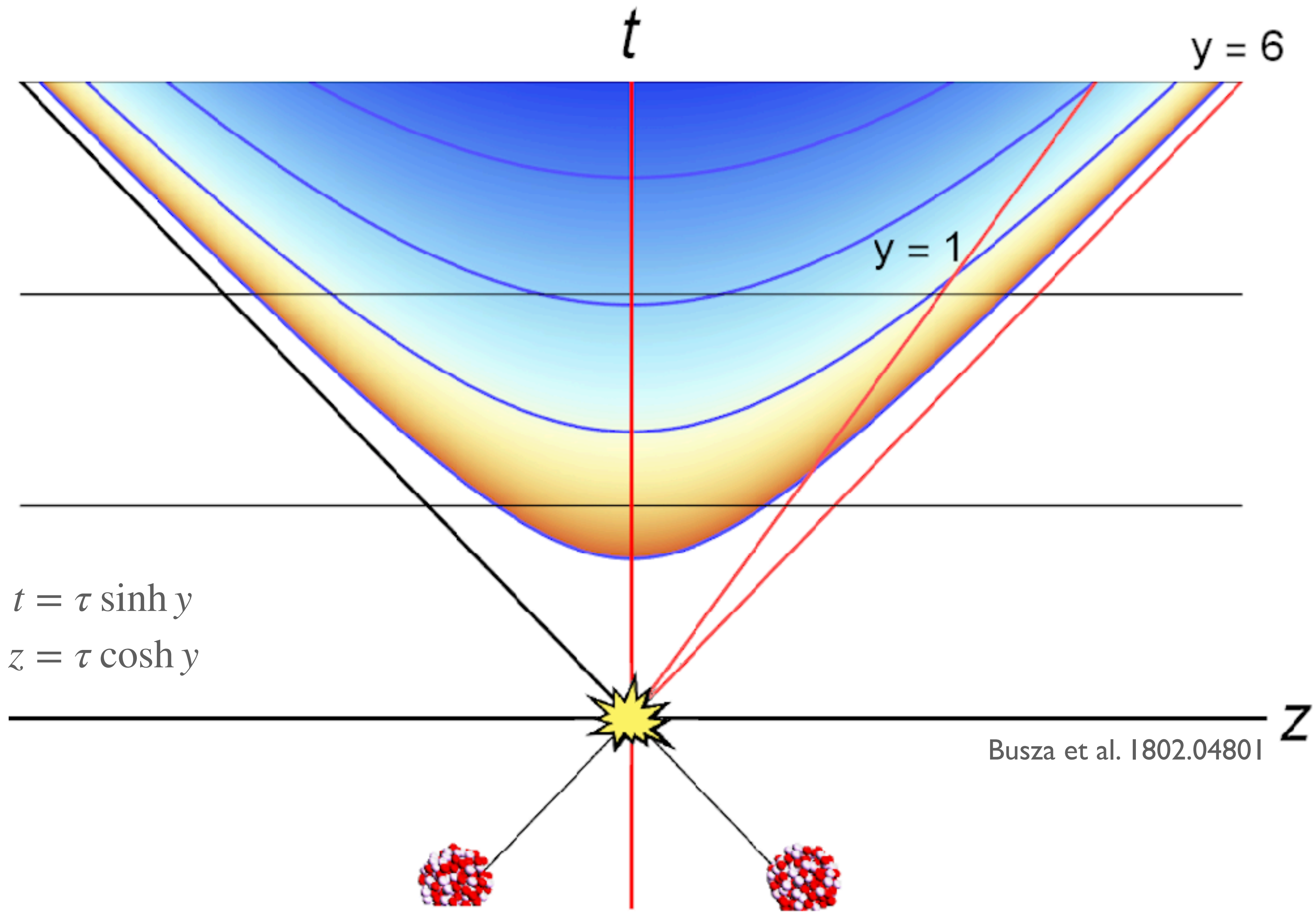
- The early-time puzzle in ultrarelativistic heavy ion collisions
- Pre-equilibrium attractors in Bjorken flow
    - Mueller-Israel-Stewart theory
    - Kinetic theory [Aniceto, MS, Noronha 2401.06750]
- Incorporating transverse dynamics [An, MS 2312.17237]

The early-time puzzle

The early-time puzzle

- The success of MIS-like models far from equilibrium is puzzling, since MIS is not QCD
- This suggests the emergence of some universality far from the hydrodynamic domain – a pre-hydrodynamic attractor
- The origin of this attractor is likely due to a confluence of factors
  - boost invariance (dynamics)
  - dominance of the longitudinal expansion (kinematics)

Energy-momentum in Bjorken flow



$$(T^\mu_\nu) = \mathrm{diag}(-\mathscr{E}(\tau), \mathscr{P}_L(\tau), \mathscr{P}_T(\tau), \mathscr{P}_T(\tau))$$

Conformal Bjorken flow

For conformal theories

$$\mathcal{E} = \mathcal{P}_L + 2\mathcal{P}_T \equiv 3\mathcal{P}$$

The energy-momentum tensor can thus be parameterised in terms of the energy density and the pressure anisotropy:

$$\mathcal{A} \equiv \frac{\mathcal{P}_T - \mathcal{P}_L}{\mathcal{P}}$$

This is a measure of distance from equilibrium.

We will also use the dimensionless evolution parameter

$$w \equiv \tau T \sim \tau/\tau_R, \qquad \mathcal{E}(\tau) \sim T(\tau)^4$$

Attractors in Bjorken flow

The conservation equation reads

$$\frac{d\log T}{d\log w} = \frac{\mathscr{A} - 6}{\mathscr{A} + 12}$$

Solutions depend on one integration constant.

The remaining initial data is contained in the pressure anisotropy.

Often the pressure anisotropy follows an attractor: a universal curve which coincides with the prediction of hydrodynamics at late times, but which “attracts" generic solutions already at early time.

Attractors describe a partial "loss of memory” of initial states.

Bjorken flow in MIS

MIS theory

We consider the simplest variant of conformal MIS:

$$u \cdot \nabla \mathscr{E} = -(\mathscr{E} + p)\nabla \cdot u + u^\nu \nabla^\mu \pi_{\mu\nu},$$

$$(\mathscr{E} + p) u \cdot \nabla u_\mu = -\Delta_{\mu\nu} \nabla^\nu p - \Delta_{\mu\nu} \nabla_\lambda \pi^{\nu\lambda},$$

$$\Delta_{\mu\alpha}\Delta_{\nu\beta} u \cdot \nabla \pi^{\alpha\beta} = -\left(1 + \frac{4}{3}\tau_\pi \nabla \cdot u\right)\pi_{\mu\nu} - 2\eta\sigma_{\mu\nu},$$

Equation of state and transport coefficients:

$$\mathscr{E} = \frac{1}{3}p = C_e T^4, \quad \eta = \frac{4}{3} C_e C_\eta T^3, \quad \tau_\pi = C_\tau T^{-1}$$

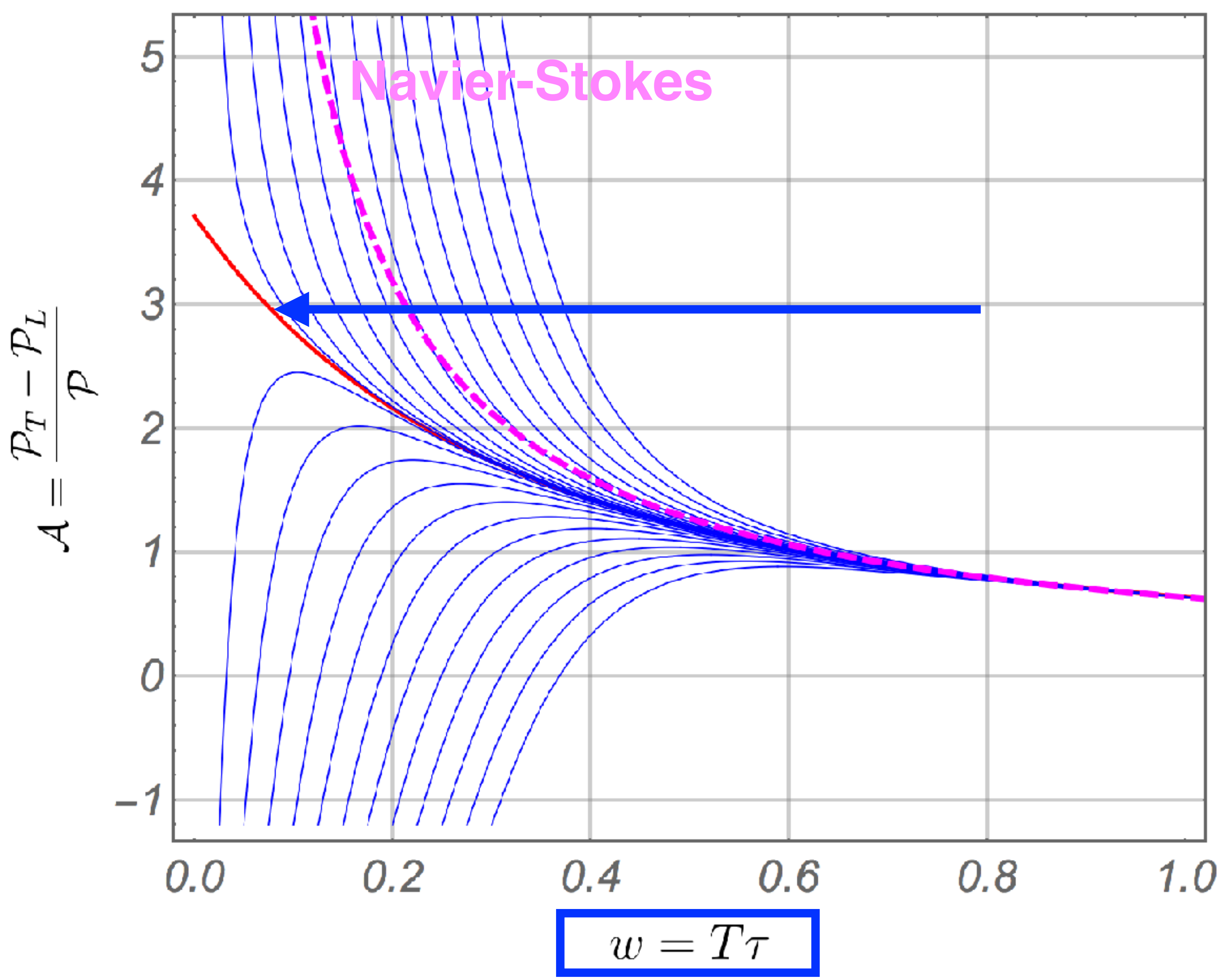
N=4 SYM values of the constants:

$$C_e = 8\pi^2/15, \qquad C_\eta = 1/4\pi, \qquad C_\tau = (2 - \ln 2)/2\pi$$

The Bjorken attractor in conformal MIS
by numerical integration

$$C_\tau \left(1 + \frac{\mathcal{A}}{12}\right) \mathcal{A}' + \frac{C_\tau}{3w}\mathcal{A}^2 = \frac{3}{2}\left(\frac{8C_\eta}{w} - \mathcal{A}\right)$$

The pressure anisotropy satisfies this first order ODE, where
$C_\eta \equiv \eta/s, \quad C_\tau \equiv \tau_R T$



An attractor connects the early, far-from-equilibrium domain to the hydrodynamic region at late times

There is a rapid reduction of complexity initially, followed by a period of more moderate loss of memory

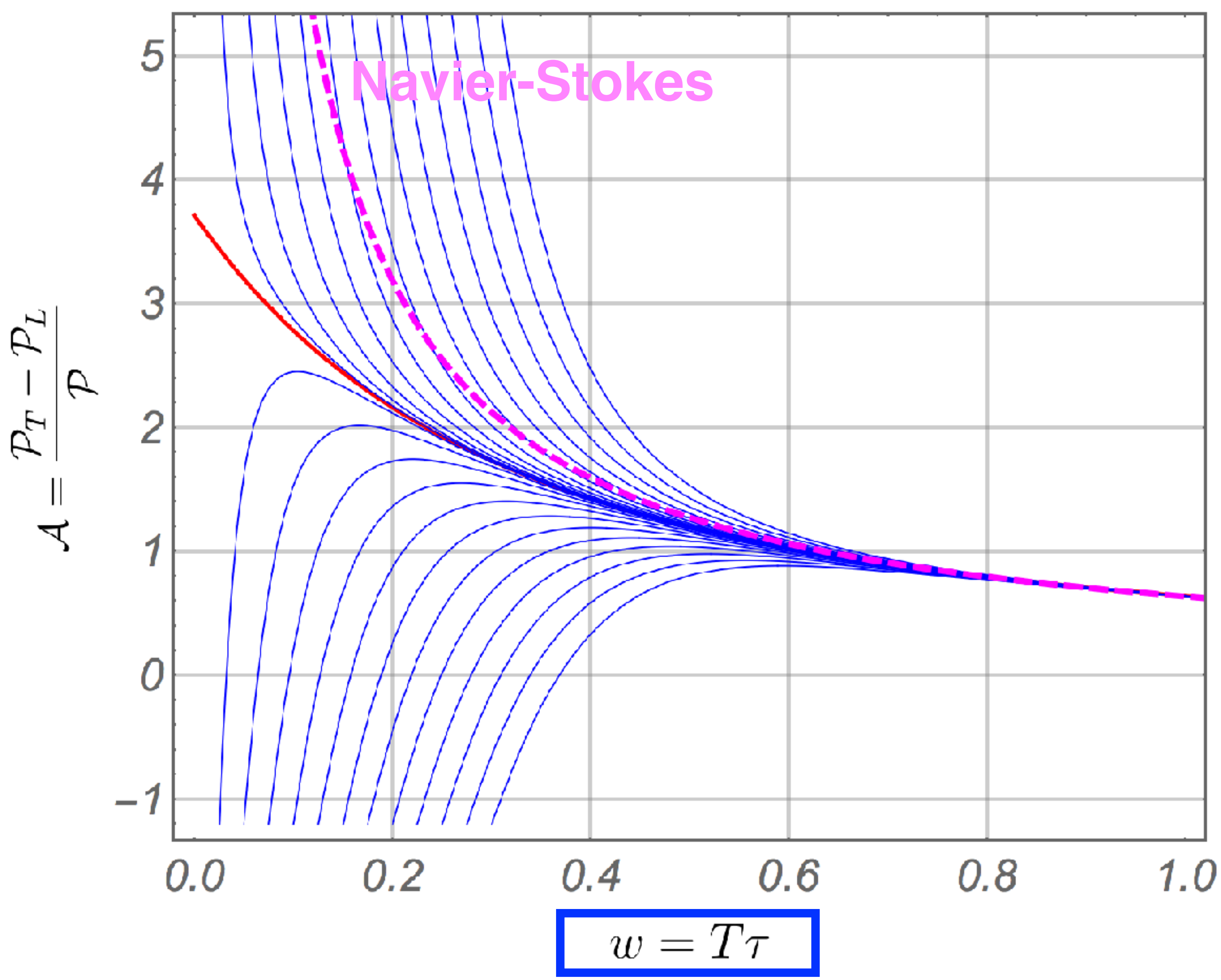
Solutions starting off the attractor reach its vicinity even if the pressure anisotropy is large so the system is still far from equilibrium.

The Bjorken attractor in conformal MIS

the late-time asymptotic view

$$\mathcal{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16C_\eta C_\tau}{3w^2}}_{\text{2nd order}} + \ldots = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\text{gradient expansion}}$$

The expansion coefficients do not depend on initial conditions

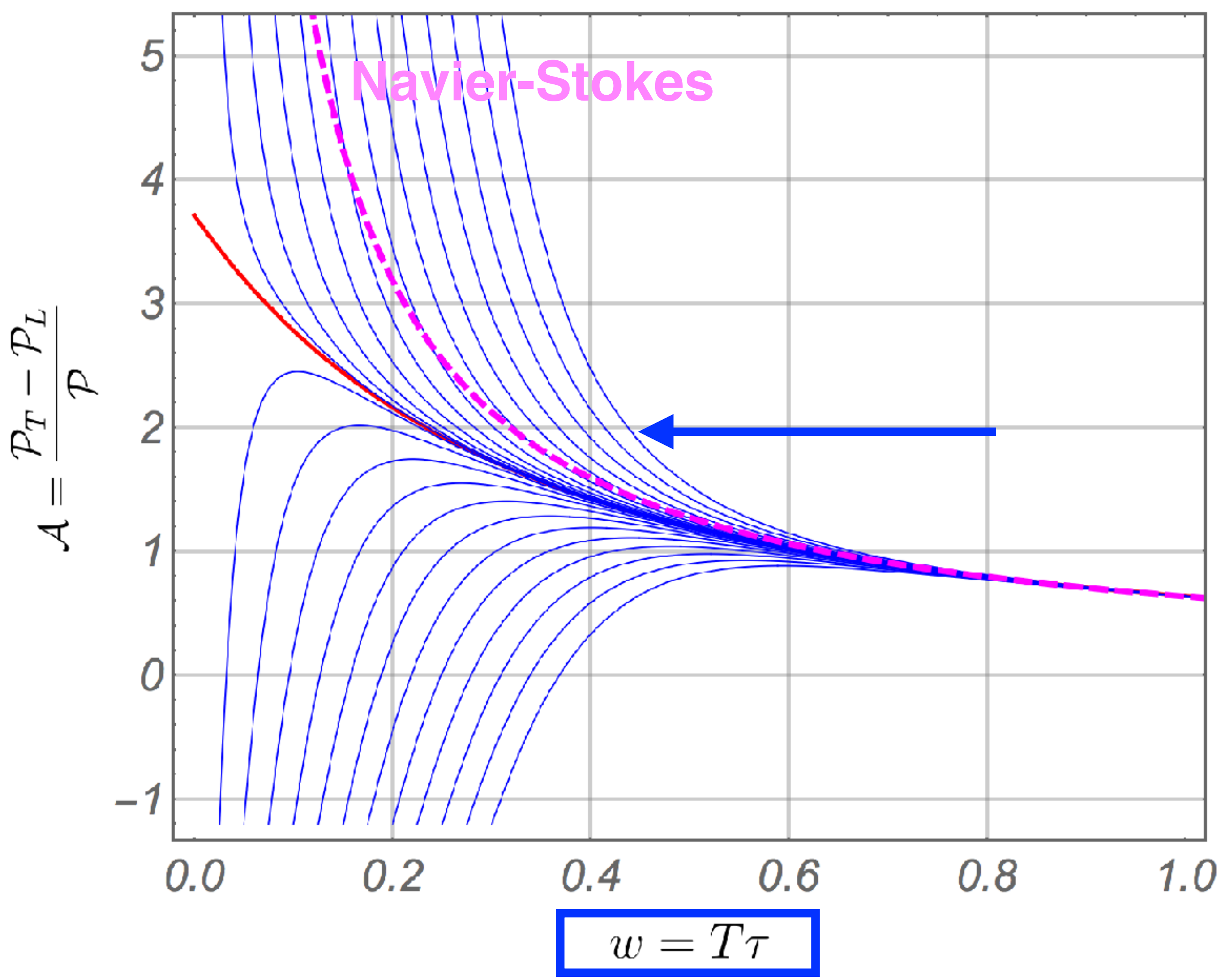


At asymptotically late times there is no memory of the initial conditions

The Bjorken attractor in conformal MIS

the transseries view

$$\mathscr{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16C_\eta C_\tau}{3w^2}}_{\text{2nd order}} + \ldots = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\text{gradient expansion}} + \underbrace{\left(\sigma\, w^{\frac{C_\eta}{C_\tau}} e^{-\frac{3}{2C_\tau}w}\right) \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n}}_{\text{1st transseries sector}} + \ldots$$



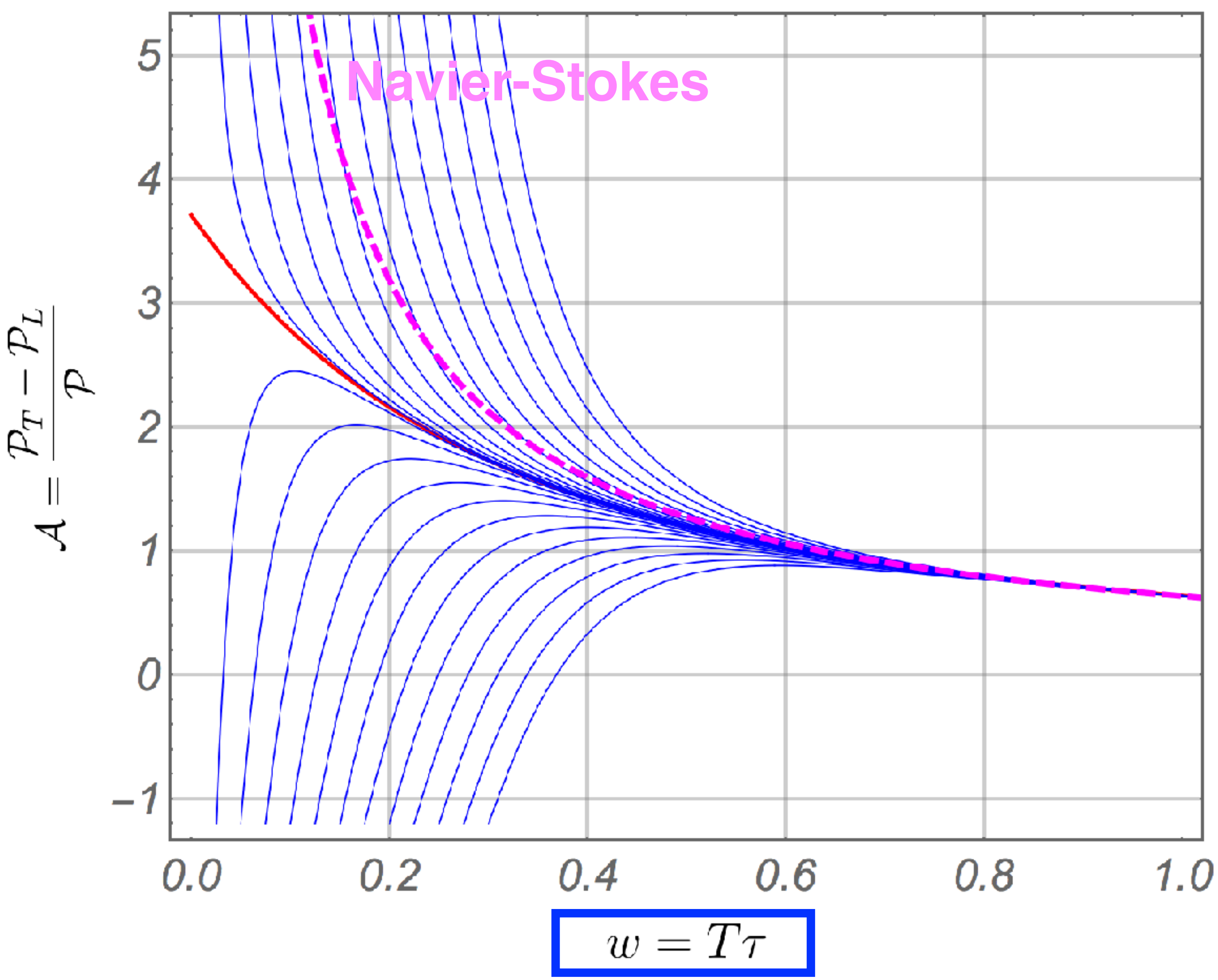
Different initial conditions are captured by exponentially-suppressed corrections to the asymptotic gradient series through the transseries parameter.

The scale of the exponential damping is set by the relaxation time, which is the non-hydrodynamic mode frequency.

The Bjorken attractor in conformal MIS

three stages

- Expansion-dominated early-time stage
- Pre-hydrodynamic stage (non-hydrodynamic mode decay)
- Asymptotic (hydrodynamic) stage



The expansion-dominated stage depends weakly on model parameters which points to its kinematic origin

The pre-hydrodynamic stage depends on both the model parameters and the initial state

The asymptotic stage is independent of initial conditions

Divergence of the gradient expansion

- The gradient expansion
  - describes the asymptotic near-equilibrium limit
  - has a vanishing radius of convergence
  - contains all the information about the theory
- The transseries
  - contains all the initial data
  - each sector can be obtained from the perturbative sector

Bjorken flow in BERTA

The BERTA and moments
for Bjorken flow

The Boltzmann Equation in the Relaxation Time Approximation:

$$\left(\frac{\partial}{\partial \tau} - \frac{p_z}{\tau}\frac{\partial}{\partial p_z}\right) f(\tau, \mathbf{p}) = \frac{f_{\mathrm{eq}}(p/T) - f(\tau, \mathbf{p})}{\tau_R}, \qquad \tau_R = \gamma T(\tau)^{-1}$$

There is an early-time attractor easily seen numerically and studied analytically by various means.

One approach is to use moments

$$\mathscr{L}_n(\tau) \equiv \int \frac{d^3p}{(2\pi)^3} p P_{2n}(\cos\psi)\, f(\tau, \mathbf{p}), \qquad \forall n \geq 0, \qquad \cos\psi = p_z/p_0$$

It will be convenient to use the dimensionless moments

$$\mathscr{M}_n \equiv \frac{\mathscr{L}_n}{\mathscr{L}_0}, \qquad n \geq 1 \qquad \Longrightarrow \qquad \mathscr{M}_1 = -\frac{1}{3}\mathscr{A}$$

Truncations of the moment hierarchy

The BERTA can be expressed as a hierarchy of coupled ODEs.

Truncation at level N:

$$\mathscr{M}_n = 0, \qquad n \geq N$$

leads to a nonlinear ODE of order N-1.

- At level 2 one obtains an equation similar to conformal MIS:

$$w(\mathscr{A} + 12)\mathscr{A}' + 4\mathscr{A}^2 + \frac{6}{7}(21w + 10)\mathscr{A} - \frac{144}{5} = 0$$

- At level 3:

$$w^2(\mathscr{A}+12)^2\mathscr{A}'' + w^2(A + 12)\mathscr{A}'^2 + \frac{1}{11}w(\mathscr{A} + 12)(143\mathscr{A} + 396w + 312)\mathscr{A}' + \\ + \frac{108}{11}\left(33w^2 + 52w + 8\right)\mathscr{A} + 16\mathscr{A}^3 + \frac{18}{11}(99w + 40)\mathscr{A}^2 - \frac{864}{385}(231w + 100) = 0$$

Truncations of the moment hierarchy

The ODE obtained at any level N

- has a pre-hydrodynamic attractor
- the stable fixed point at the origin approximates free streaming
- the gradient expansion correctly reproduces N terms

$$\mathscr{A} \sim \sum_{n=1}^{\infty} \frac{a_n}{w^n} = \frac{8/5}{w} + \frac{32/105}{w^2} + \dots \qquad \Longleftrightarrow \qquad \eta/s = 1/5$$

- the series has a vanishing radius of convergence

The generating function technique

Calculations of these series on an industrial scale is time consuming, but can be vastly improved using a generating function:

$$G_{\mathscr{M}}(x,w) = \sum_{n=0}^{+\infty} x^n \, \mathscr{M}_n(w)$$

- The generating function satisfies a PDE
- The moment series can be found very efficiently at early/late times

The attractor of BERTA
the series at early times

The attractor can also be calculated as series in positive powers of w.

This series is convergent, so it defines the attractor of BERTA directly, without going through ODEs obtained by truncations.

This early time series can be analytically continued to late times

$$\mathscr{A} = \sum_{n=0}^{\infty} c_n w^n \quad \Longrightarrow \quad \mathrm{Pade}[\mathscr{A}]_{(70,71)} \sim \frac{1.60004}{w} + \frac{0.27348}{w^2} + \ldots$$

This procedure reproduces the value of the shear viscosity of BERTA defined through the gradient expansion

$$\mathscr{A} \sim \frac{8C_\eta}{w} + \ldots \quad \Longleftrightarrow \quad \eta/s = 1/5$$

Aniceto, Noronha, MS 2401.06750

Transverse dynamics

Transverse dynamics as perturbations

a semi-analytic extension of the Bjorken model

- The symmetries of Bjorken flow are so restrictive that it cannot capture much of the interesting physics
- Can one extend the Bjorken model to incorporate the dynamics in the transverse plane in a perturbative way?
- Initially the transverse flow is negligible: we will treat it as a perturbation, at the linear level
- Main questions:
  - Is the attractor stable?
  - Does this approximation capture flow at least qualitatively?

An, MS 2312.07703

Transverse dynamics as perturbations

a semi-analytic extension of the Bjorken model

- Linearise around a Bjorken background:

$$
\begin{aligned}
T(\tau, \mathbf{x}) &= T(\tau) + \delta T(\tau, \mathbf{x}) \\
u^\mu(\tau, \mathbf{x}) &= u^\mu + \delta u^\mu(\tau, \mathbf{x}) \\
\pi_{ij}(\tau, \mathbf{x}) &= \pi_{ij}(\tau) + \delta\pi_{ij}(\tau, \mathbf{x})
\end{aligned}
$$

- Normalised modes:

$$
\delta\hat{T}(\tau, \mathbf{x}) = \frac{\delta T(\tau, \mathbf{x})}{T(\tau)}, \qquad \delta\hat{\pi}_{ij}(\tau, \mathbf{x}) = \frac{\delta\pi_{ij}(\tau, \mathbf{x})}{C_e T(\tau)^4}
$$

- Fourier modes

$$
\hat{\phi}(\tau, \mathbf{x}) = \int \frac{d^2k}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\phi}(\tau, \mathbf{k})
$$

- Linearised equations: a set of 6 coupled ODEs for each k

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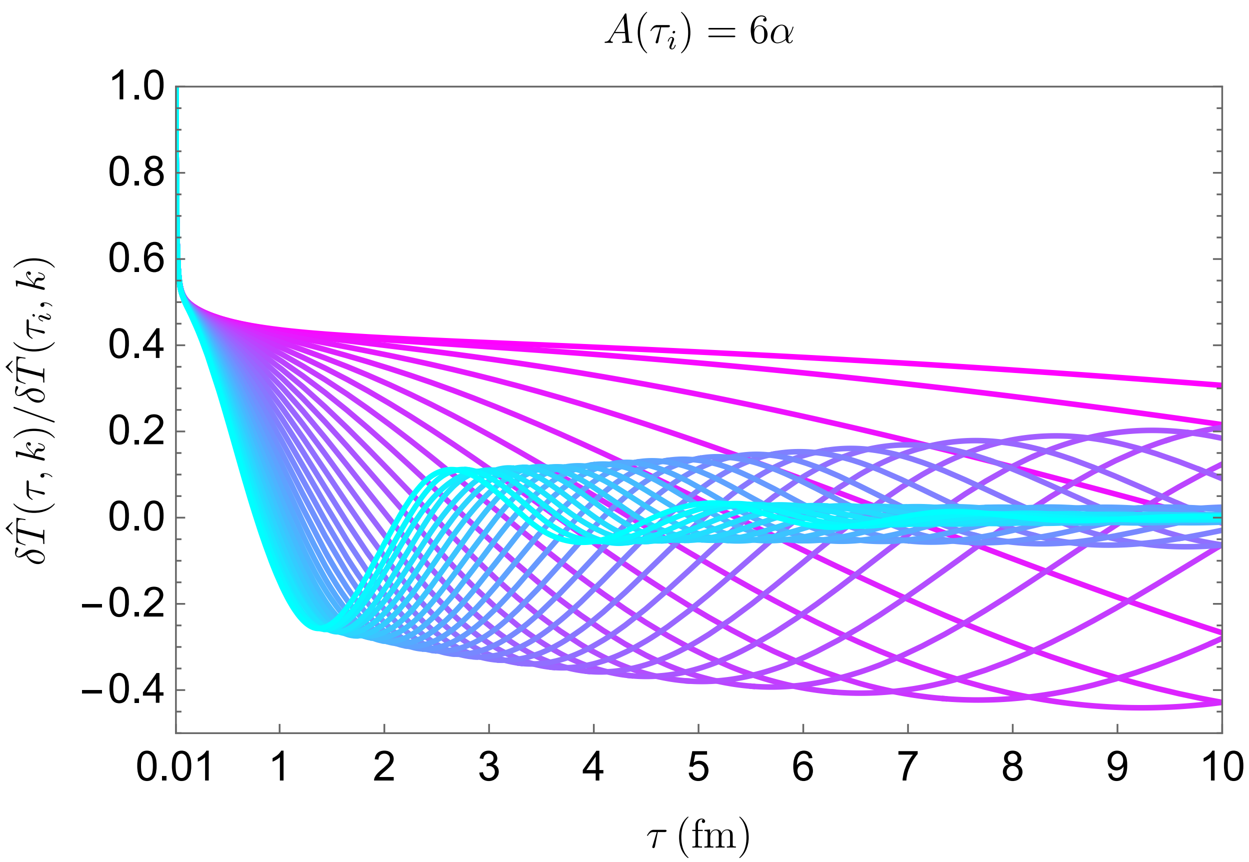
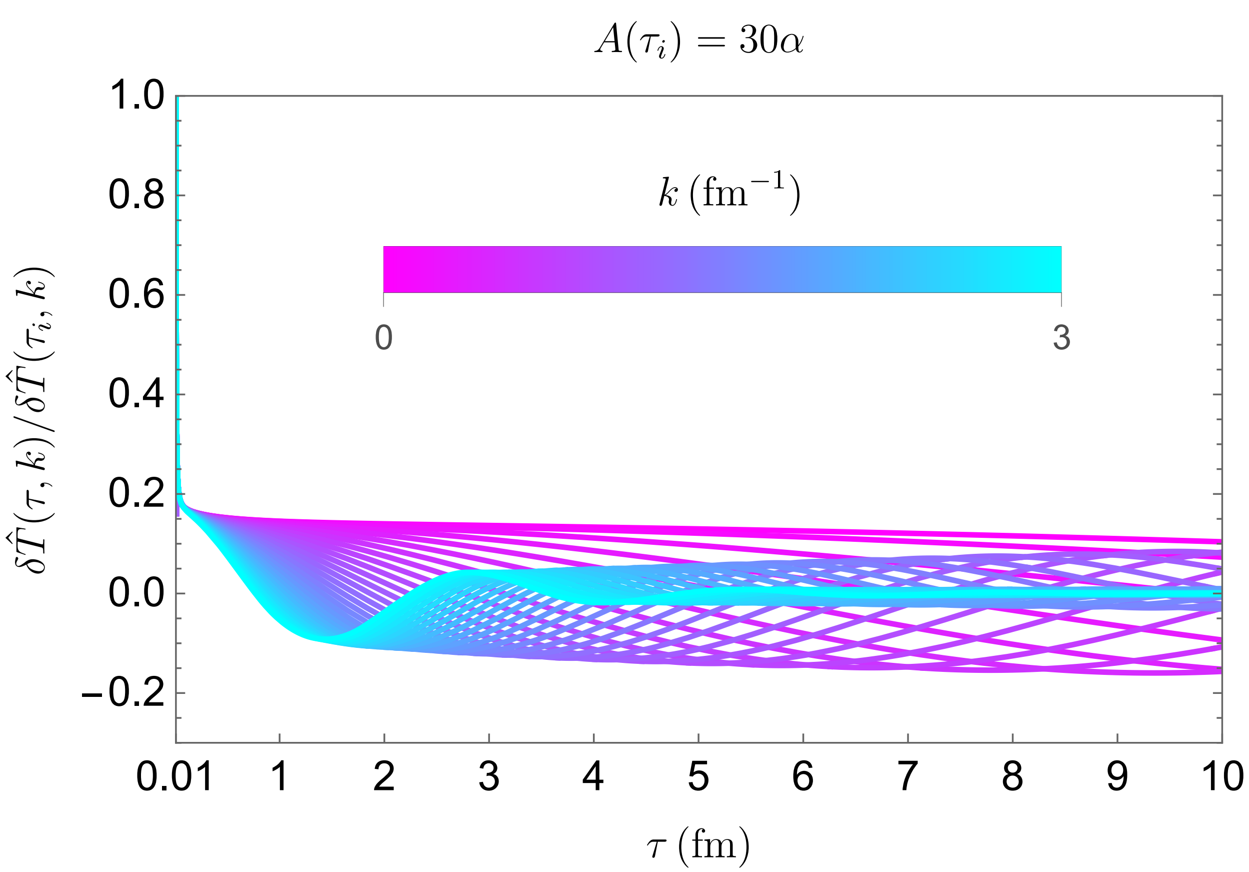
Transverse dynamics as perturbations

stability of perturbations around the attractor (numerics)



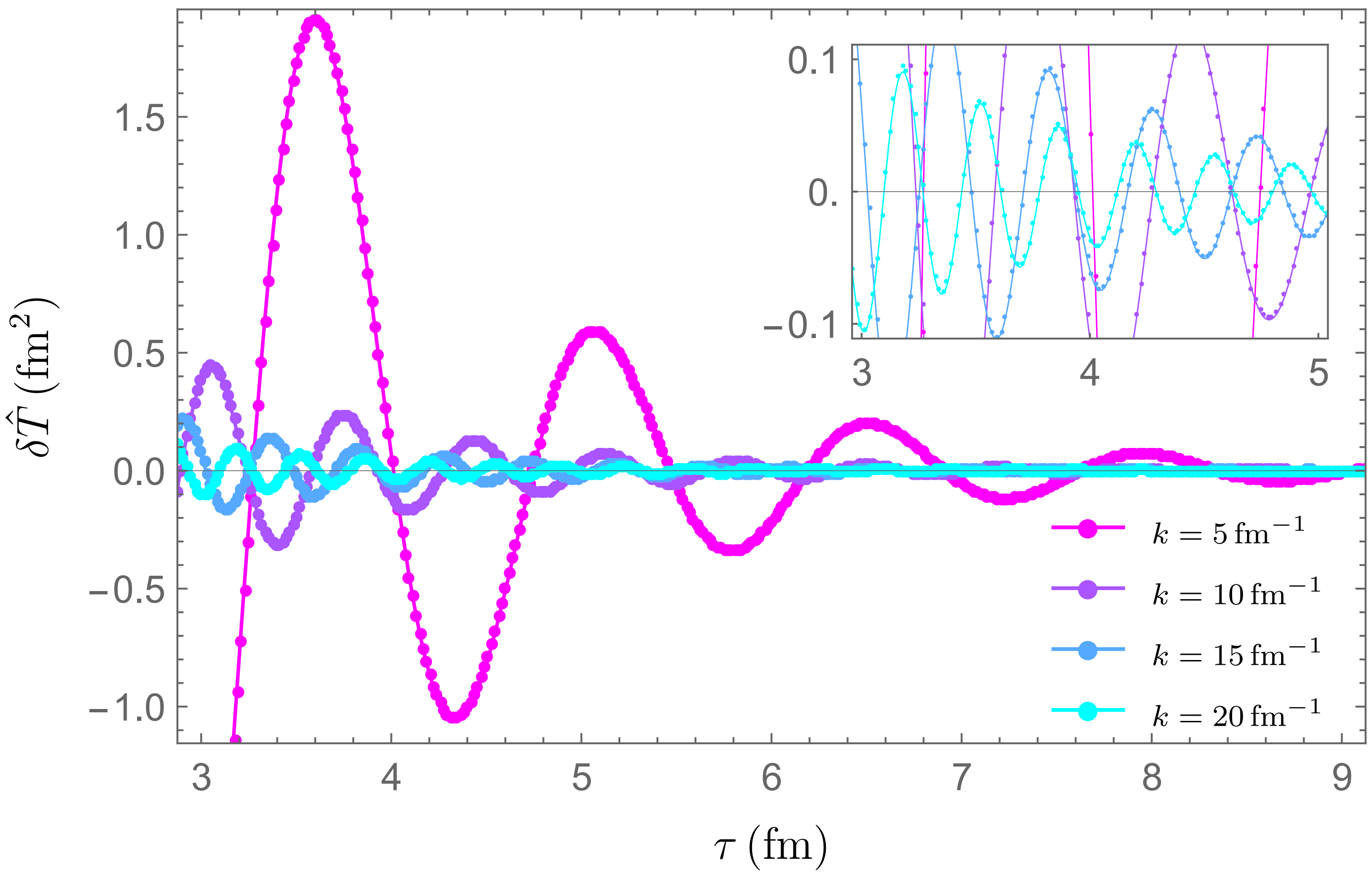
Perturbations initialised on the attractor



Perturbations initialised off the attractor

Transverse dynamics as perturbations

stability of perturbations around the attractor (late time asymptotics)



$$\delta\hat{T} = \sum_{i=1}^{4} \sigma_i (\Lambda\tau)^{\beta_i} e^{-i\omega_i\tau - A_i(\Lambda\tau)^{2/3}}(1 + \ldots)$$

Different initial conditions are reflected by the amplitudes which determine the physics at freeze-out time

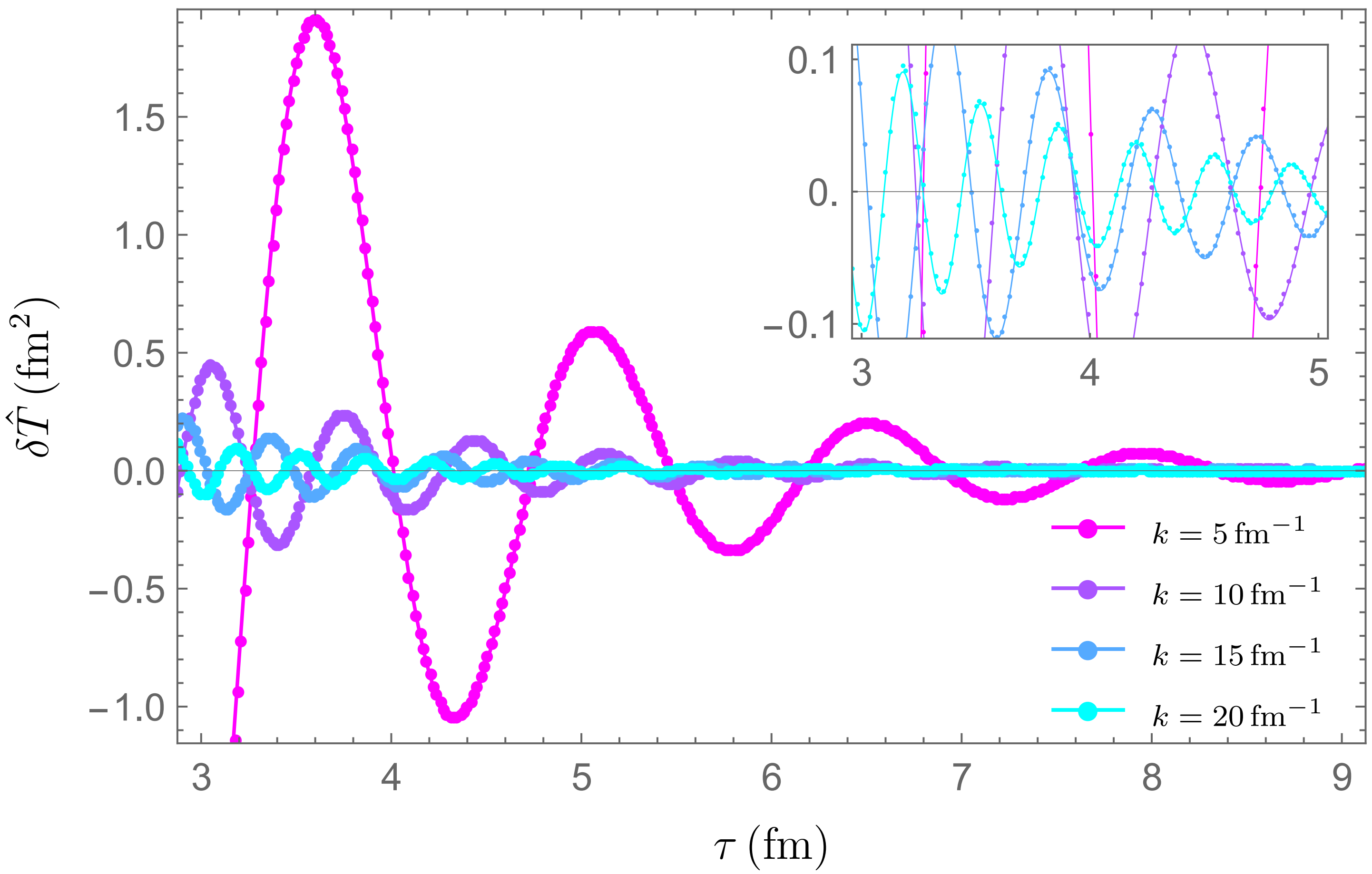
$$A_1 = A_2 = \frac{\alpha^2}{C_\tau c_\infty^2}, \quad A_3 = \frac{3}{2C_\tau}, \quad A_4 = \frac{1}{2C_\tau c_\infty^2}, \quad A_5 = A_6 = \frac{3}{4C_\tau},$$

$$\omega_1 = -\,\omega_2 = c_\infty k \left[1 + \frac{2\alpha^2}{3c_\infty^2}\left(2C_\tau(1-\alpha^2) - \frac{(1+\alpha^2)\Lambda^2}{C_\tau^2 c_\infty^4 k^2}\right)(\Lambda\tau)^{-2/3}\right], \quad \omega_3 = \omega_4 = 0$$

$$c_\infty \equiv \sqrt{\frac{1}{3}\left(1 + 4\frac{C_\eta}{C_\tau}\right)}, \qquad \alpha \equiv \sqrt{\frac{C_\eta}{C_\tau}}$$

Transverse dynamics as perturbations

stability of perturbations around the attractor (late time asymptotics)



$$\delta\hat{T} = \sum_{i=1}^{4} \sigma_i (\Lambda\tau)^{\beta_i} e^{-i\omega_i\tau - A_i(\Lambda\tau)^{2/3}}(1 + \ldots)$$

Large wave vector modes are damped more strongly than small wavelength modes

$$\beta_1 = \beta_2 = \frac{1}{54c_\infty^4}\left(1 + 8\alpha^2 + 64\alpha^4 + 32\alpha^6 + \frac{4\alpha^2\Lambda^2}{C_\tau^3 c_\infty^4 k^2}\right),$$

$$\beta_3 = -\frac{2}{3}(1-\alpha^2),$$

$$\beta_4 = \frac{2\alpha^2}{27c_\infty^4}\left(1 - 16\alpha^2 - \frac{2\Lambda^2}{C_\tau^3 c_\infty^4 k^2}\right)$$

The exponential corrections depend on parameters of the non-hydrodynamic sector

Transverse dynamics as perturbations

flow observables

- Flow is usually quantified in terms of coefficients in the expansion

$$\frac{dN(p_\perp,\phi)}{p_\perp dp_\perp d\phi dy} = v_0(p_\perp)\left(1+\sum_{n=1}^{\infty} 2v_n(p_\perp)\cos(n\phi)\right)$$

- These coefficients are can be expressed in terms of transverse averages of the perturbations (in real space), e.g.

$$v_0(\hat{p}_\perp) = \frac{m_\perp \tau_f}{(2\pi)^3}\Sigma_\perp\left[F_0 + F_1\langle\delta\hat{T}\rangle_\perp + F_{11}\langle\delta\hat{T}\delta\hat{T}\rangle_\perp + \frac{1}{2}\hat{p}_\perp^2\left(F_3\langle\delta\hat{\pi}_{ii}\rangle_\perp + F_{13}\langle\delta\hat{\pi}_{ii}\delta\hat{T}\rangle_\perp + F_{22}\langle\delta u_i\delta u_i\rangle_\perp\right)\right],$$

$$v_2(\hat{p}_\perp) = \frac{\hat{p}_\perp^2\left(F_3\langle\delta\hat{\pi}_{11}-\delta\hat{\pi}_{22}\rangle_\perp + F_{13}\langle(\delta\hat{\pi}_{11}-\delta\hat{\pi}_{22})\delta\hat{T}\rangle_\perp + F_{22}\langle\delta u_1^2-\delta u_2^2\rangle_\perp\right)}{4(F_0+F_1\langle\delta\hat{T}\rangle_\perp + F_{11}\langle\delta\hat{T}\delta\hat{T}\rangle_\perp) + 2\hat{p}_\perp^2\left(F_3\langle\delta\hat{\pi}_{ii}\rangle_\perp + F_{13}\langle\delta\hat{\pi}_{ii}\delta\hat{T}\rangle_\perp + F_{22}\langle\delta u_i\delta u_i\rangle_\perp\right)},$$

- Flow originates entirely from the exponentially-suppressed corrections which are still not negligible at freeze-out

Transverse dynamics as perturbations

flow observables

- Sadly, the Fourier transform is inverted numerically
- Elliptic flow is captured at a qualitative level
- There is some sensitivity the value of the relaxation time

Summary

- Early-time attractors provide a window of opportunity for hydrodynamic models
- Asymptotic methods serve as an essential toolkit allowing for some analytic studies
- Physics at freeze-out is captured by exponentially-suppressed corrections to the Bjorken attractor