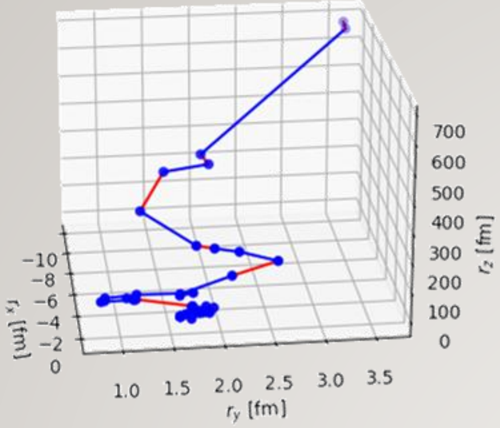
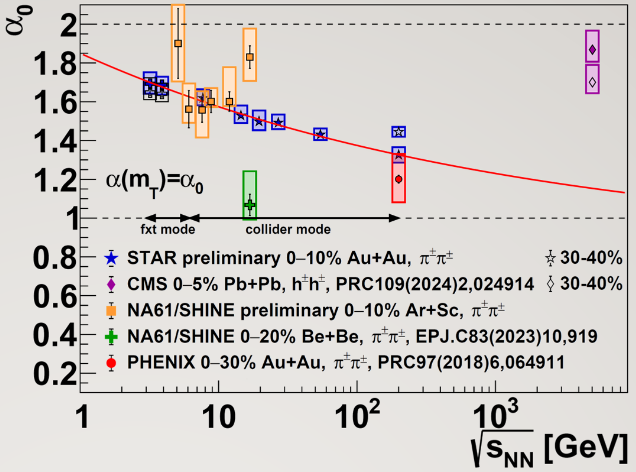
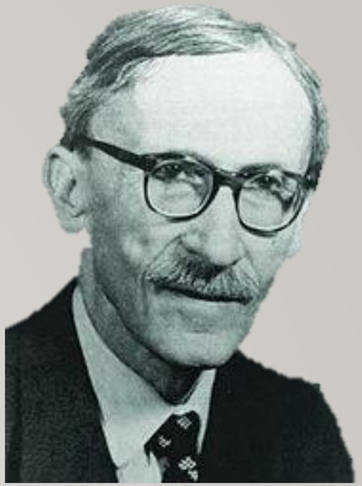






LÉVY WALK IN HEAVY-ION COLLISIONS

IDEAS, FACTS, QUESTIONS (A BYOE TALK)

MÁTÉ CSANÁD (ELTE EÖTVÖS LORÁND UNIVERSITY, BUDAPEST)

HYDRO WORKSHOP, GALILEO GALILEI INSTITUTE, FIRENZE, APRIL 24, 2025

CONTENTS

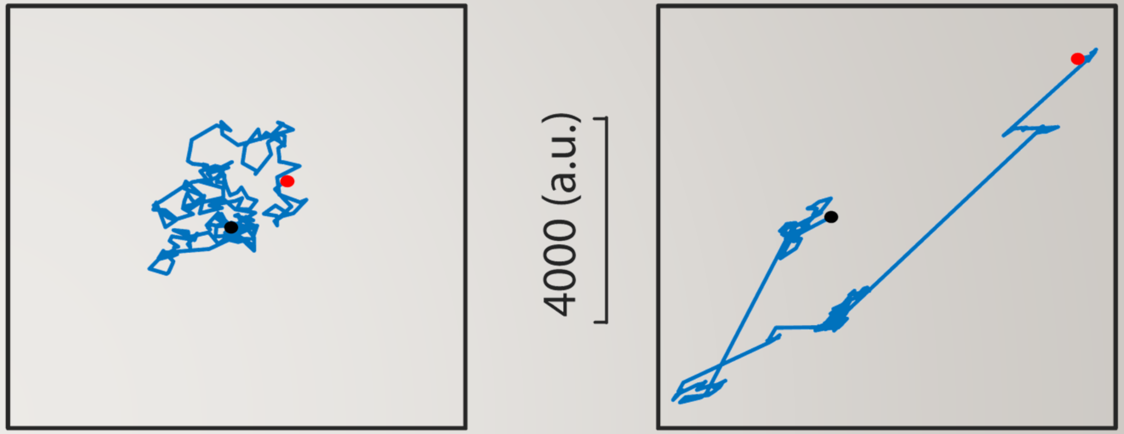
- Ideas
  - Lévy walk, femtoscopy, simulations
- Facts
  - Measurements, comparisons
- Questions
  - How to understand all this?

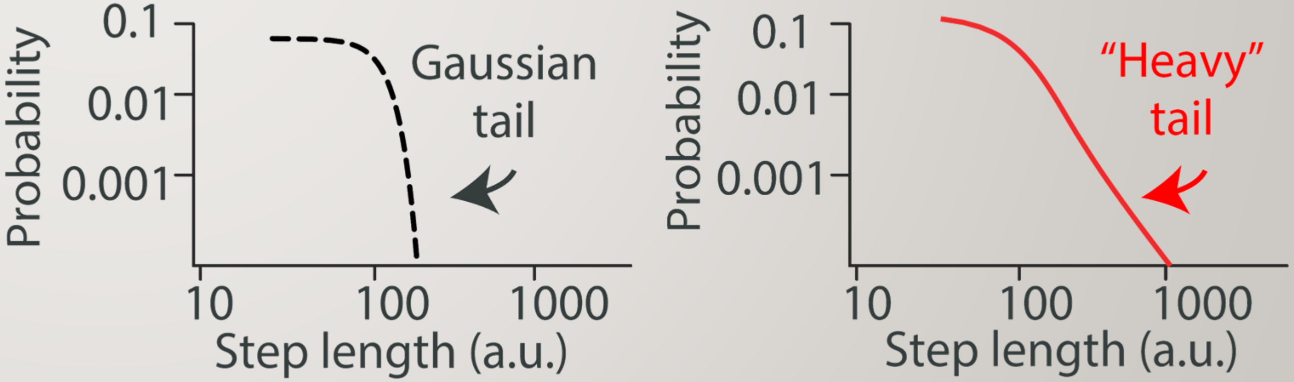
CONTENTS

- Ideas
  - Lévy walk, femtoscopy, simulations
- Facts
  - Measurements, comparisons
- Questions
  - How to understand all this?

LÉVY WALK IN NATURE

- Random variables with no finite 2nd moment
  → central limit theorem does not apply
- Generalized central limit theorem does
  → sum follows Lévy-stable distribution
- Found in chemical, biological, physical processes





Nature 451(2008)1098
Nature 453(2008)495
Nature 449(2007)1044
Nature 465(2010)1066
Nature 486(2012)545

https://www.nature.com/articles/s42003-021-02256-1

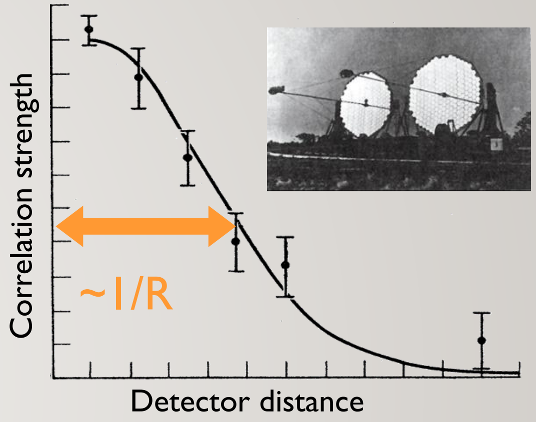
HBT OR FEMTOSCOPY IN HIGH ENERGY PHYSICS

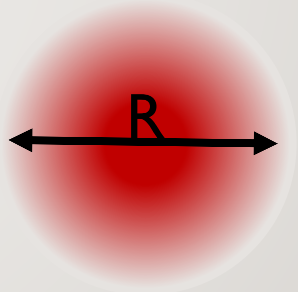
- R. Hanbury Brown, R. Q. Twiss - observing Sirius with radio telescopes
  - Intensity correlations vs detector distance ⇒ source size
  - Measure the sizes of apparently point-like sources!
- Goldhaber et al: applicable in high energy physics
- Understanding: Glauber, Fano, Baym, …
  Phys. Rev. Lett. 10, 84; Rev. Mod. Phys. 78 1267, …
  - Momentum correlation $C(q)$ related to source $S(r)$

    $C(q) \cong 1 + \left|\int S(r) e^{iqr} dr\right|^2$

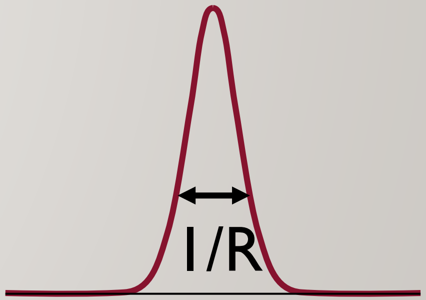
    (under some assumptions)
  - Can be expressed with distance distribution $D(r)$:

    $C(q) \cong 1 + \int D(r) e^{iqr} dr$
  - Neglected: pair reconstruction, final state interactions, multi-particle correlations, coherence, …
- What is the source shape? Can be explored via femtoscopy





source function S(r)



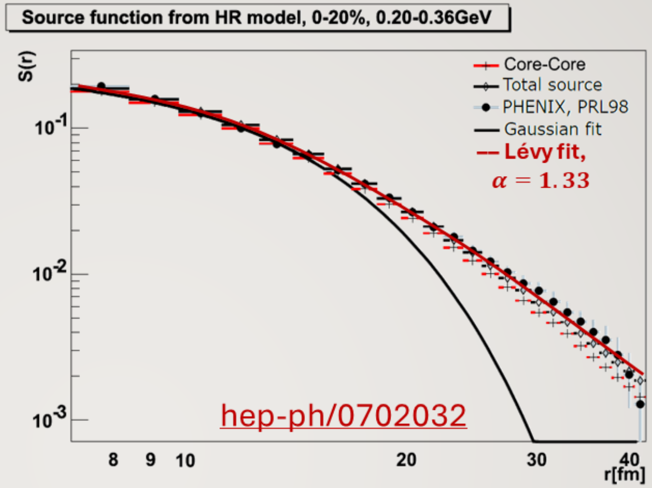
correlation funct. C(q)

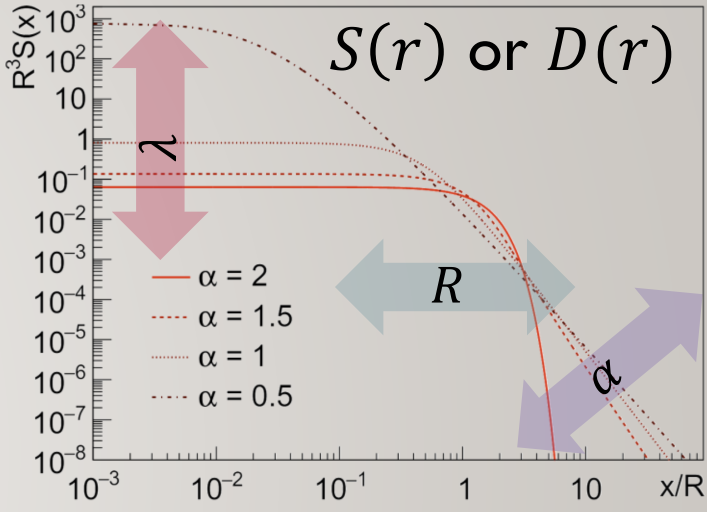
LÉVY DISTRIBUTIONS IN HEAVY-ION PHYSICS

- Central limit theorem, diffusion, and thermodynamics lead to Gaussians
- Measurements suggest phenomena beyond Gaussian distribution
- Lévy-stable distribution (symmetric):

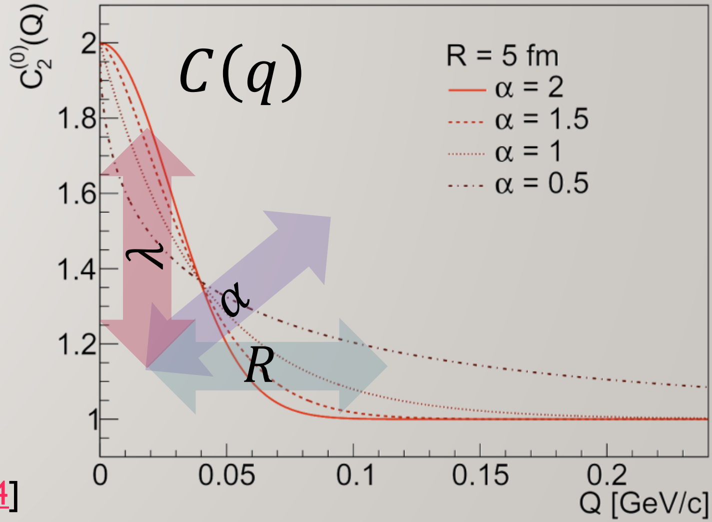
$$\mathcal{L}(\alpha, R; r) = \frac{1}{2\pi}\int d^3q e^{iqr} e^{-\frac{1}{2}|qR|^\alpha}$$

  - From generalized central limit theorem
  - Power-law tail $\sim r^{-1-\alpha}$ if $\alpha < 2$
  - Special cases: α = 2 Gaussian, α = 1 Cauchy





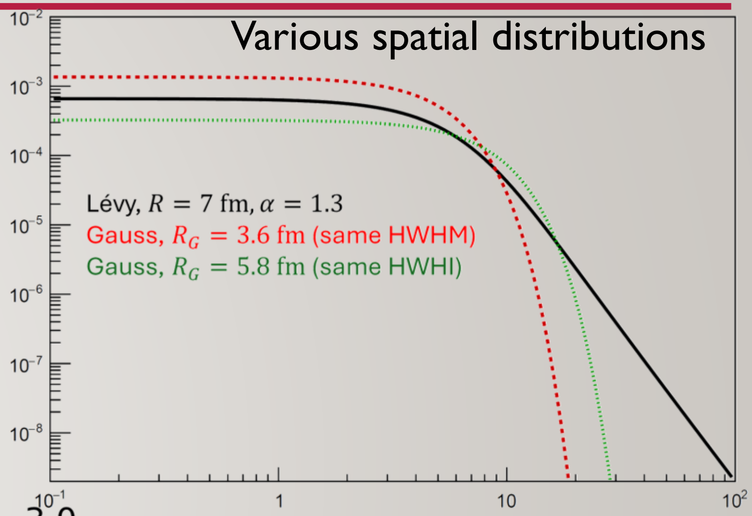
- Shape of the correlation functions with Lévy source: $C_2(q) = 1 + \lambda \cdot e^{-|qR|^\alpha}$
  Csörgő, Hegyi, Zajc, Eur.Phys.J. C36 (2004) 67-78
- **Parameters: strength λ, scale R, shape α** ⟶
- Lévy source seen & exponent measured from SPS through RHIC to LHC
  NA61 [EPJC83(2023)919], PHENIX [PRC97(2018)064911 & PRC110(2024)064909], CMS [PRC109(2024)024914]

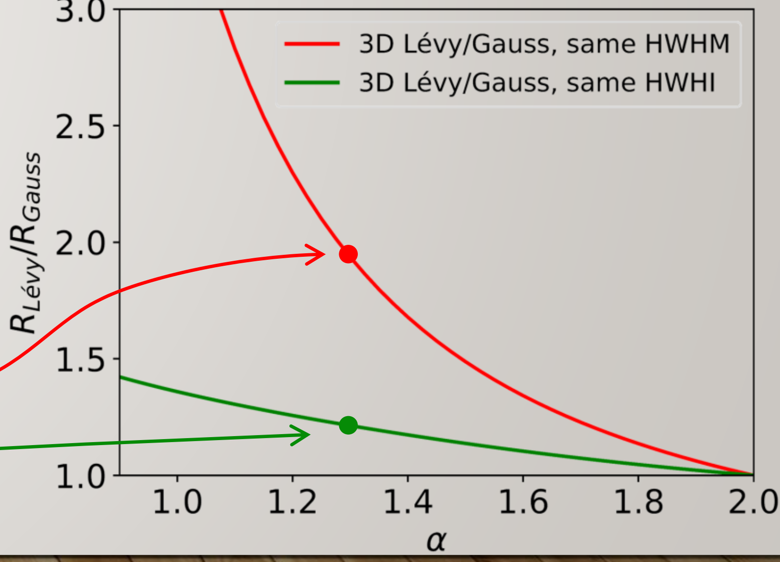


COMPARING DIFFERENT SOURCE SIZE MEASURES

- No tail if $\alpha = 2$, power law if $\alpha < 2$, RMS = ∞: value depends on cutoff
- What do Gaussian HBT radii mean if the source has a power-law tail?
  - Important also w.r.t. critical point search and QGP exploration
- Alternative measures (see J.Phys.G52(2025)025102 for details)
  - HWHM: (half) width at half maximum
  - HWHI: (half) width at half integral

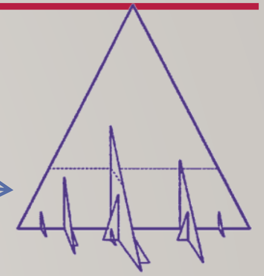
  } width/R: nontrivial α-dependence
- Relations for 3D Gauss: HWHM $\approx 1.17 \cdot R_\mathrm{G}$, HWHI $\approx 1.54 \cdot R_\mathrm{G}$
- For (e.g.) Lévy $\alpha = 1.3$: HWHM $\approx 0.61 \cdot R_\mathrm{L}$, HWHI $\approx 1.27 \cdot R_\mathrm{L}$
- **Thus (e.g.) $\alpha = 1.3$ and $R_L = 7$ fm "means":**
  - Same HWHM Gaussian: $R_\mathrm{G} \approx 3.61$ fm ← $R_\mathrm{Gauss} \approx R_\mathrm{Lévy}/1.94$
  - Same HWHI Gaussian: $R_\mathrm{G} \approx 5.77$ fm ← $R_\mathrm{Gauss} \approx R_\mathrm{Lévy}/1.21$

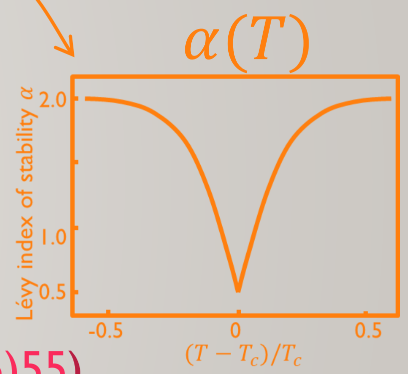


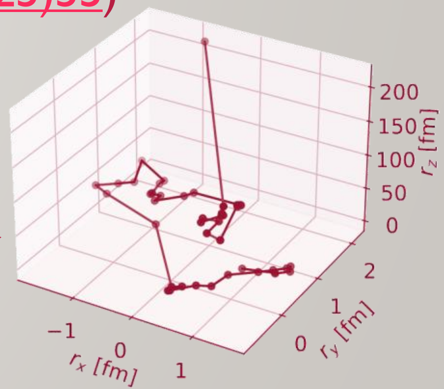


WHY DO LÉVY SHAPES APPEAR, WHY IS IT IMPORTANT?

- A more comprehensive list of possible reasons:
  - Jet fragmentation (Csörgő, Hegyi, Novák, Zajc, Acta Phys.Polon. B36 (2005) 329-337)
    - See also Caucal, Mehtar-Tani, JHEP 09 (2022) 023
    - Important in e^+e^-, see L3 Collaboration, Eur.Phys.J.C 71 (2011) 164
  - Critical phenomena (Csörgő, Hegyi, Novák, Zajc, AIP Conf.Proc. 828 (2006) no.1, 525-532)
    - Role in the few GeV region? Affected by finite size effects?
  - Directional or event averaging (Cimerman et al., Phys.Part.Nucl. 51 (2020) 282)
    - Ruled out by event-by-event and 3D analyses
  - Lévy walk (BJP37(2007); PRB103(2021), Entropy24(2022); PLB847(2023); Comm.Phys.8(2025)55)
    - Only plausible explanation (so far!) at high energies and large systems
- Importance of utilizing Lévy sources, leaving α as parameter:
  - Measuring α and R: quark-hadron transition, critical point, etc.
  - Measuring λ: In-medium mass modification, coherent pion production







LÉVY WALK IN SCATTERING

- Lévy walk and Lévy flight: known in ecology, climatology, etc.
- In HIC: increasing mean free path, step size increases
  - Seen in expansion under Coulomb potential in solid-state physics
- Observed in UrQMD and EPOS (Commun. Phys. 8 (2025) 55)
  - Scatterings, decays, coalescence contribute to Lévy walk (as discussed later)
  - Interestingly, long-range Coulomb not implemented usually

E. I. Kiselev, Phys. Rev. B 103, 235116 (2021)

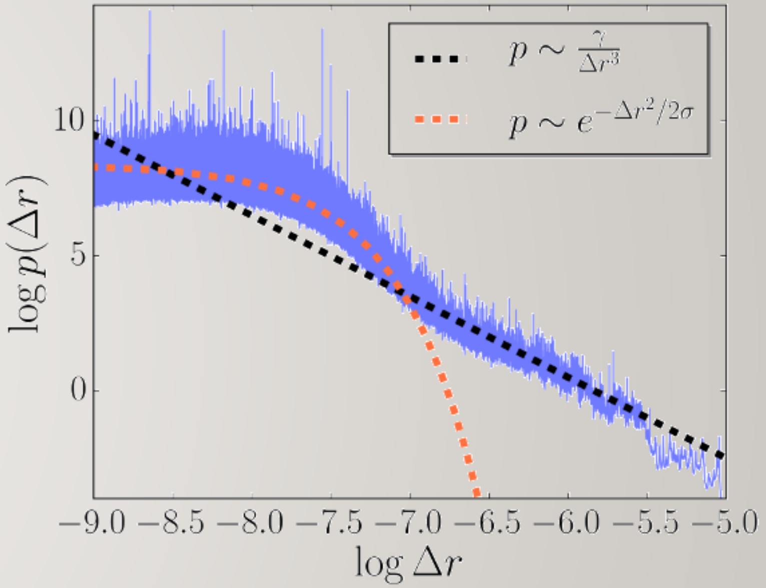
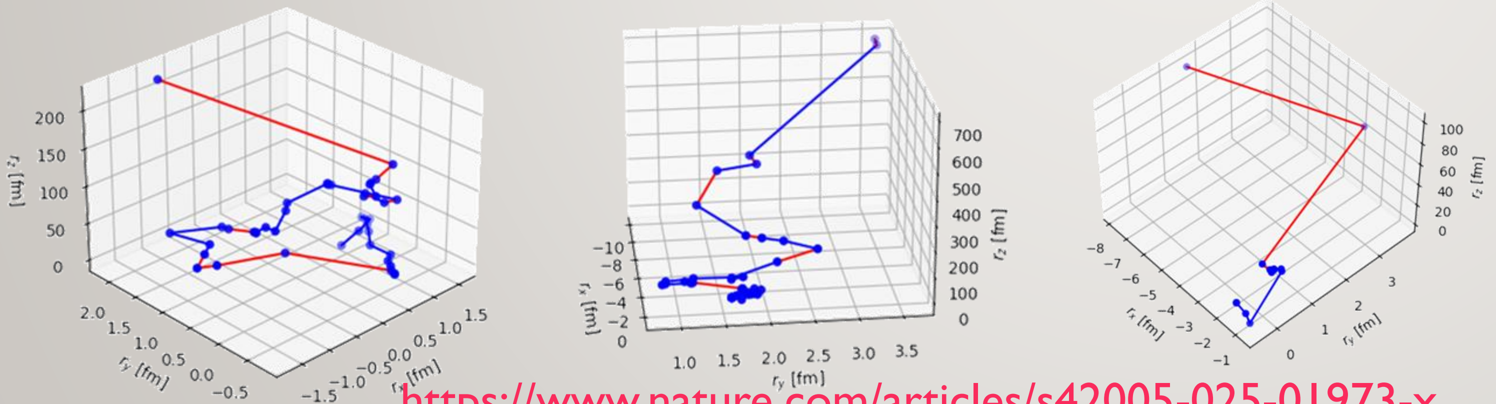


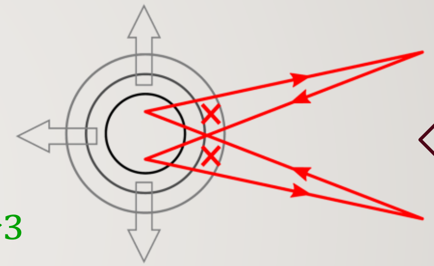
Figure 1. The Figure shows the step size distribution $p(\Delta r)$ of a random walk as performed by Coulomb interacting, diffusing particles in two dimensions. At large step sizes, the distribution clearly follows the $p \sim \Delta r^{-3}$ power-law which leads to the superdiffusive dynamics described by Eq. (1). The data was obtained by integrating the system of coupled Langevin equations of Eq. (56).



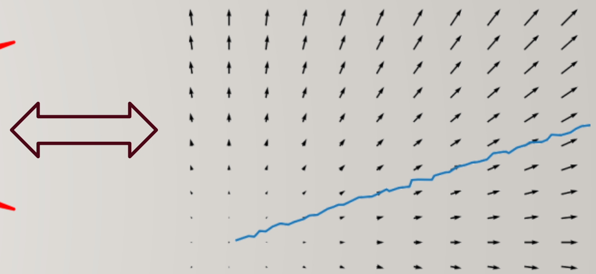
https://www.nature.com/articles/s42005-025-01973-x

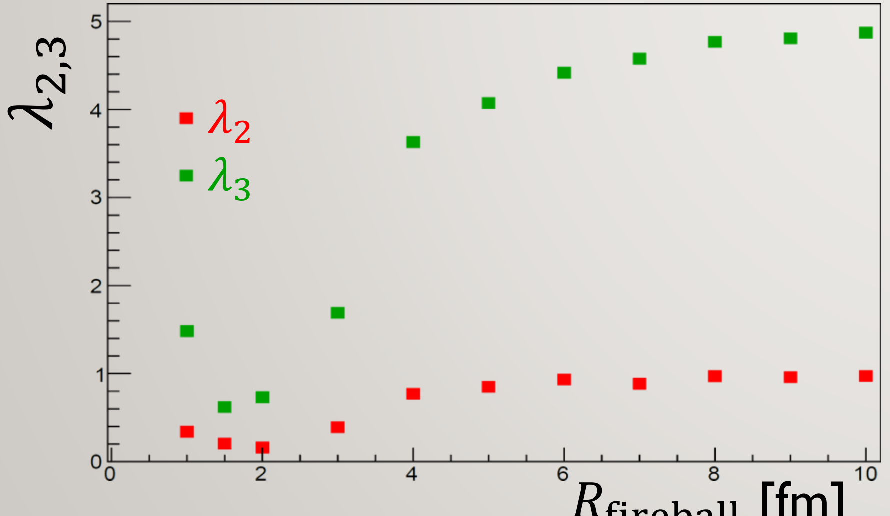
CHARGED HADRON CLOUD: A SIDE-NOTE

- Coulomb potential: infinite range, affecting evolution for a long time
- Solid-state physics (as mentioned on previous slide): may cause Lévy flight and power-law tails
- Another interesting effect: distortion of flight paths after kinetic freeze-out
  - Phase shift, similar to an Aharonov-Bohm effect (Gribov-90 (2021) 261 and IJMP A 40 (2025) 2444007)
- Phase shift decreases 2- & 3-particle corr. strengths λ_2 & λ_3



exaggerated illustration



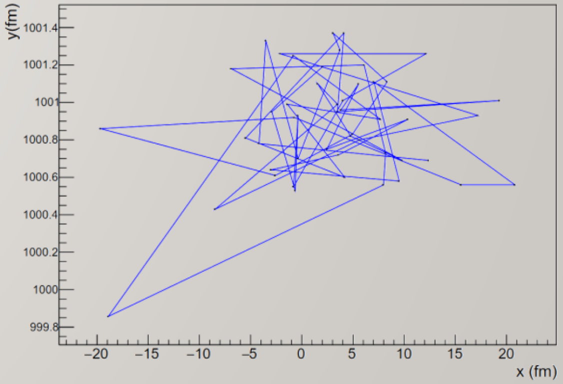


$\lambda_3 = 5$

No Aharonov-Bohm effect, pure core, fully chaotic source

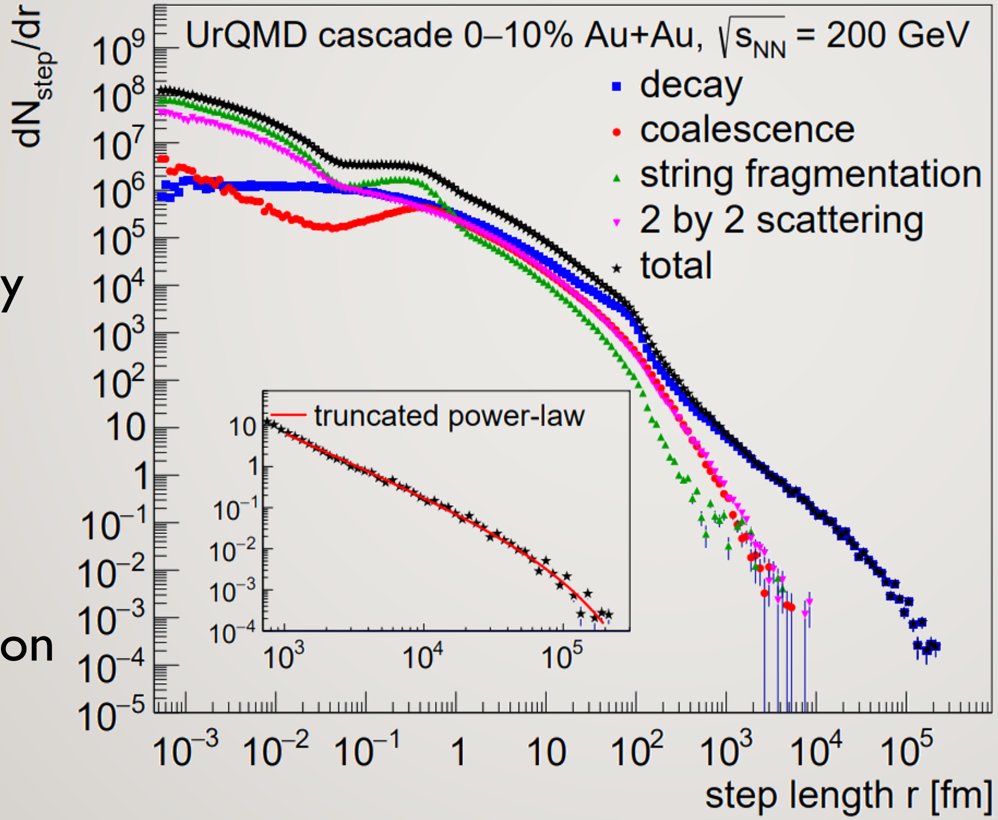
$\lambda_2 = 1$

simulated transverse path

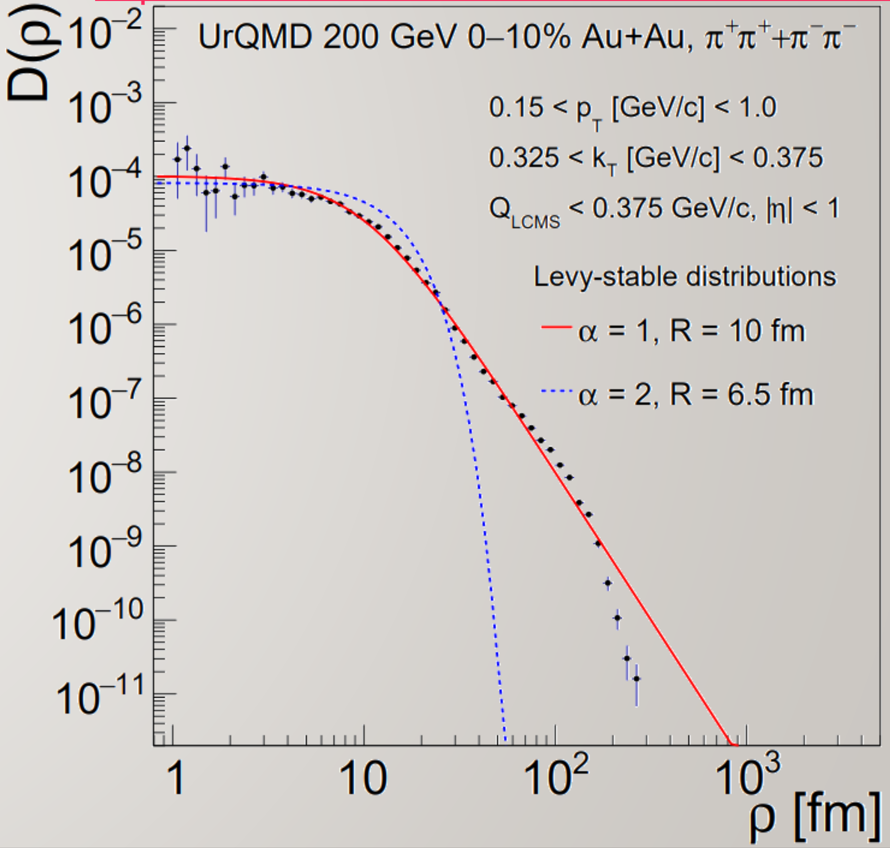


HOW DO THE TAILS EMERGE IN HADRON SCATTERING?

- UrQMD: 4 type of processes, scattering (2→2), decay (1→N), coalescence (2→1), string fragm. (1→N)
- Step before the given process; sum of steps: freeze-out coordinate distribution
- Step length: power-law tail with $\sim r^{-1.53}$
- Sum: follows Lévy
- Distance distribution: autoconvolution
- Lévy walk in action



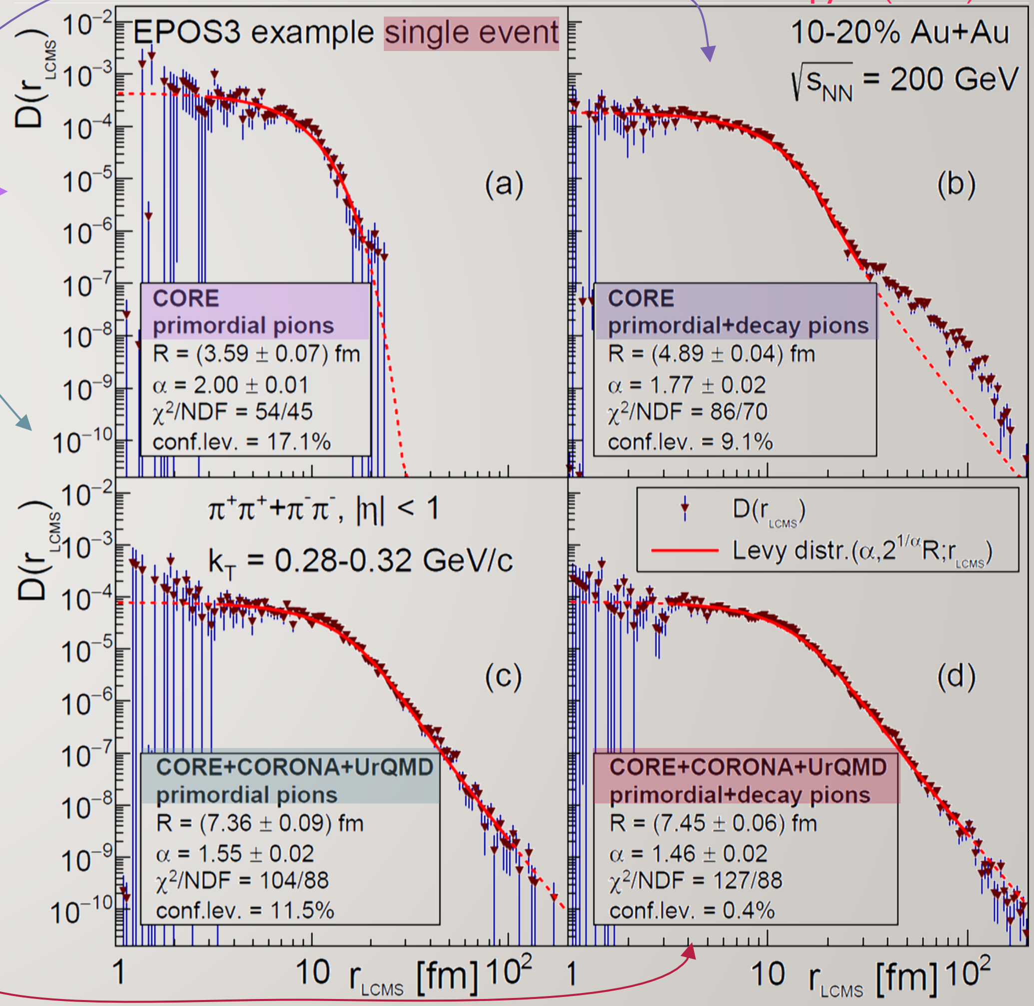
https://www.nature.com/articles/s42005-025-01973-x



COUPLE HYDRO WITH URQMD: EPOS, EVENT-BY-EVENT

- EPOS model: parton-based Gribov-Regge theory (PBGRT)
  - Werner et al., PRC82 (2010) 044904, PRC89 (2014) 064903, …
- Source observed in four stages:
  - a) CORE, primordial pions: close to Gaussian
  - b) CORE, with decay products: power-law structures
  - c) CORE+CORONA+UrQMD, primordial pions: Lévy shape
  - d) CORE+CORONA+UrQMD, with decay products: Lévy shape
  - Radii in the four stages (one example event)
    3.59 fm → 4.89 fm → 7.36 fm → 7.45 fm
  - Shape (α) change: 2.00 → 1.77 → 1.55 → 1.46
- **Scattering stage needed for Lévy shaped sources?**
- Can one relate the observed HBT radii to the hydro phase homogeneity lengths?

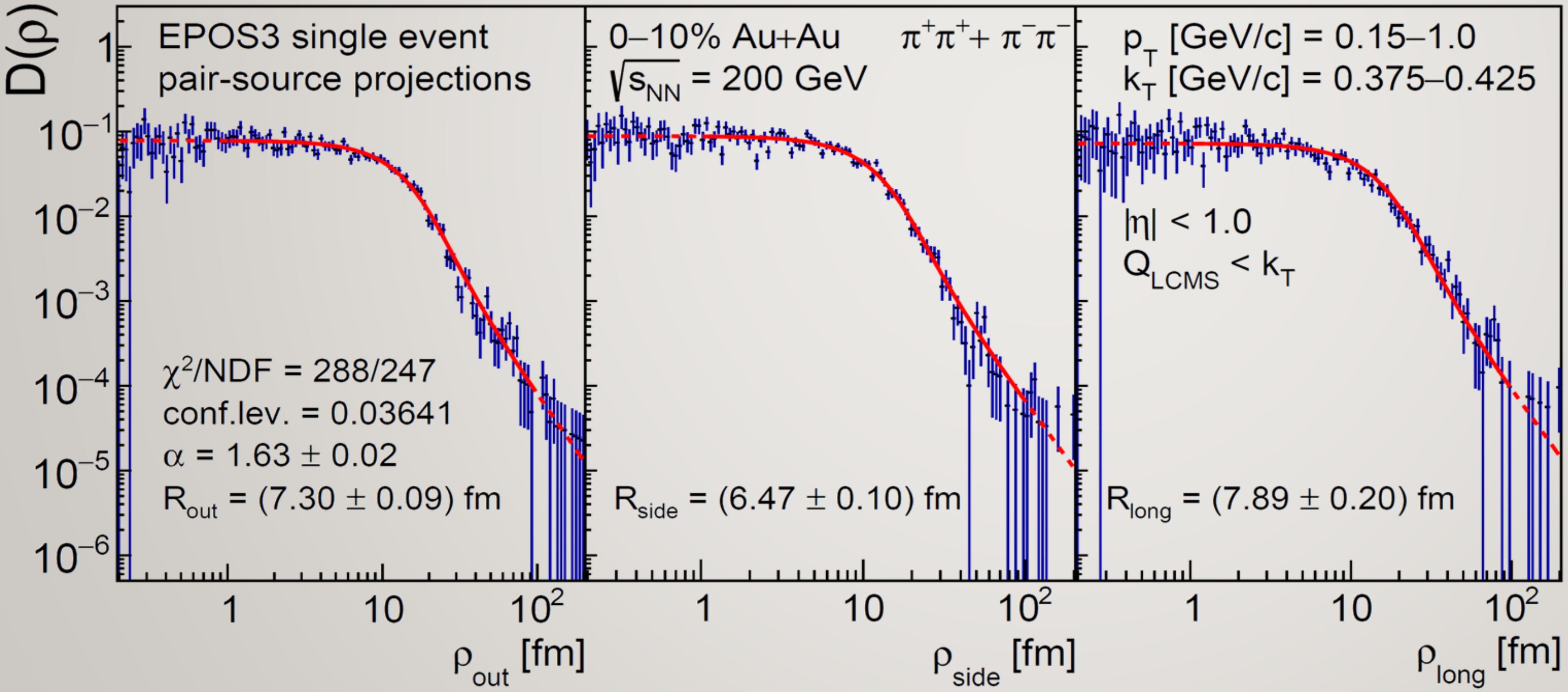
Entropy24(2022)308

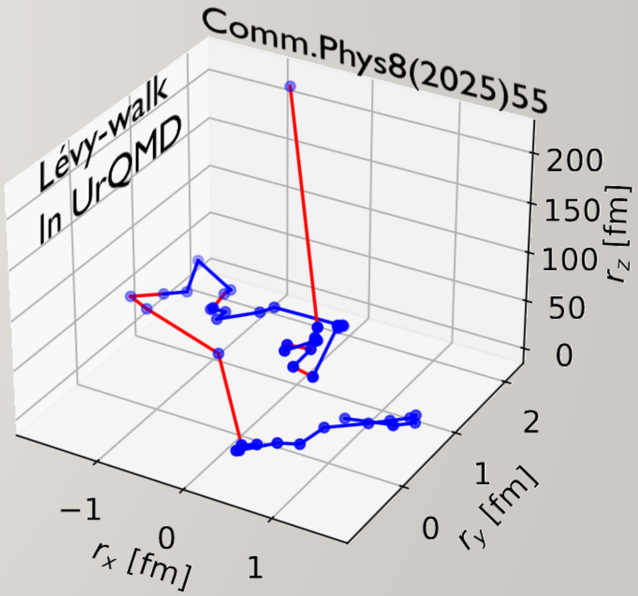


LÉVY SHAPES IN SINGLE 3D EPOS EVENTS, 3D

- What if the Lévy shapes appeared only because of directional averaging?
- Let's check 3D event shapes in EPOS! → 3D Lévy works, with similar α and radii (as those in 1D)
- Clear physical reason: Lévy walk

Comm.Phys.8(2025)55, https://www.nature.com/articles/s42005-025-01973-x



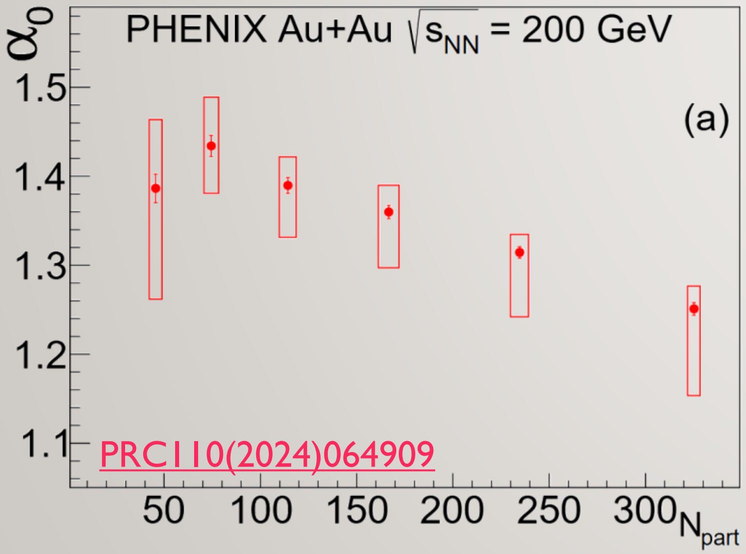


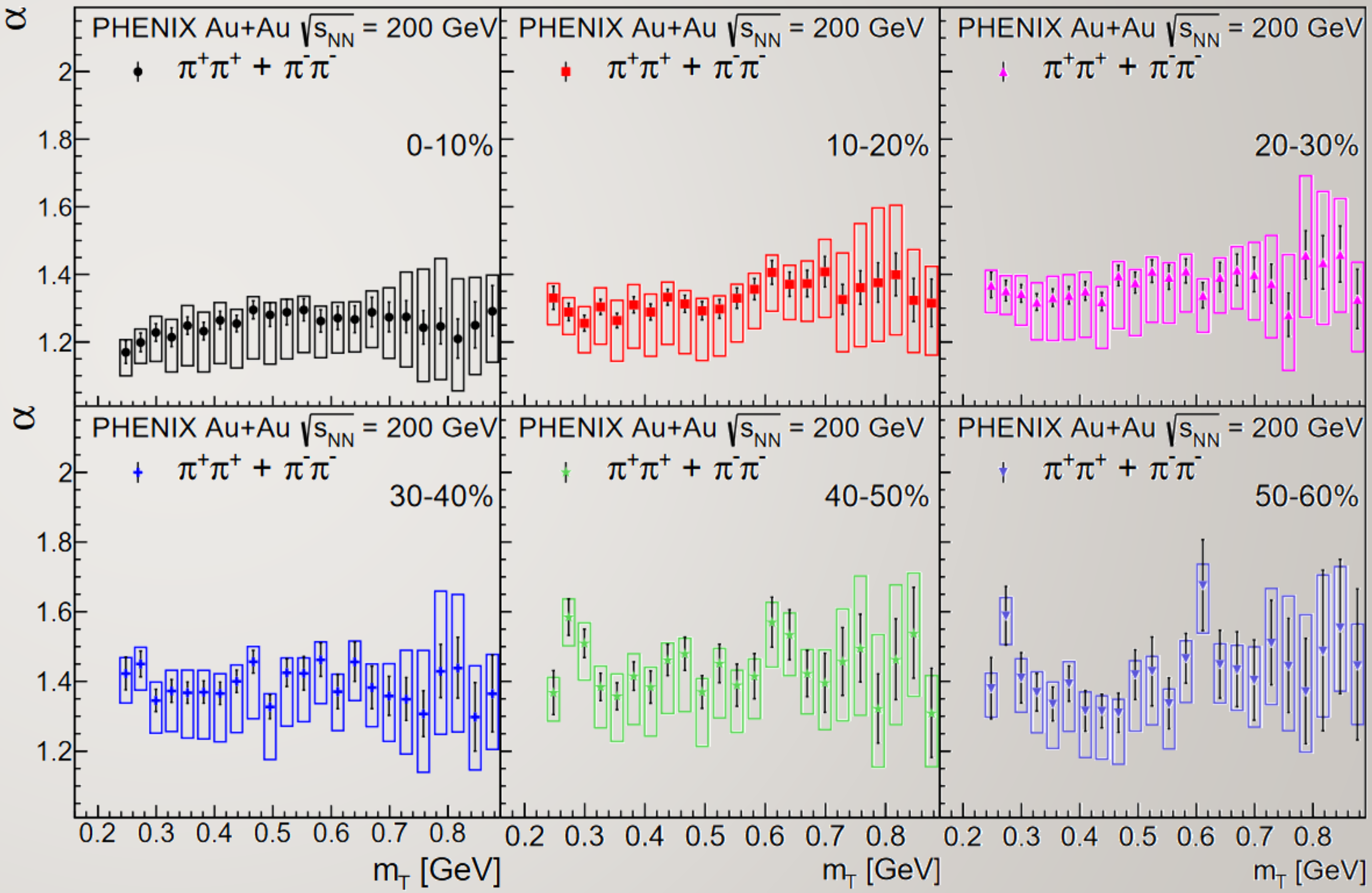
CONTENTS

- Ideas
  - Lévy walk, femtoscopy, simulations
- Facts
  - Measurements, comparisons
- Questions
  - How to understand all this?

CENTRALITY DEPENDENCE IN 200 GEV AU+AU, PHENIX

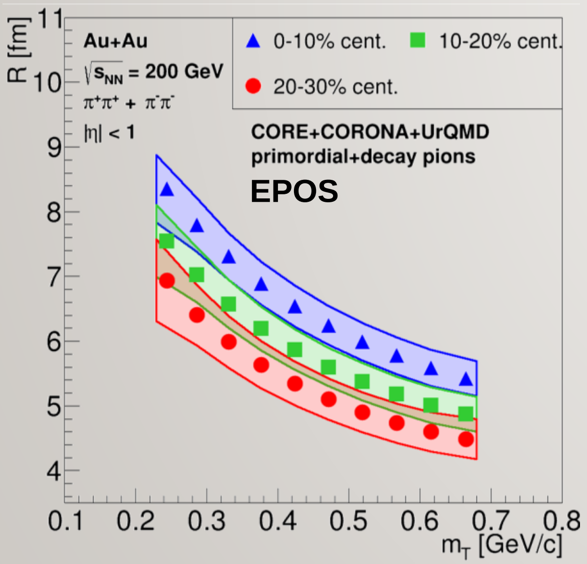
- Lévy-index α measured in 200 GeV Au+Au collisions at PHENIX, approximately constant in m_T
- $\alpha_0 = \langle\alpha(m_T)\rangle$ versus $N_{\rm part}$: decrease for central collisions
  - Due to longer time to develop tails?
- PHENIX paper: PRC110(2024)064909

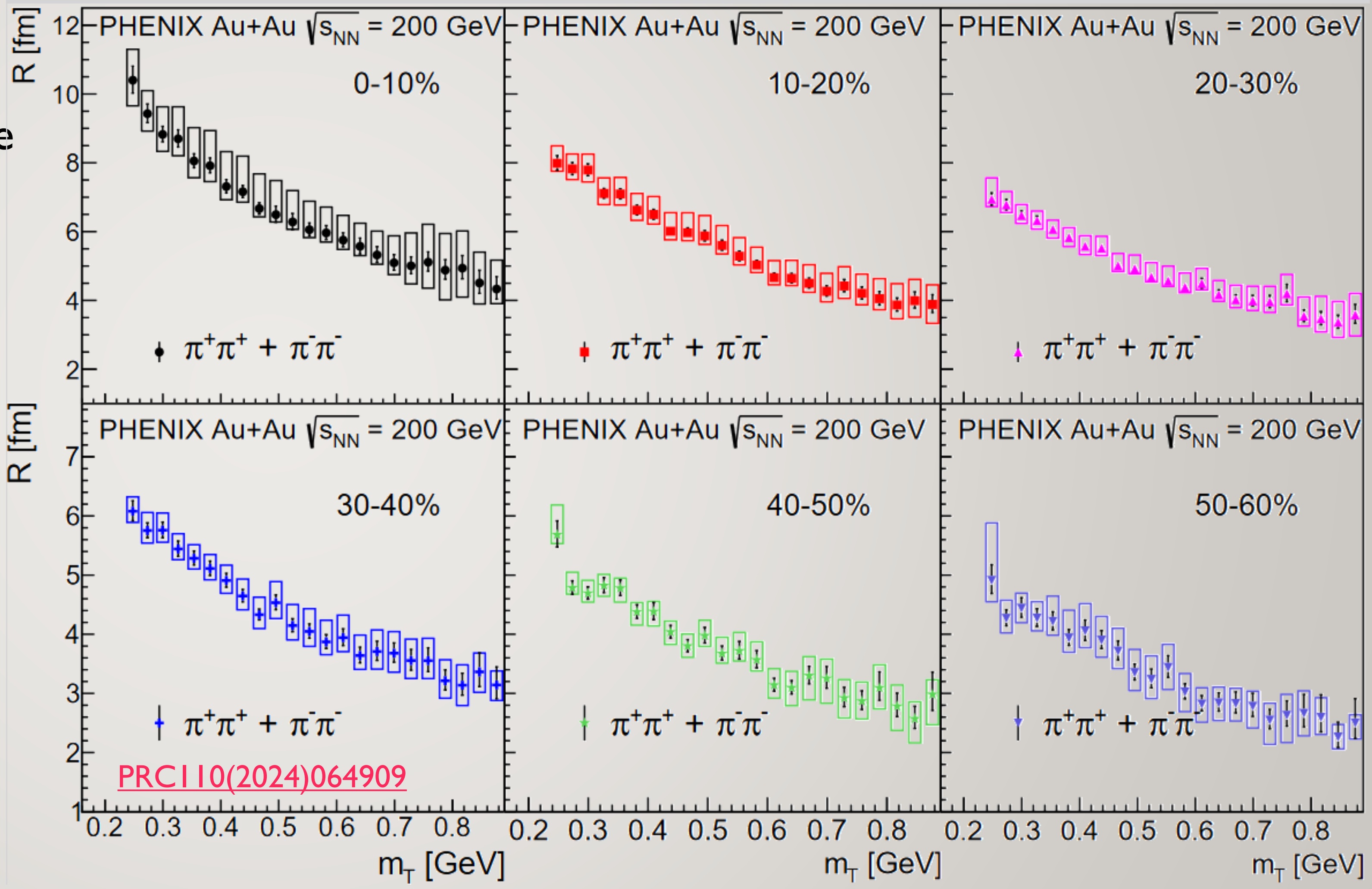




CENTRALITY DEPENDENCE IN 200 GEV AU+AU, PHENIX

- Monotonic decrease, $R \sim 1/\sqrt{m_T}$
- As predicted for Gaussian source
  - Why does it work here?
  - Does hydro drive radii?
- Note: data close to EPOS result

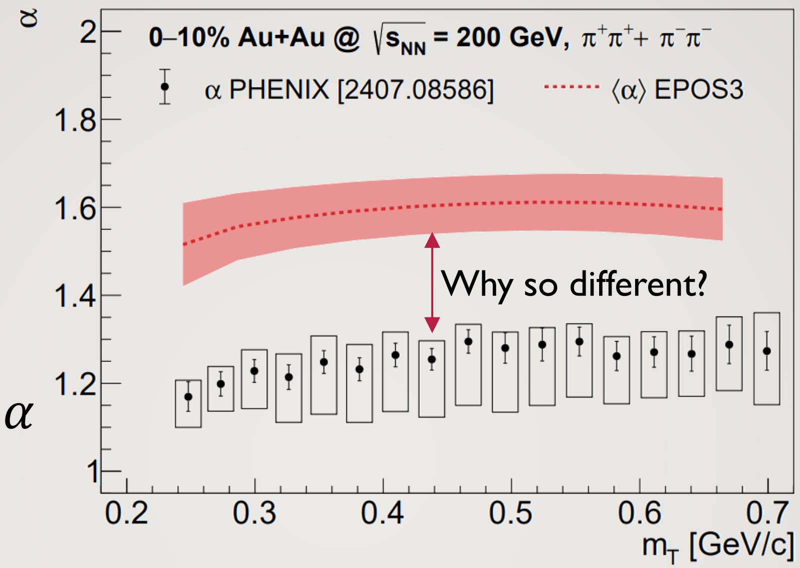


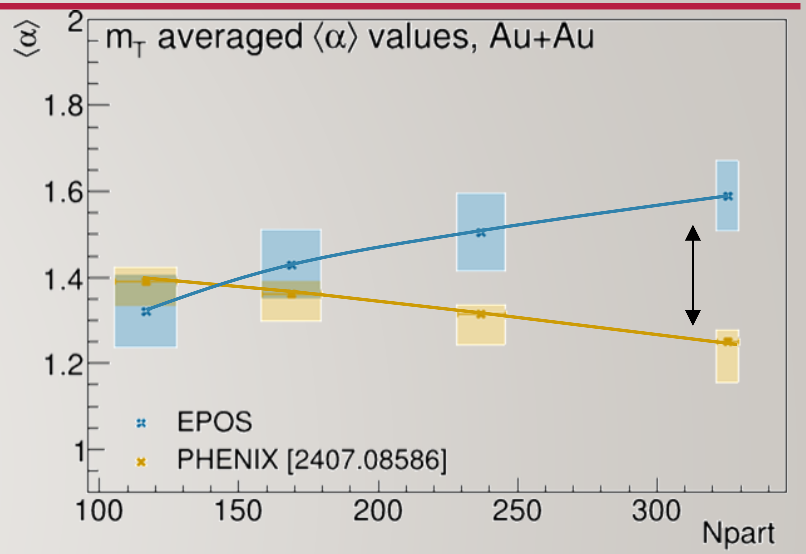


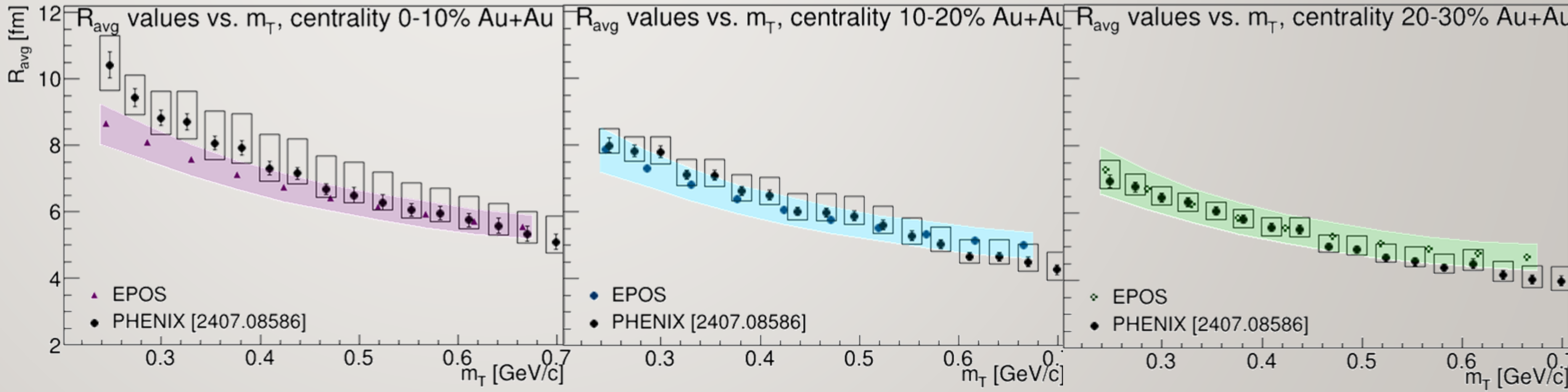
PRC110(2024)064909

DETAILED EPOS VS DATA COMPARISION AT 200 GEV

- More detailed comparison to EPOS: disagreement for α, agreement for R
- Especially for central collisions
- Denser system in EPOS: larger α
- Maybe due to more normal diffusion?
- Or long-range Coulomb scattering missing in simulations?

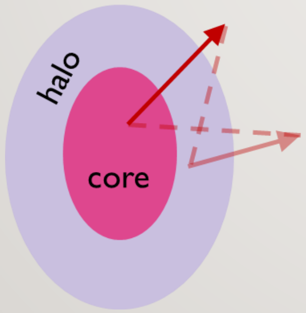


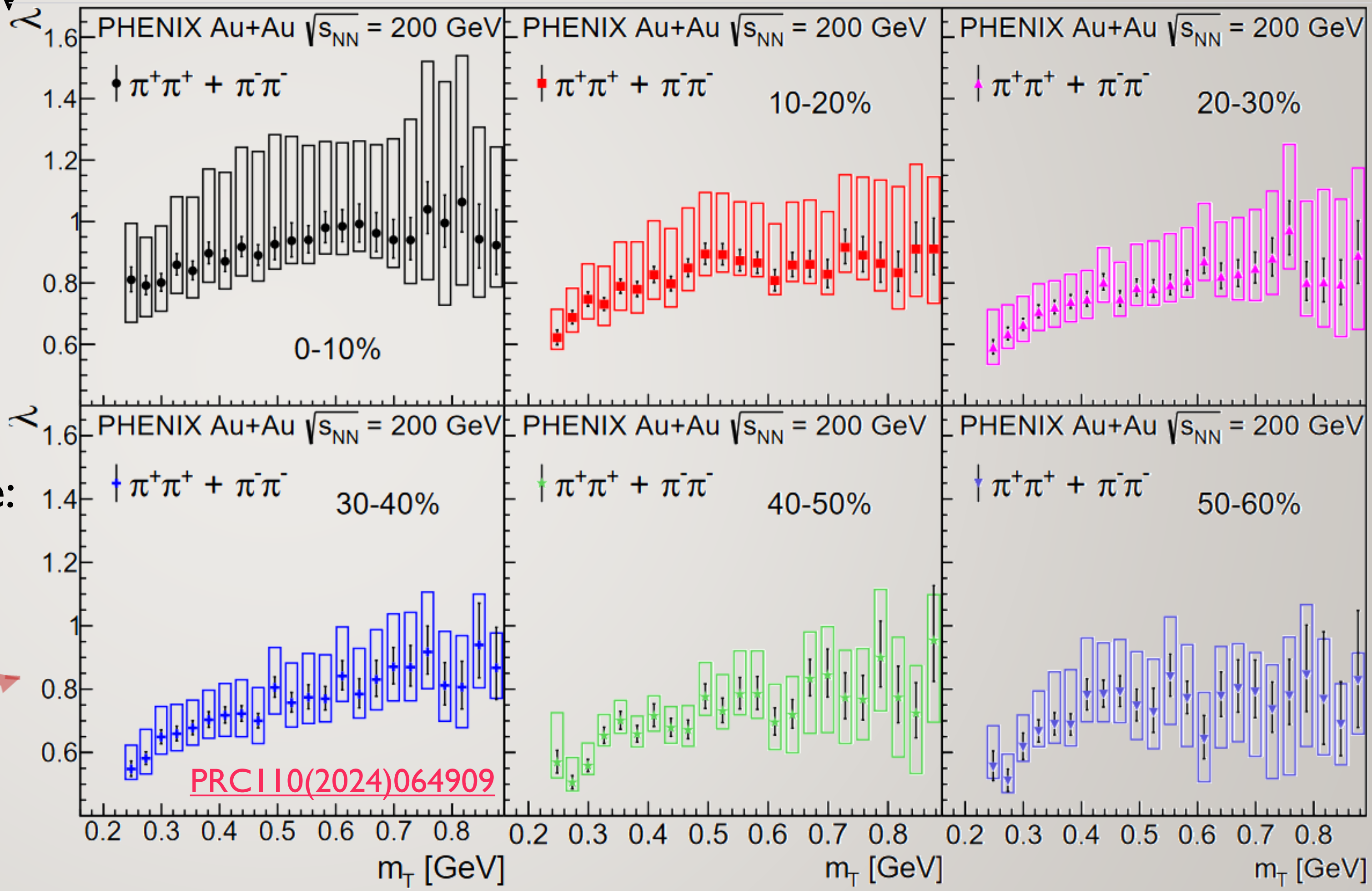




THE λ PARAMETER IN 200 GEV AU+AU AT PHENIX

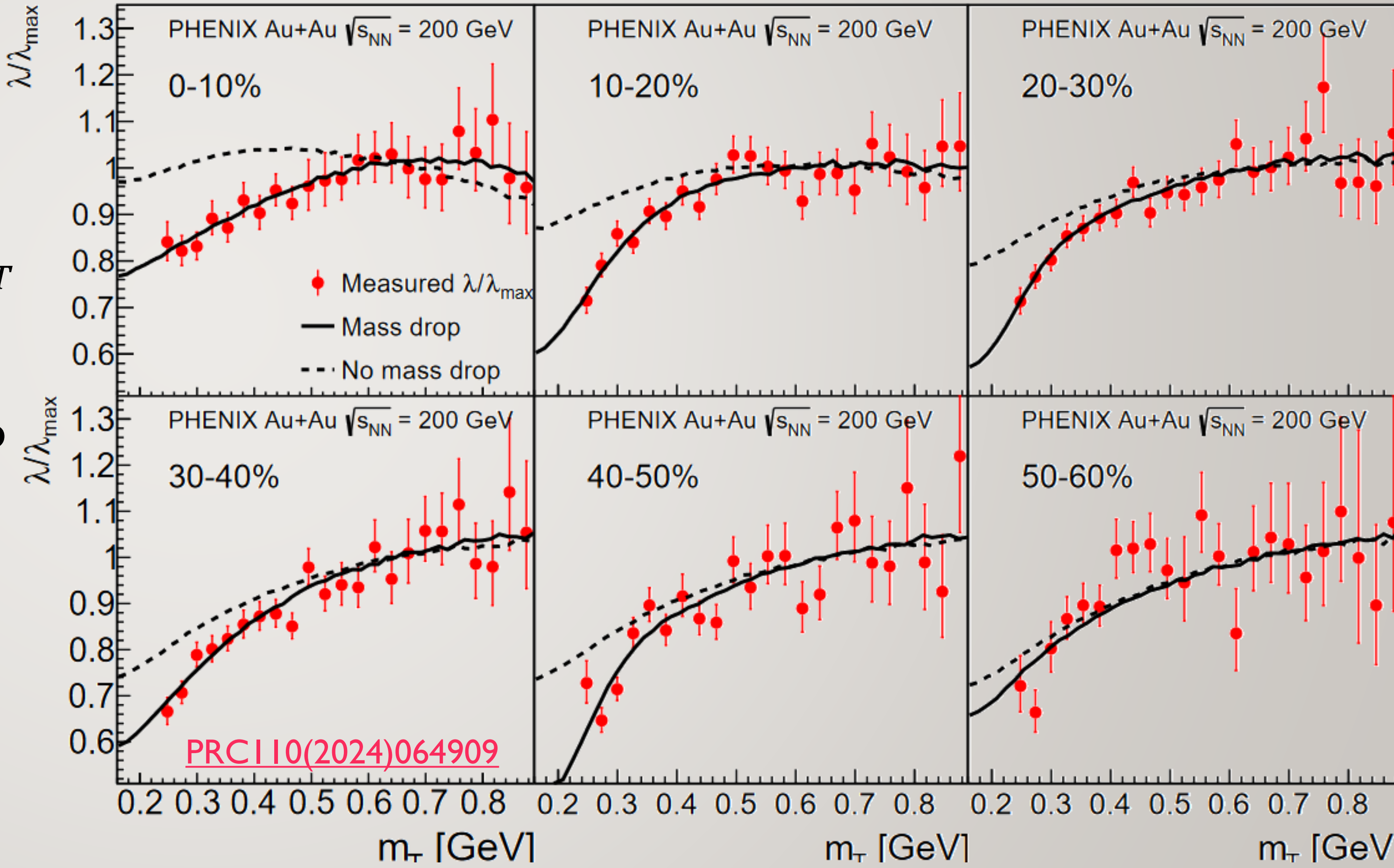
- Saturation region: $m_T \gtrsim 600$ MeV
- Large systematic uncertainties
  - Due to pair reconstruction and other experimental effects
- Can be scaled out if dividing by $\lambda_{\max} = \langle \lambda(m_T) \rangle_{m_T \text{ large}}$
- Meaning of λ in core-halo picture: $\sqrt{\lambda} = N_{\text{core}}/N_{\text{total}}$
- Measures resonance fraction among πs





RESCALED λ VS MONTE-CARLO MODELS: MASS DROP?

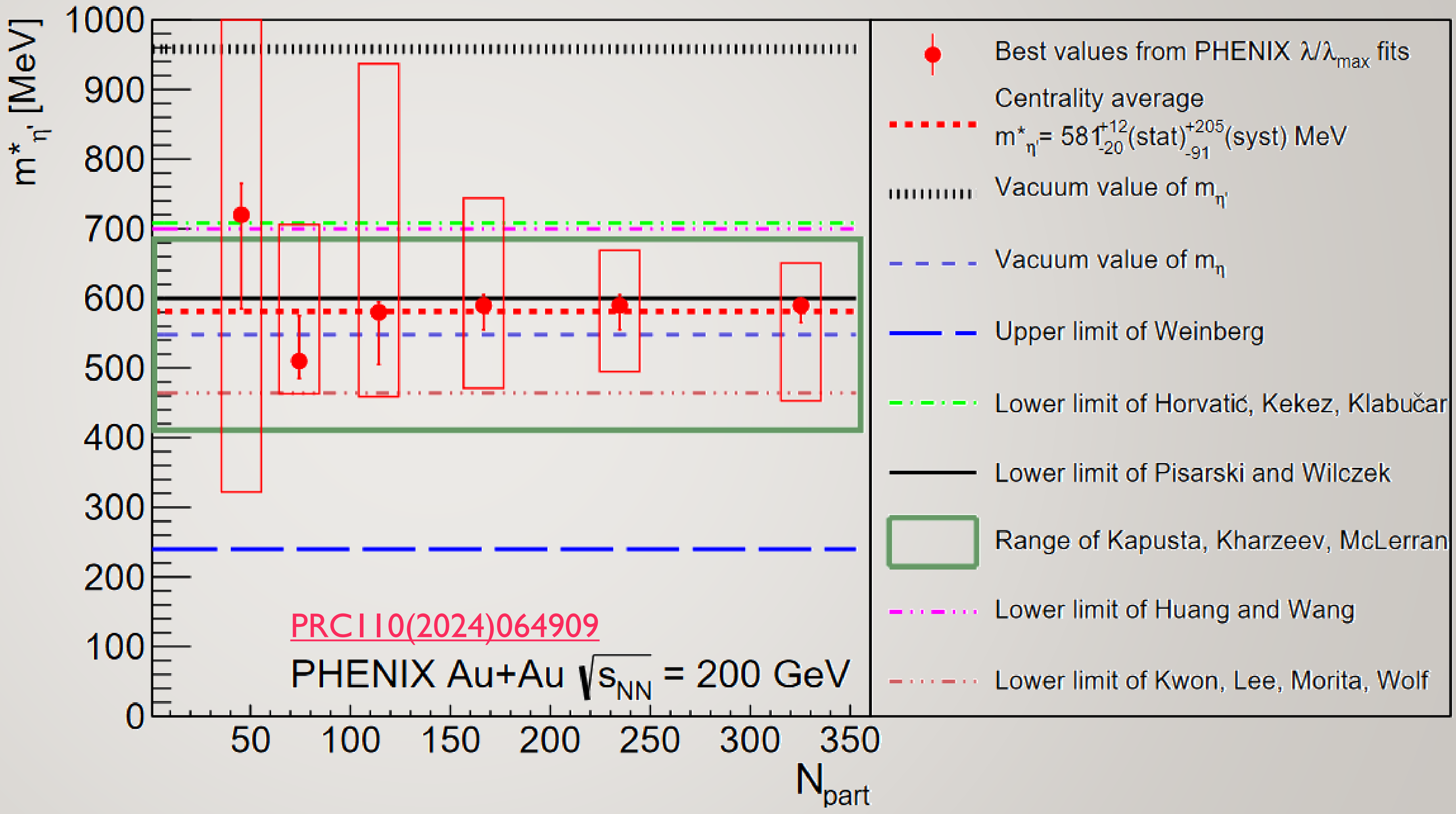
- MC Models based on thermal resonance production, λ measures primordial vs decay pion ratio
- Pions affected through decay channel $\eta' \to \eta + \pi^+ + \pi^- \to 2\pi^+ + 2\pi^- + \pi^0$
- Smaller η' mass → larger η' content → more decay $\pi^\pm\pi^\pm$ pairs at low m_T → smaller λ at low m_T
- Data incompatible with no mass drop
  - Within present framework!
- Best fitting η' mass can be found
- Model dependence studied
  - Thermal model, flow, temperature



PRC110(2024)064909

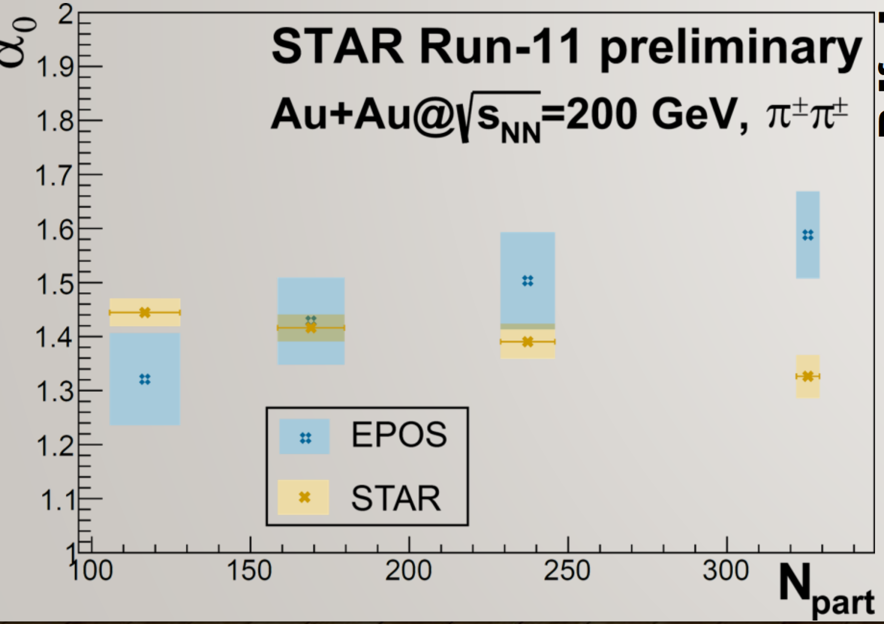
CENTRALITY DEPENDENCE OF BEST ETA’ MASS

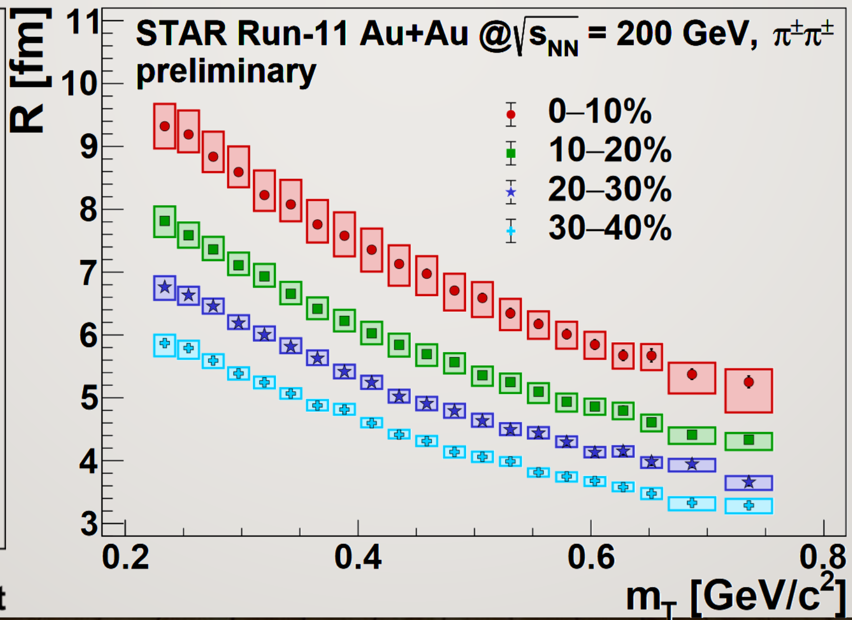
- Significant decrease in all centrality classes, except most peripheral
- Result: $m^*_{\eta'} \approx m_\eta$
- Implies a second transition?
- *„Nuclei, as heavy as bulls, through collision generate **new states** of matter”* (T. D. Lee)

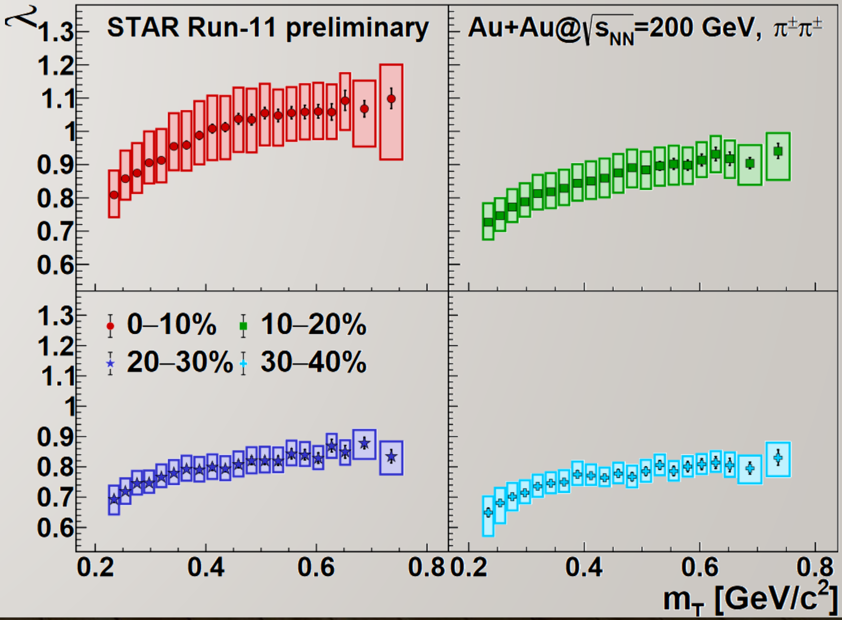


CENTRALITY DEPENDENCE AT 200 GEV WITH STAR

- Lévy scale R: decreasing trend with m_T and with centrality, similar to PHENIX results
  - Connection to flow and initial geometry, similarly to Gaussian radii
- Lévy exponent α: EPOS quantitatively close, largest discrepancy for central collisions, similar to PHENIX results
  - Effect of Coulomb scattering? PRB103(2021)235116, IJMPA40(2025)2444007
- Correlation strength λ: increase from low to high m_T and from peripheral to central collisions, similar to PHENIX results
  - m_T dependence: might attributed to modified in-medium η' mass? PRL81(1998)2205, PRL105(2010)182301, PRC110(2024)064909

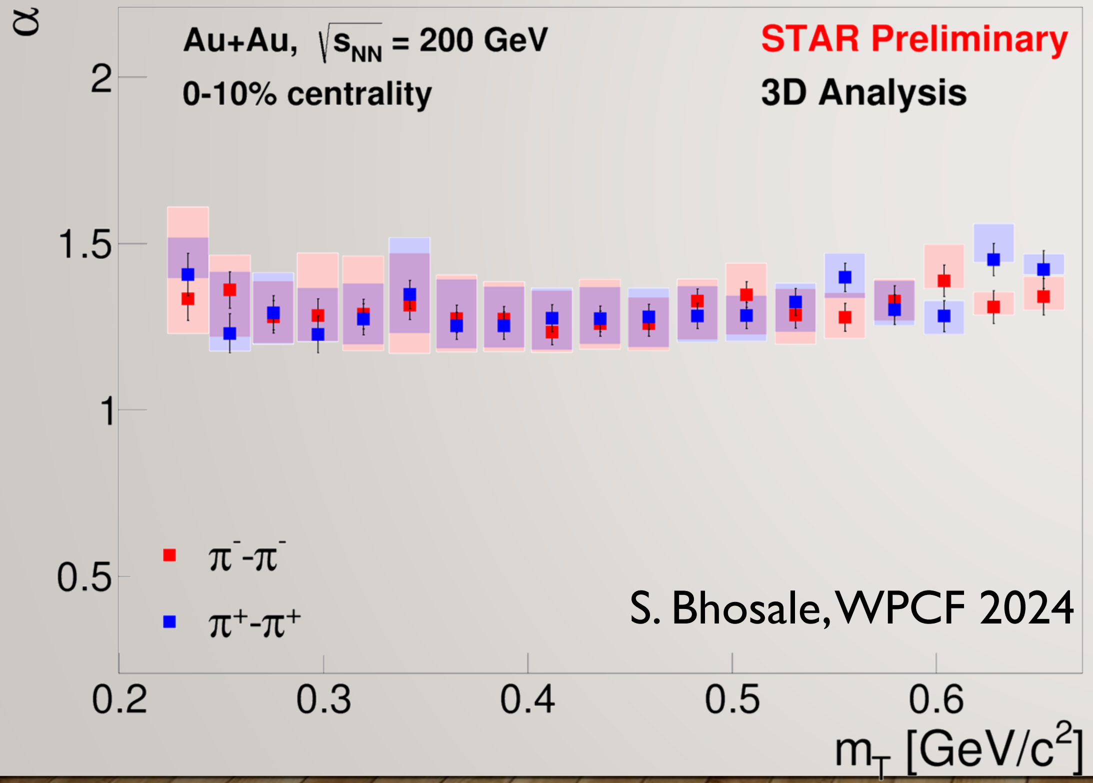


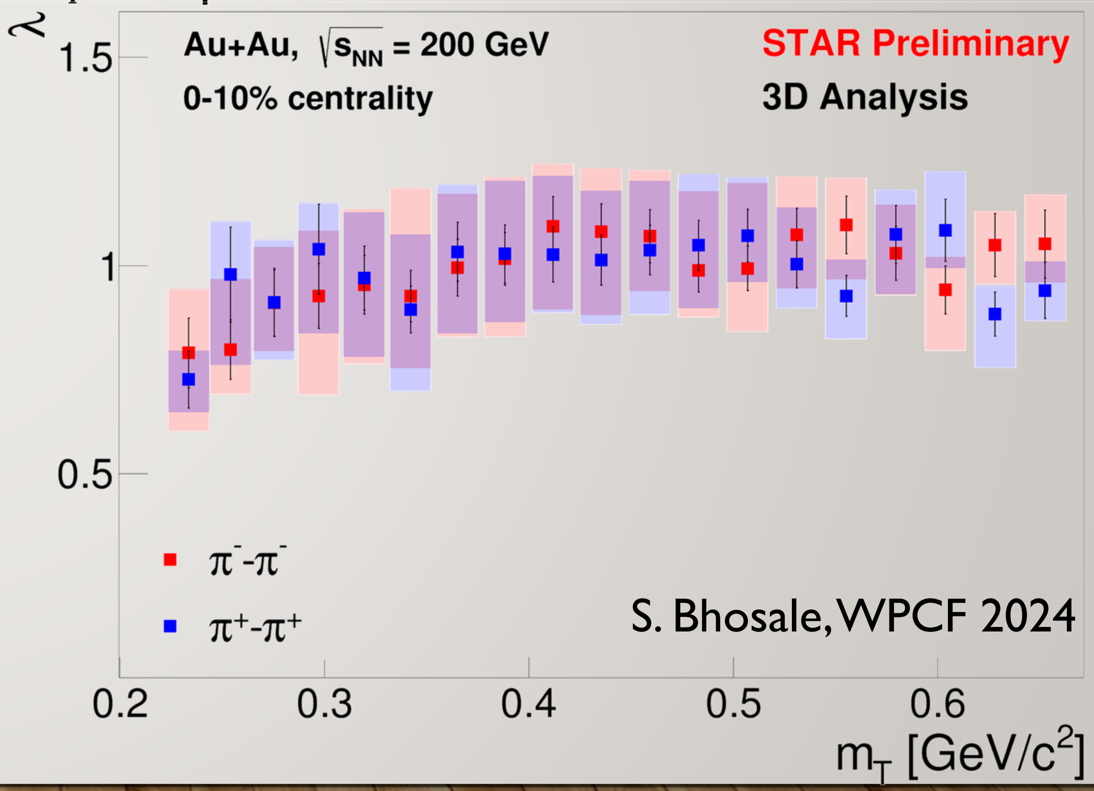




LÉVY FEMTOSCOPY IN 3D AT 200 GEV

- Anisotropic Lévy-stable distribution: $\mathcal{L}(\alpha, R; \boldsymbol{r}) = \frac{1}{2\pi} \int d^3q e^{iqr} e^{-\frac{1}{2}\left|\boldsymbol{q}\boldsymbol{R}^2\boldsymbol{q}\right|^{\alpha/2}}$, where $\boldsymbol{R}^2$: matrix of squared radii
- Lévy exponent α: negligible dependence on m_T, average value ~1.3, compatible with 1D
- Correlation strength $\boldsymbol{\lambda}$: small increase from low to high m_T, compatible with 1D





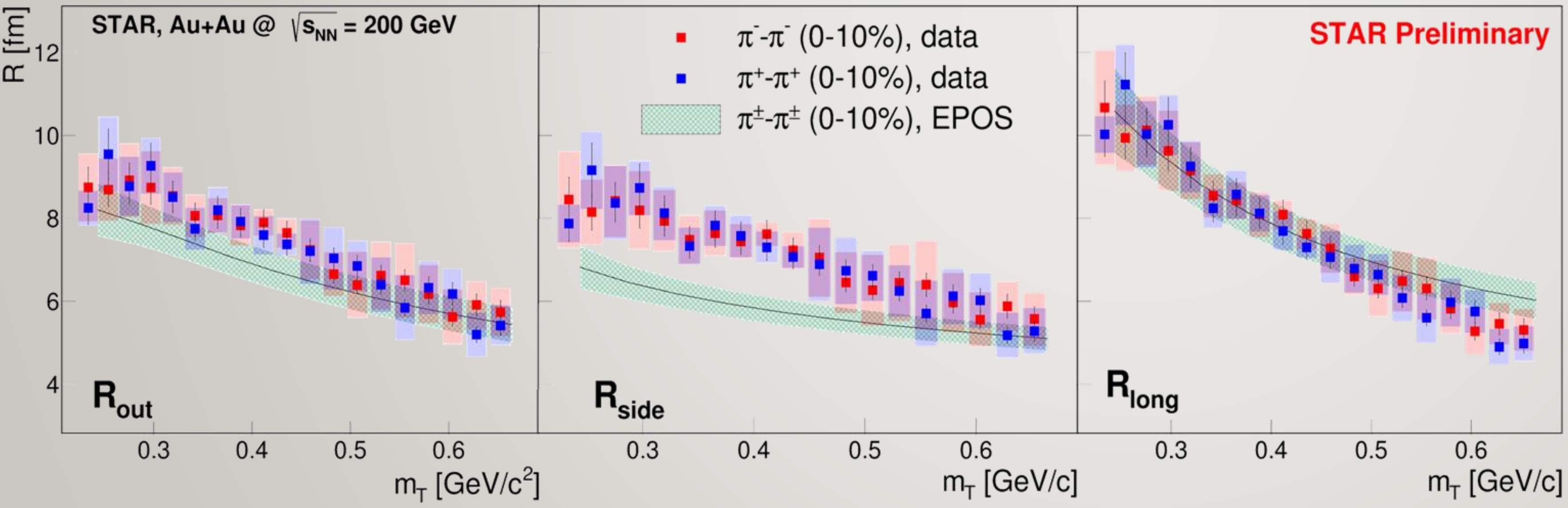
EPOS COMPARED TO STAR 3D PRELIM. DATA AT 200 GEV

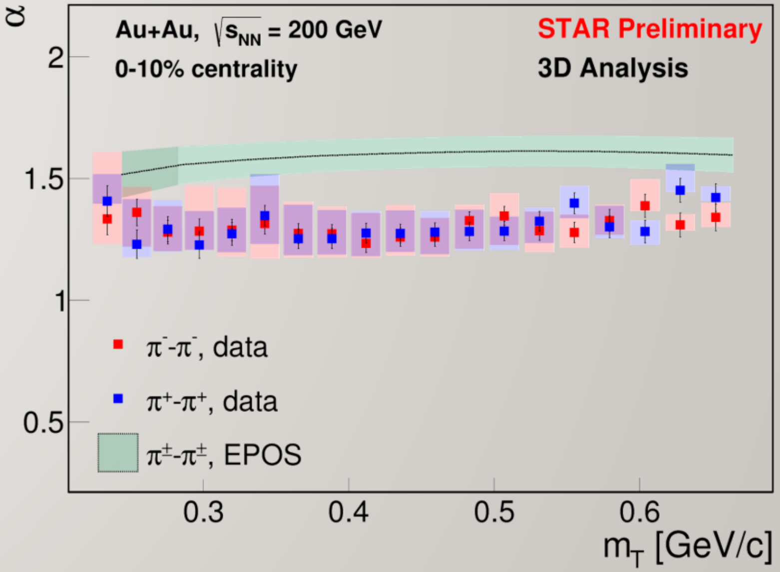
- EPOS and data (both from 3D analysis) comparison partly shows good agreement for radii
  - EPOS analysis described in Commun.Phys. 8 (2025) 1, 55
- Moderate discrepancy for $R_{\rm side}$ and α: maybe due to long-range Coulomb scattering (not in EPOS)
  - See effect of Coulomb potential in a 2D solid-state physics paper: E. I. Kiselev, Phys. Rev. B 103, 235116 (2021)

$R_{o,s,l}$ *vs* m_T

S. Bhosale, WPCF 2024

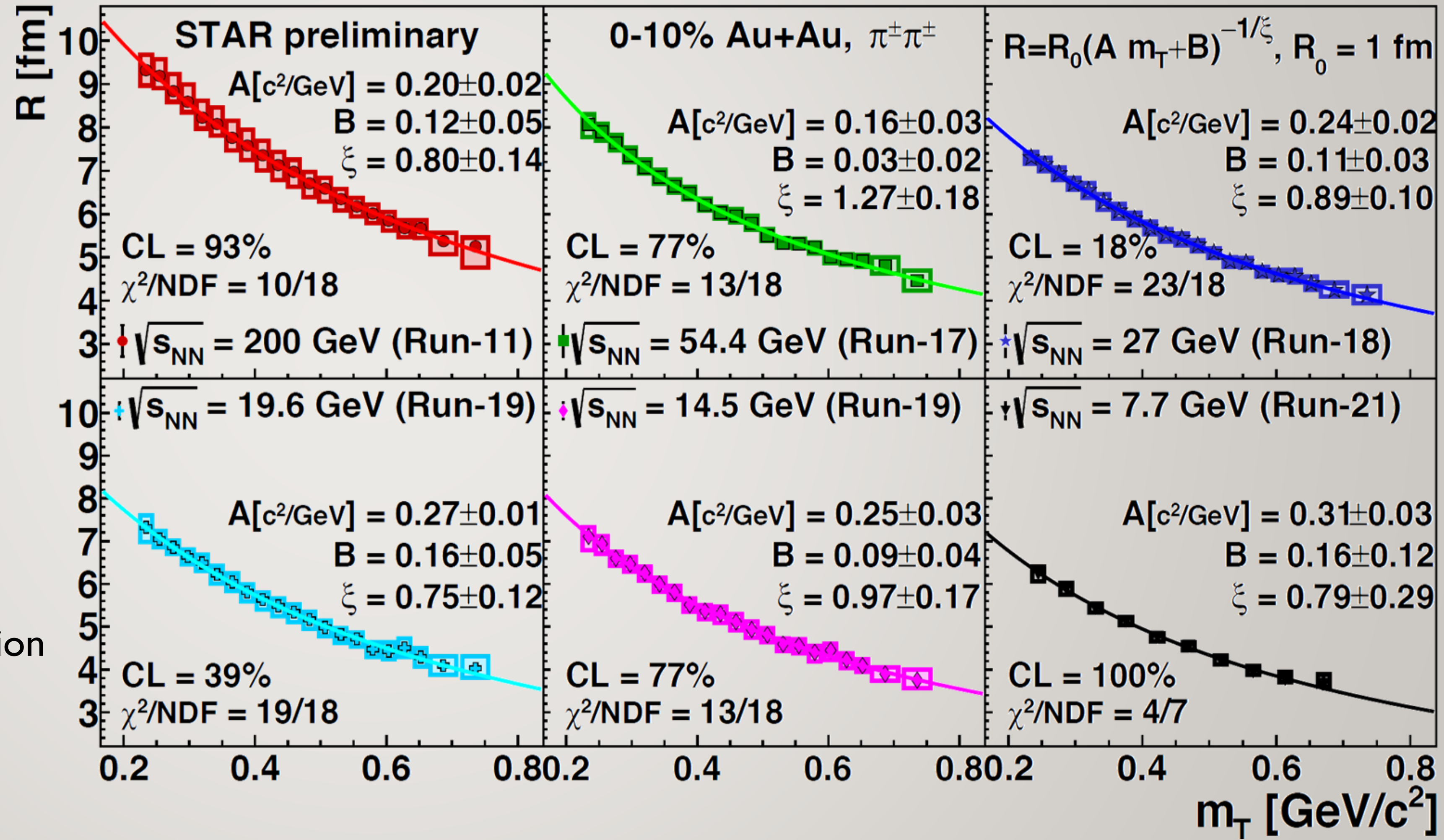
α *vs* m_T





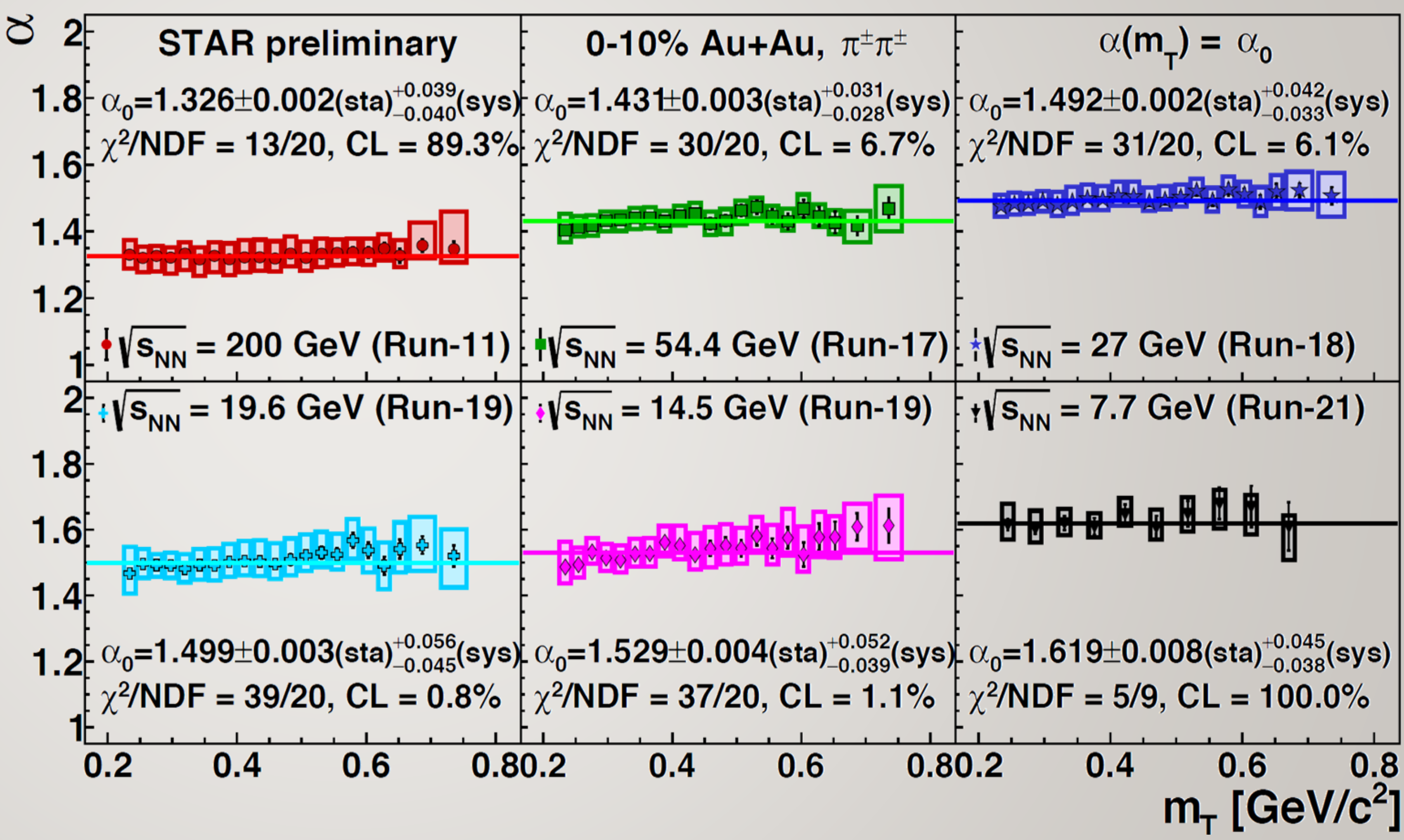
RESULTS AT COLLIDER ENERGIES DOWN TO 7.7 GEV

- What happens at lower collision energies?
- Slow decrease with $\sqrt{s_{NN}}$ from 200 to 7.7 GeV
  - Same trend as Gaussian R
- Decrease in R with m_T
  - Connection to flow
  - Not $1/\sqrt{m_T}$ like trend
  - Qualitatively similar decrease as hydro prediction



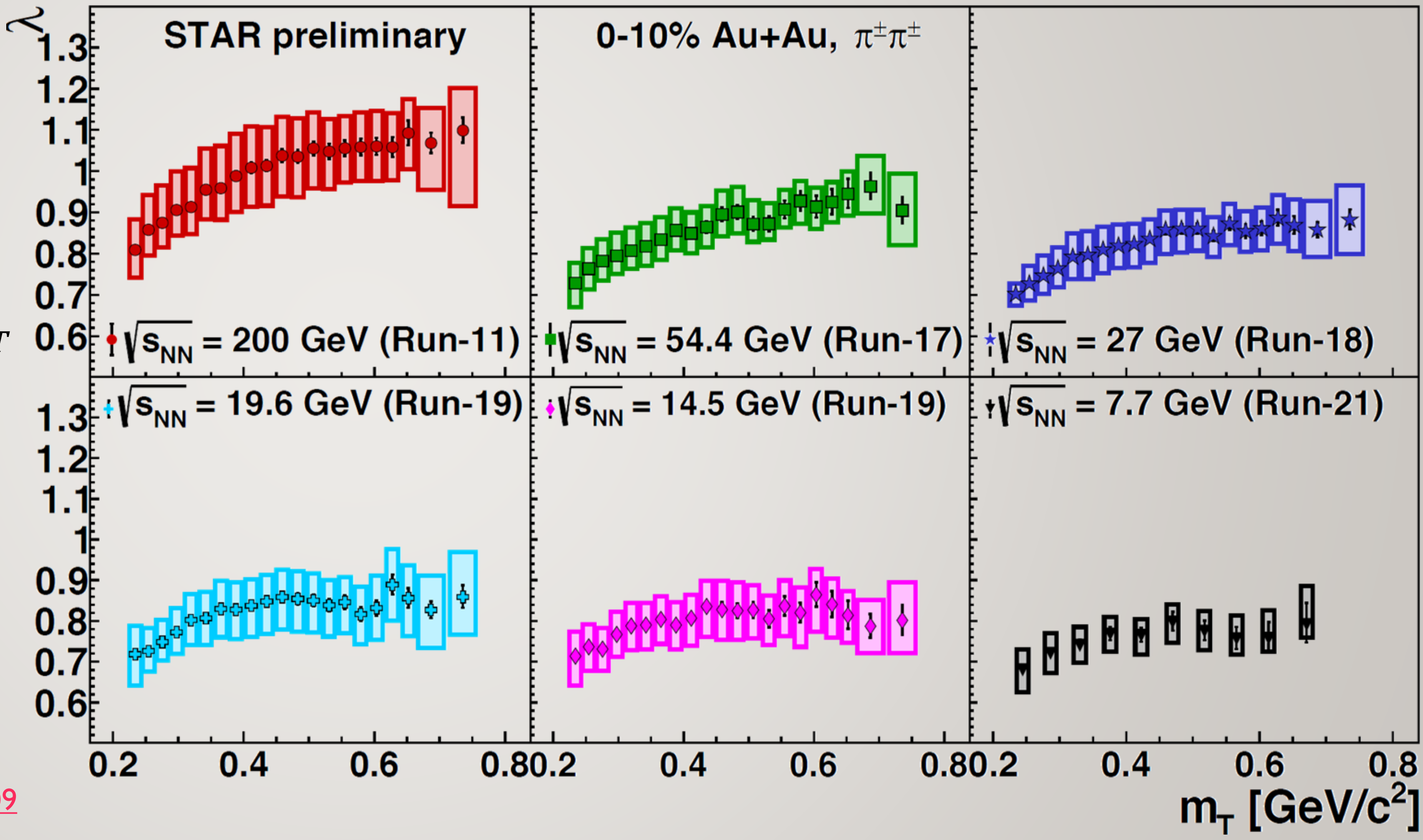
RESULTS AT COLLIDER ENERGIES: 7.7 TO 200 GEV

- No strong m_T dependence
- Average α:
  - $\approx$1.33 at 200 GeV
  - $\approx$1.62 at 7.7 GeV
- Small, smooth increase in α with $\sqrt{s_{NN}}$ from 200 to 7.7 GeV
  - Connection to decreased density? Or lifetime?
- Significantly below 2.0 and above 1.0 everywhere



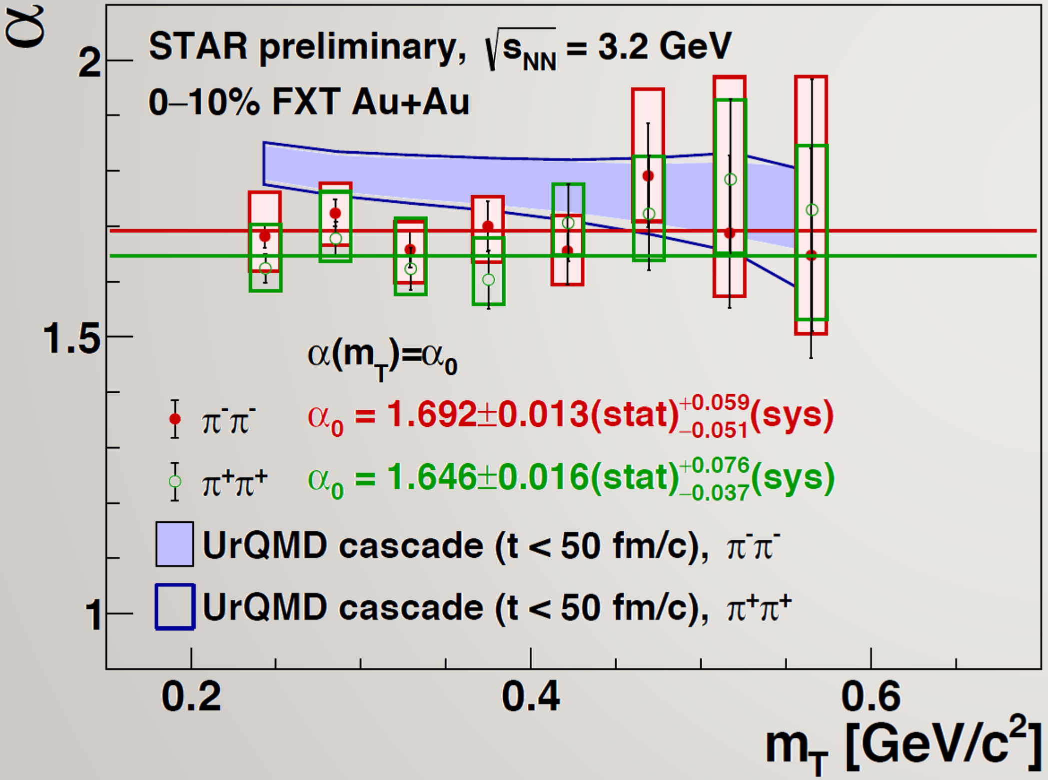
RESULTS AT COLLIDER ENERGIES: 7.7 TO 200 GEV

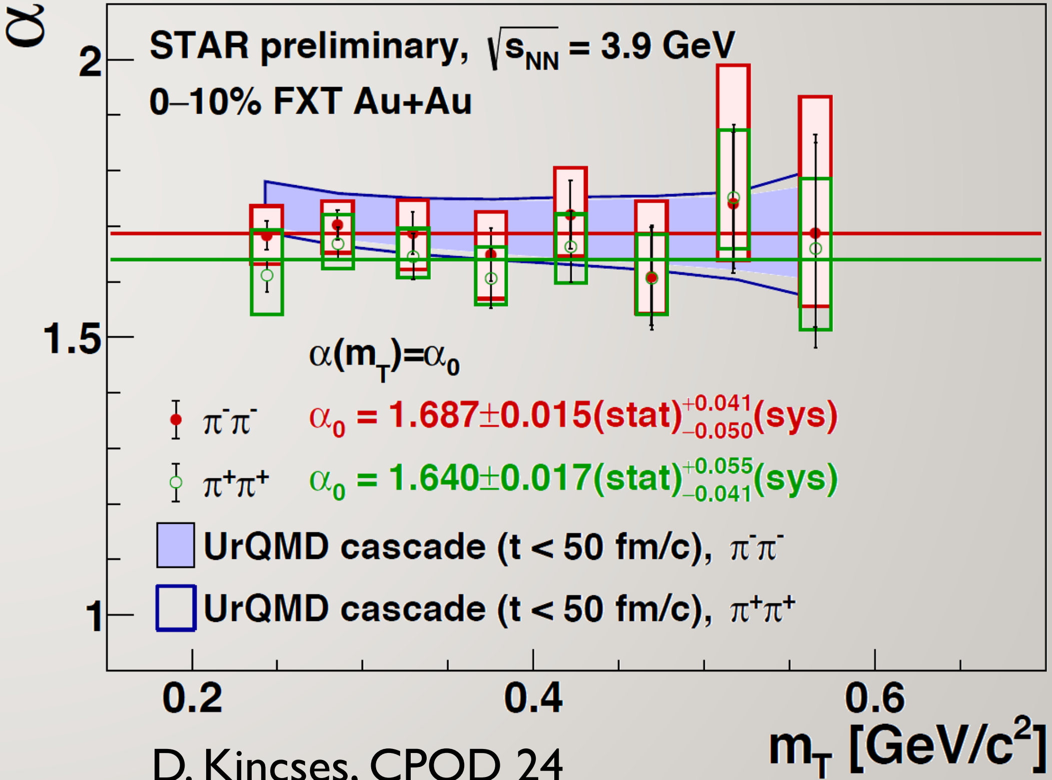
- Clear decrease in λ with $\sqrt{s_{NN}}$ from 200 to 7.7 GeV
  - Decrease in multiplicity
  - Larger role of halo
- Decrease towards small m_T
  - Increase in halo for small m_T
  - Attributed to **modified in-medium η' mass** and $U_A(1)$ restoration in the literature
    Vance, Csörgő, Kharzeev, PRL81(1998)2205
    & PHENIX, PRC110(2024)064909



FIXED TARGET ENERGIES: 3.2 AND 3.9 GEV

- Non-Gaussian values ($\alpha < 2$); small systematic difference between $\pi^-\pi^-$ and $\pi^+\pi^+$ pairs
- 3.9 and 3.2 GeV compatible with each other, no m_T dependence observed
- UrQMD within uncertainties – no other effect but rescattering and decays, good agreement (t<50 fm/c!)

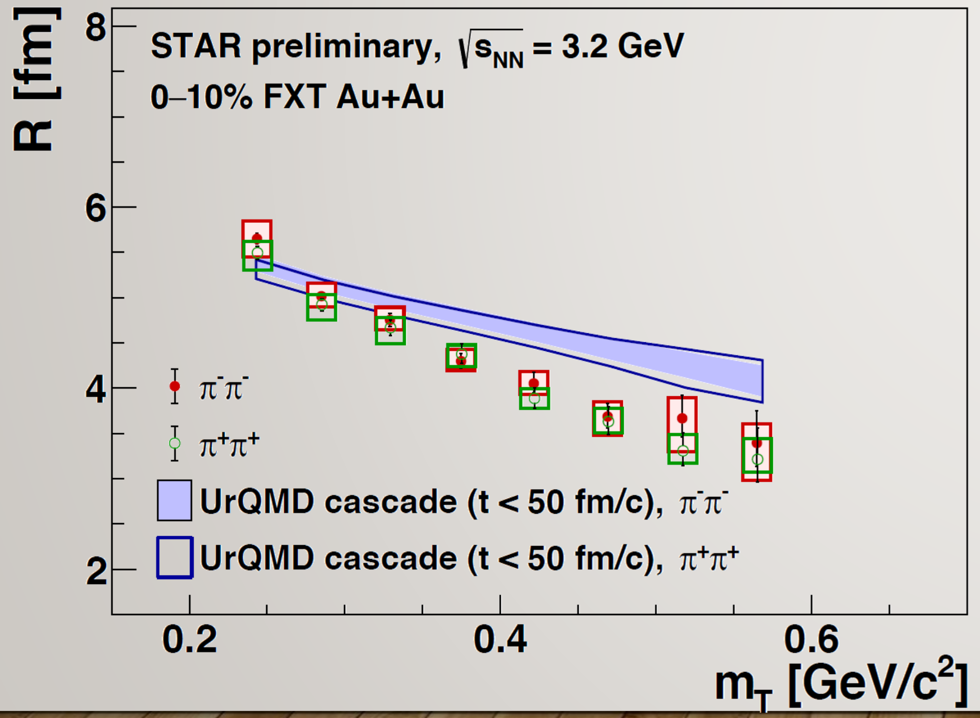


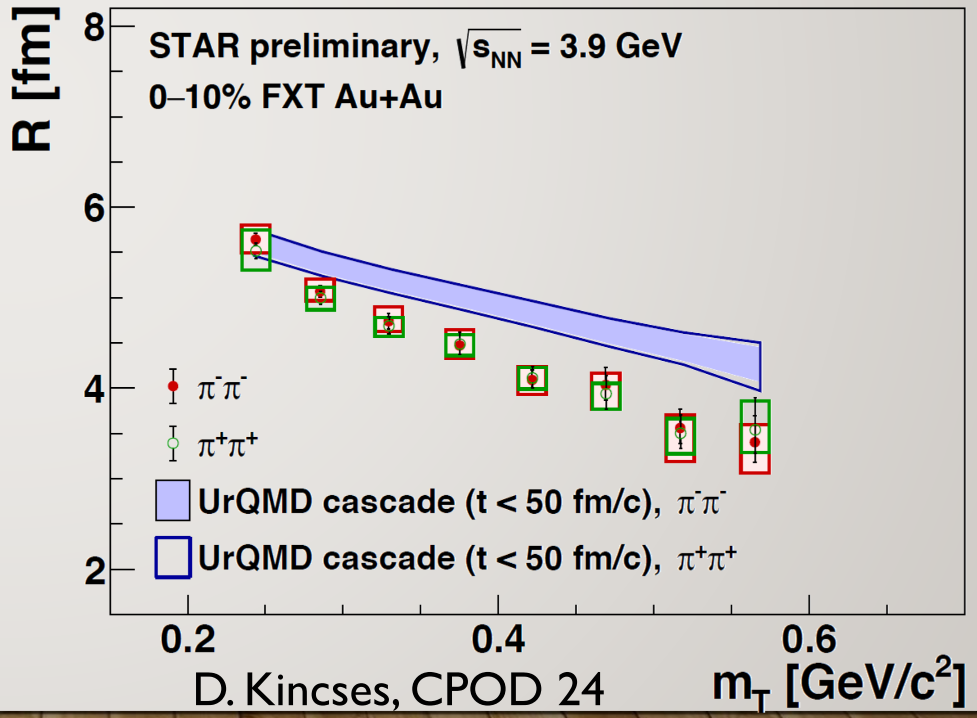


D. Kincses, CPOD 24

LÉVY SCALE R AT FXT ENERGIES

- Decreases towards higher m_T and lower energies
- Small systematic difference between $\pi^-\pi^-$ and $\pi^+\pi^+$ pairs
- Two FXT energies compatible
- UrQMD describes the trends qualitatively well, moderate quantitative mismatch, but ran only until 50 fm/c



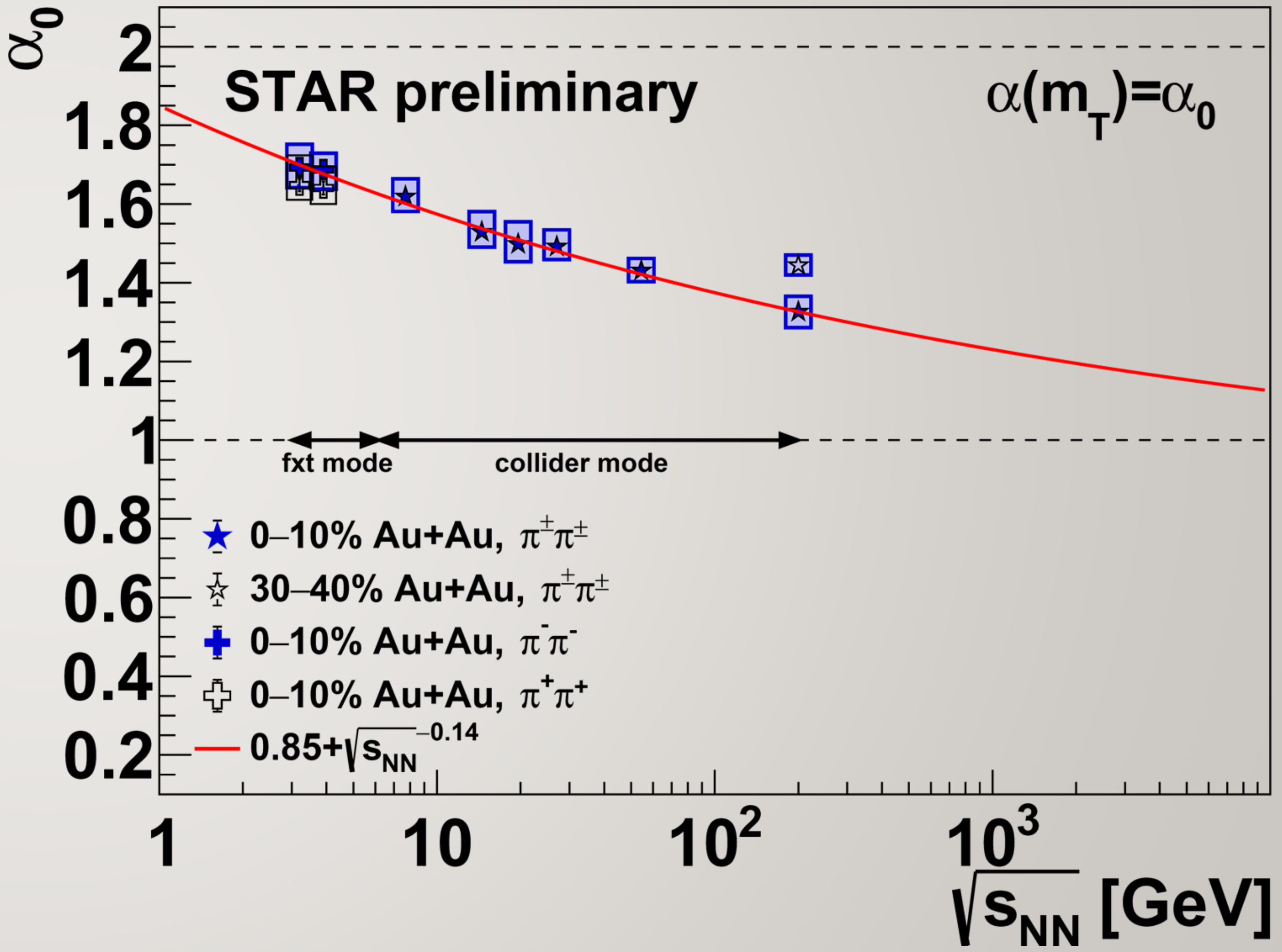


D. Kincses, CPOD 24

LÉVY EXPONENT FROM 3.2 TO 200 GEV

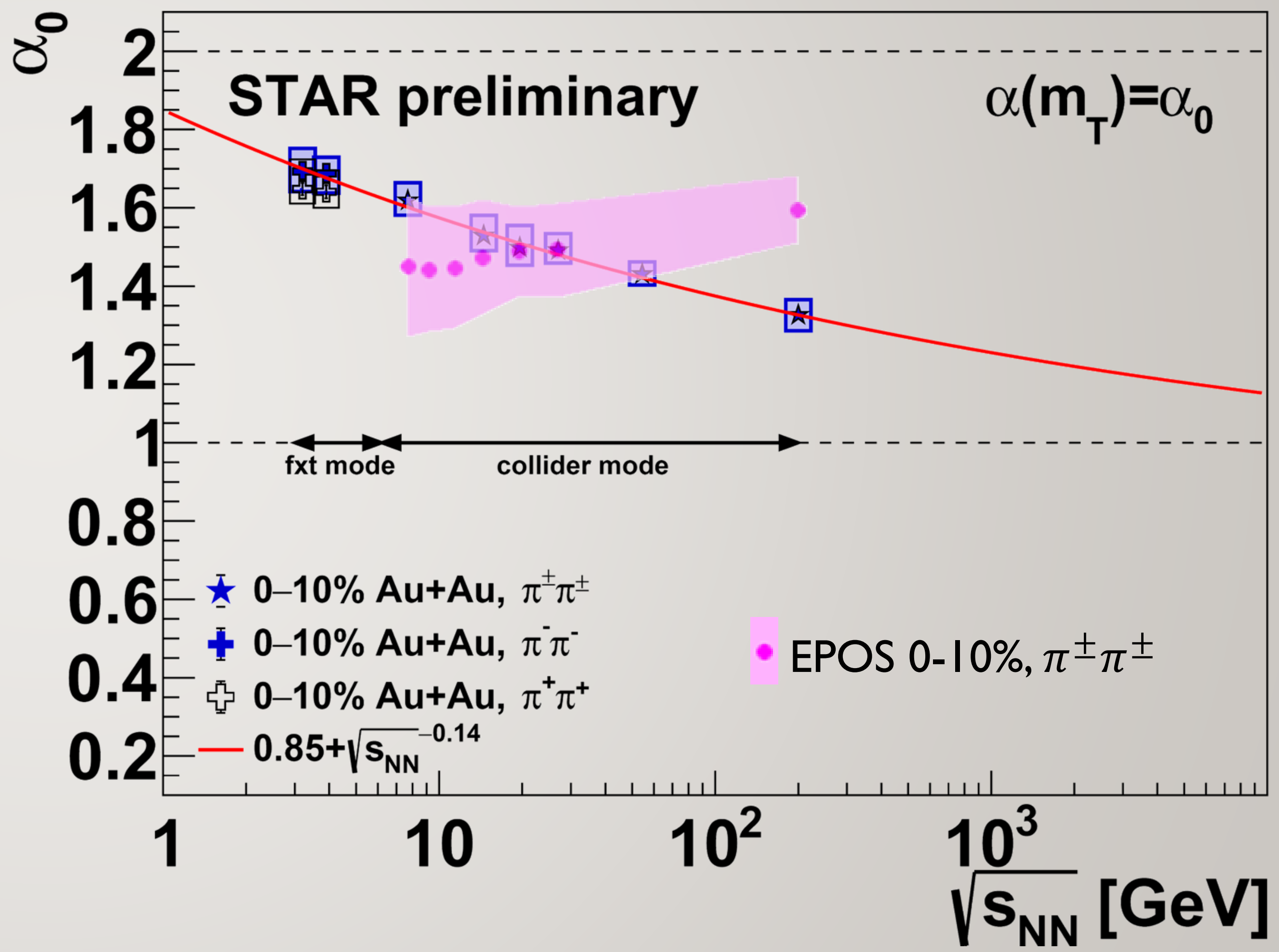
- Non-gaussian values ($\alpha \ll 2$)
- Increasing density for larger $\sqrt{s_{NN}} \rightarrow$ rescattering decreases α?
- 200 GeV centrality dependence, same trend:
  - Larger α for peripheral collisions
- Trend illustrated by power-law:
  $\alpha_0 \approx 0.85 + \sqrt{s_{NN}}^{-0.14}$
- No non-monotonic trend in α observed yet, far from conjectured critical value (0.5)
- What do Monte-Carlo models say?

D. Kincses, CPOD 24



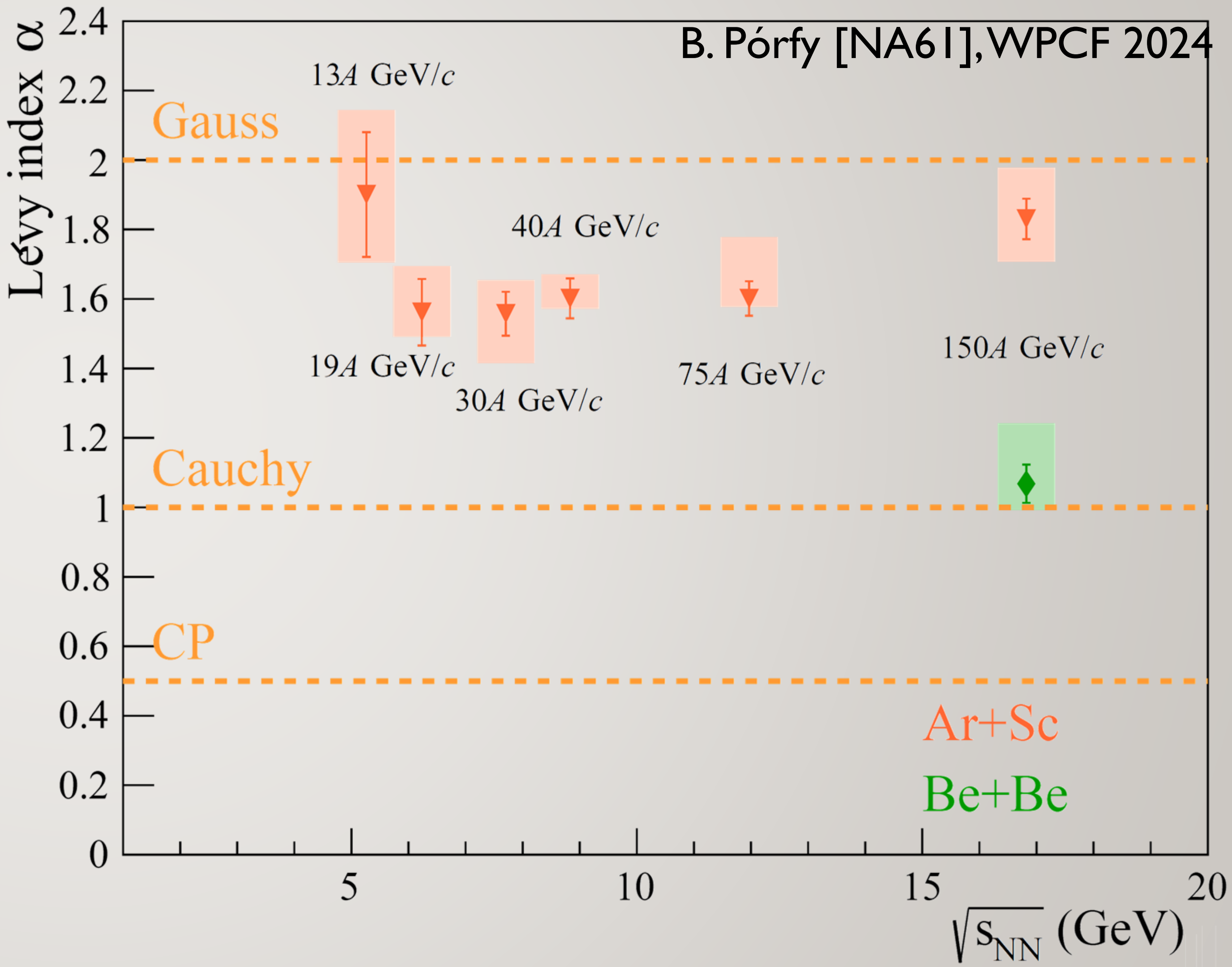
WHAT DOES EPOS SAY?

- Quantitatively agrees for 14-30 GeV
- Disagreement at 7.7 and 200 GeV
  - Band: event-by-event shape variance
- Trend qualitatively different
- Recall: centrality trend was also opposite
- Data: α **decreases** with multiplicity
  - Same for centrality and $\sqrt{s_{NN}}$
- EPOS: α **increases** with multiplicity
  - Same for centrality and $\sqrt{s_{NN}}$
- Maybe due to long-range Coulomb missing?



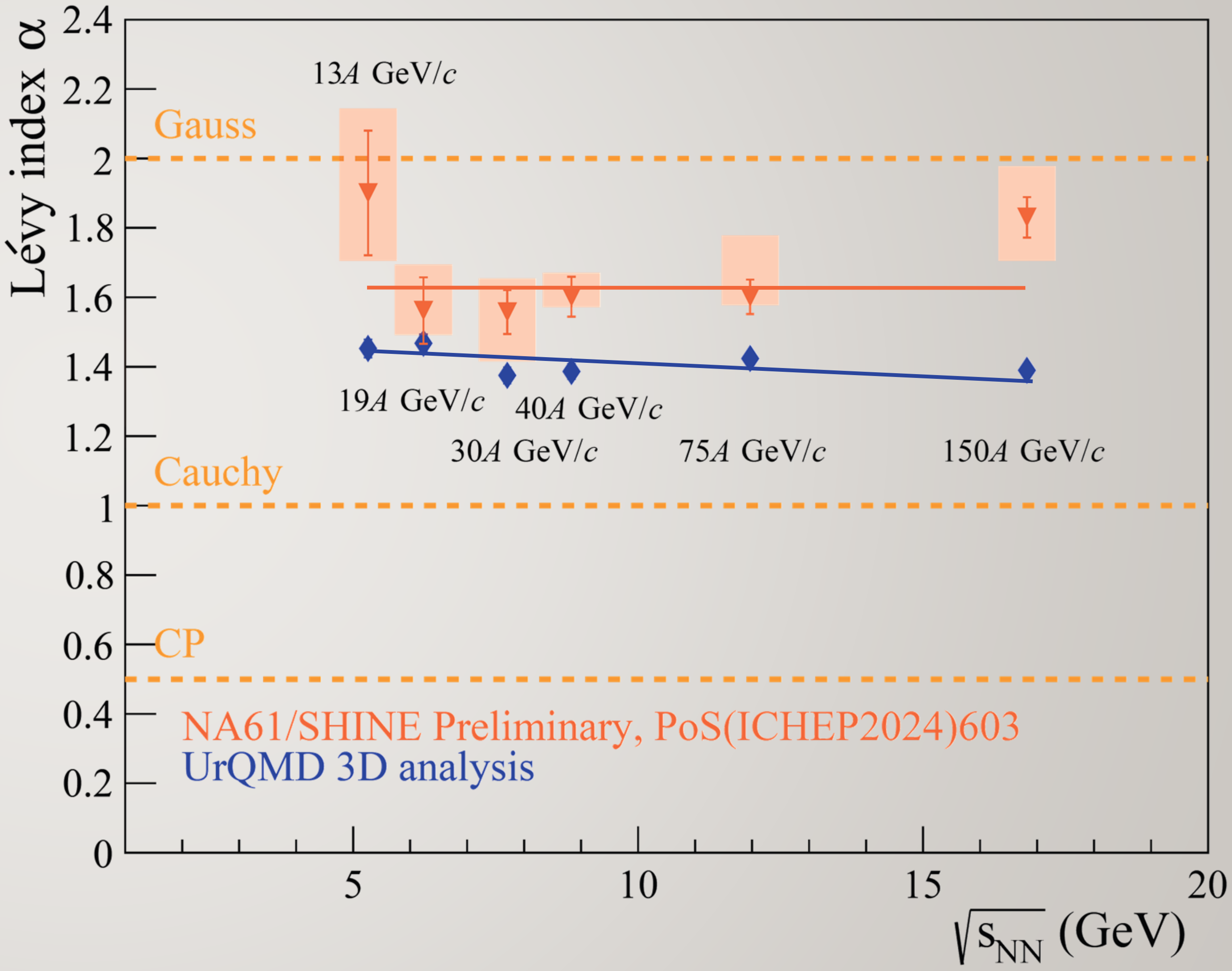
NA61/SHINE RESULTS

- At 150 AGeV: α(Be+Be) < α(Ar+Sc)
  - Corresponds to $\sqrt{s_{NN}} \approx 16.8$ GeV
- Interesting trend of α for smaller energies in Ar+Sc
  - (not incompatible with constant)
- Next step: Xe+La, 3D analysis
- General findings (not shown here)
  - $\alpha(m_T)$ approximately constant
  - $R(m_T)$ shows sign of flow
  - $\lambda(m_T)$ shows no „hole" at low m_T
  - Compare to RHIC energies



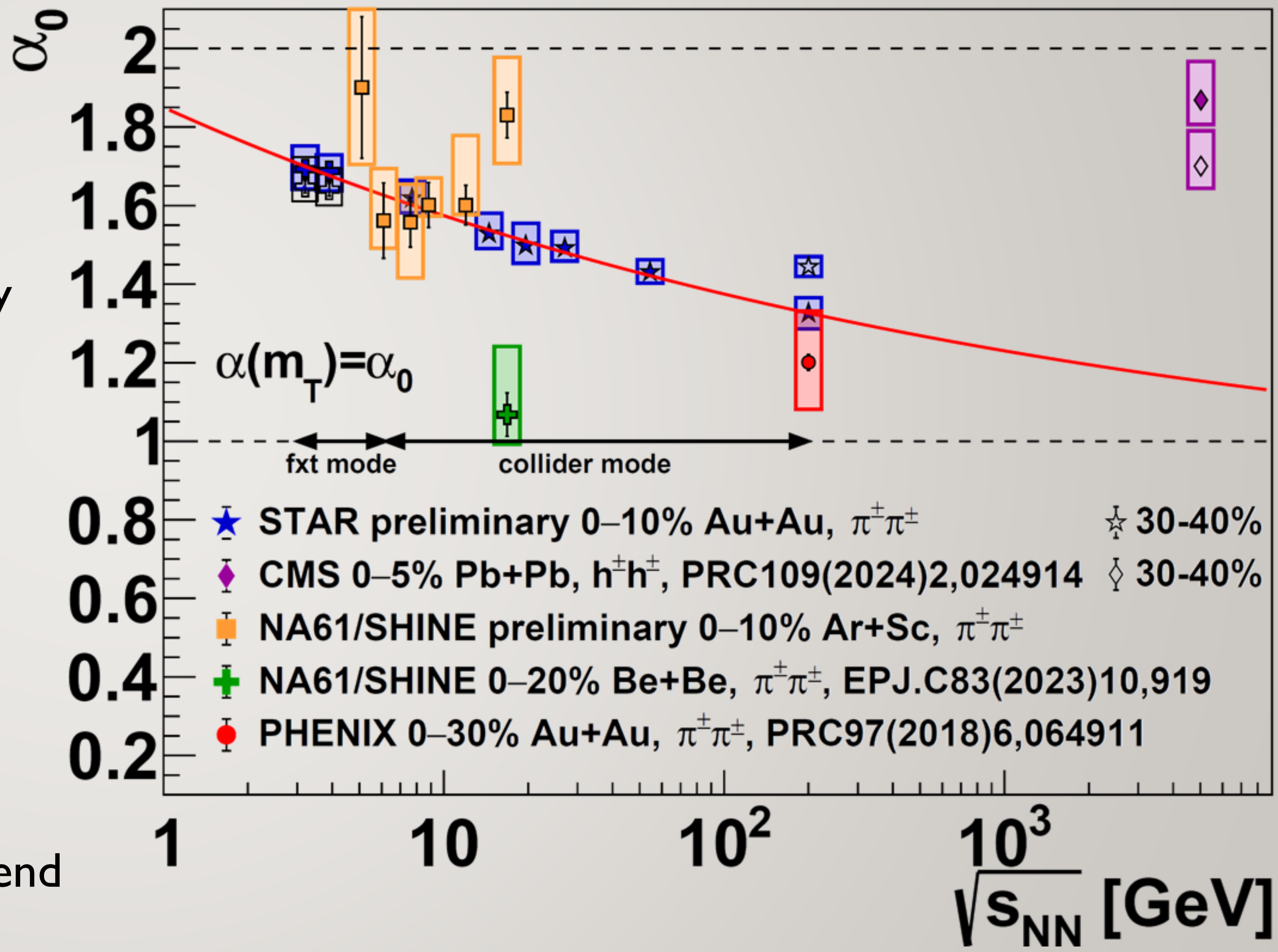
URQMD AT NA61 ENERGIES

- Quantitatively not very far from the data for $\sqrt{s_{NN}} = 6$ to 10 GeV
- Larger differences at 13 and 150 AGeV
- Seemingly different UrQMD trend compared to data
- Next analysis in Xe+La system might provide smaller uncertainties



LÉVY EXPONENT FROM 3.2 GEV TO 5 TEV

- Non-gaussian values ($\alpha \ll 2$)
- 200 GeV centrality dependence: smaller α for central collisions
- Same trend with energy: increasing density → decreased α: more time for Lévy walk?
- RHIC trend described by power-law:
  $\alpha_0 \approx 0.85 + \sqrt{s_{NN}}^{-0.14}$
- CMS result at 5 TeV: off the RHIC trend
  - Opposite centrality dependence: smaller α for peripheral collisions
- SPS: interesting, almost non-monotonic trend

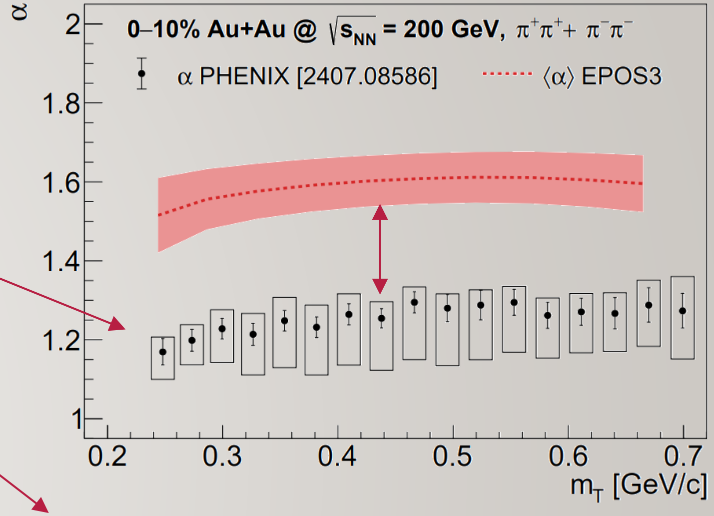


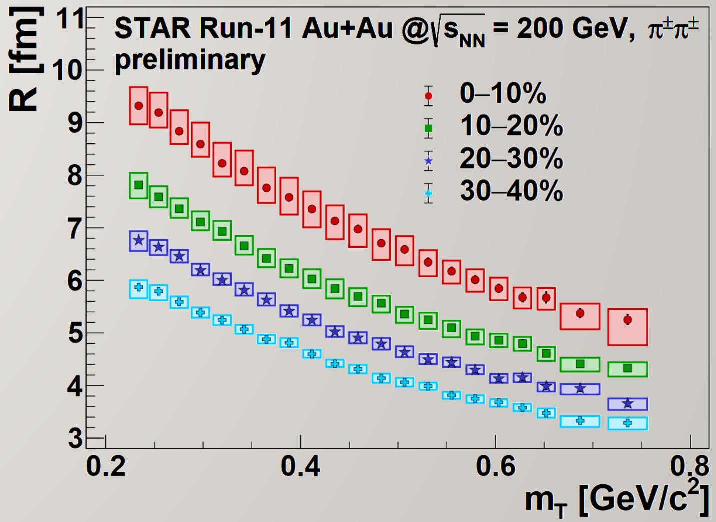
CONTENTS

- Ideas
  - Lévy walk, femtoscopy, simulations
- Facts
  - Measurements, comparisons
- Questions
  - How to understand all this?

HOW TO RECONCILE HYDRO HBT & LÉVY WALK?

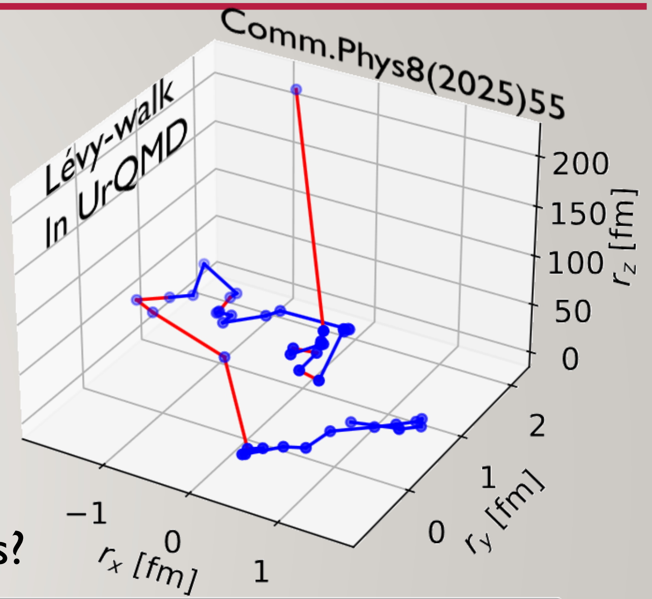
- Experimental observations:
  1. Lévy-stable source shapes, far from Gaussian ($\alpha < 2$)
  2. Radii (Lévy scales) follow hydro prediction ($R \sim 1/\sqrt{m_t}$)
- Simulation results:
  1. Hadronic scattering & decay (altogether: Lévy walk) create Lévy-stable source, modifies source size & shape
  2. Radii (Lévy scales) follow hydro prediction ($R \sim 1/\sqrt{m_t}$) and experiment
  3. Results on Lévy exponent (α) significantly differ from experiment
- How to reconcile? What do HBT radii mean if source is distorted after hydro phase?
- Experimental side: measure particle-type dependence!
- Phenomenology side: can hydro contribute to power-law tails?

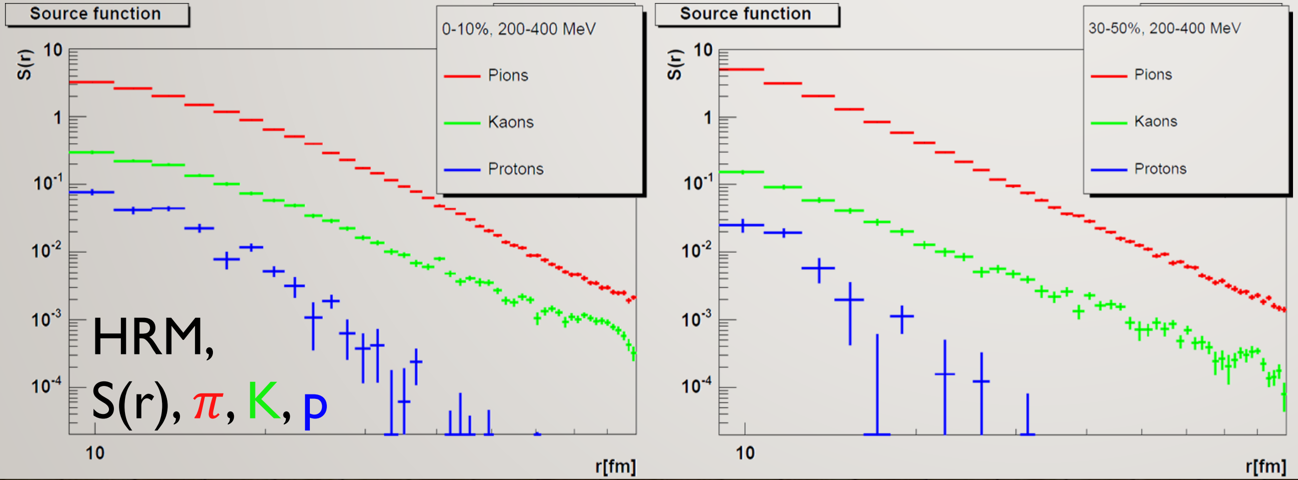


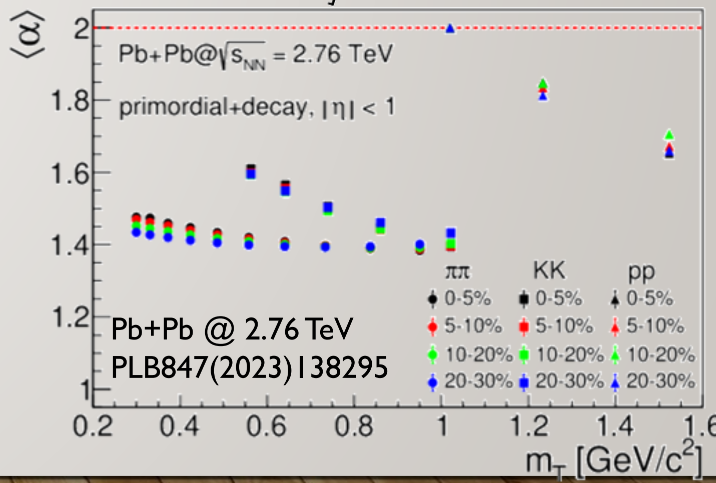


WHEN DO THE POWER-LAW TAILS FORM?

- Based on EPOS: apparently Gaussian in hydro phase
- Power-law tails due to Lévy walk: scattering processes
  - 2-by-2, decay, coalescence, all add up to a Lévy walk
- How to test? Particle type dependence!
  - Based on elastic cross-sections: $\alpha(p) > \alpha(\pi) > \alpha(K)$
    Humanic, IJMPE15(2006)197, Csanád, Csörgő, Nagy, BJP37(2007)1002
  - Not confirmed by an EPOS LHC analysis! Role of decays and inelastic collisions?

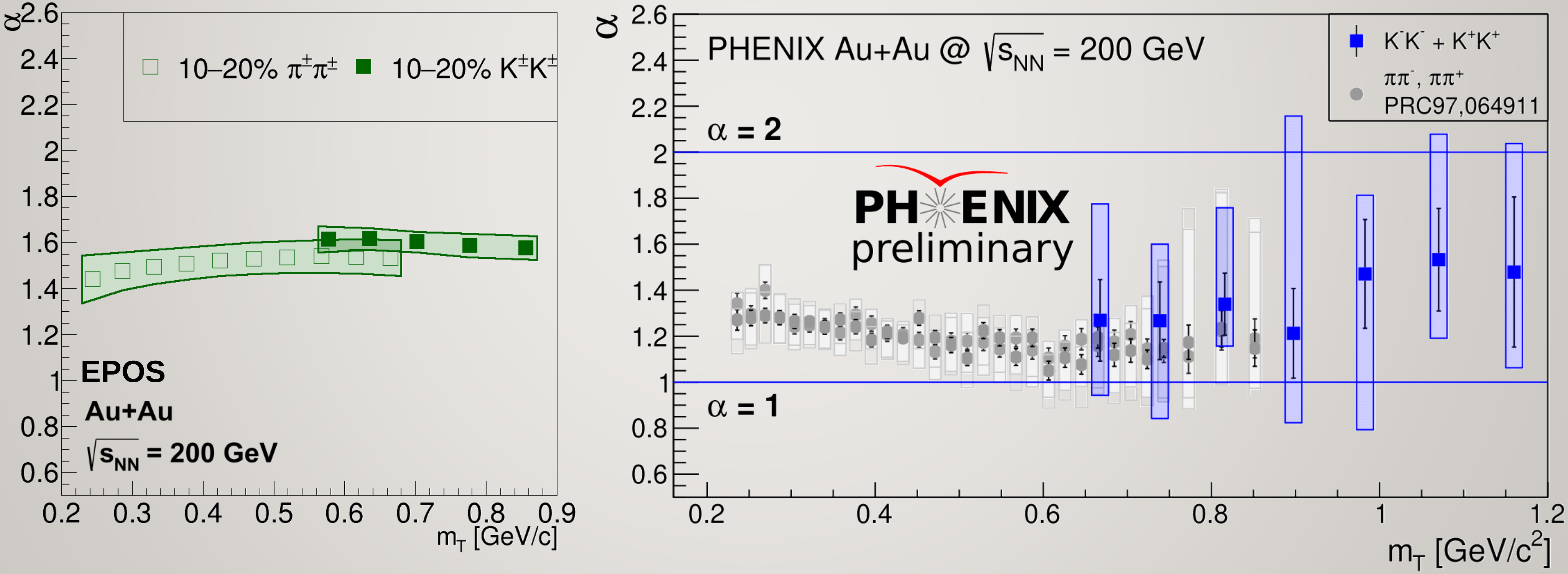






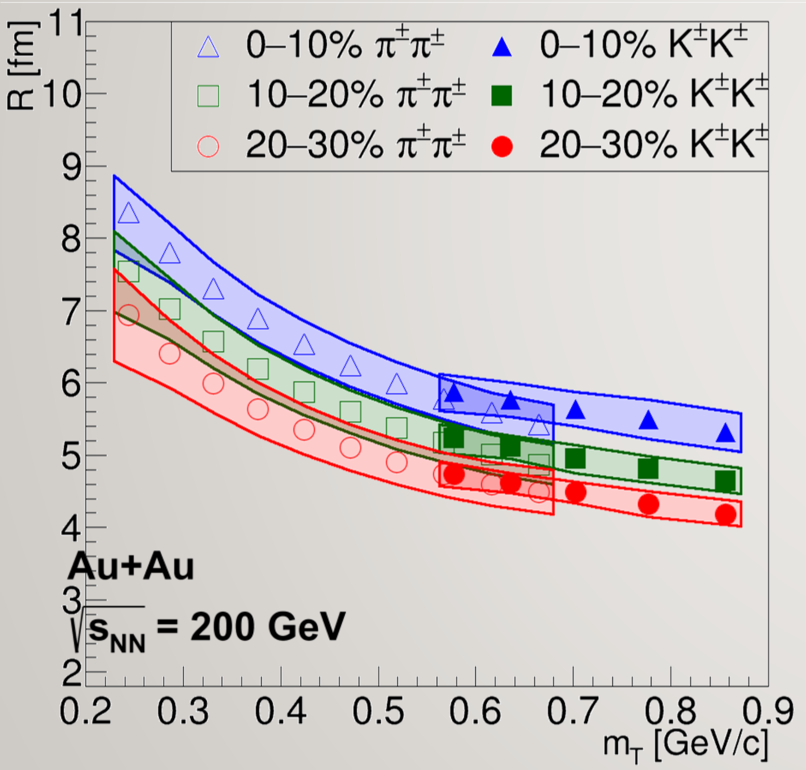
PARTICLE SPECIES COMPARISON, DATA VS EPOS, LÉVY α

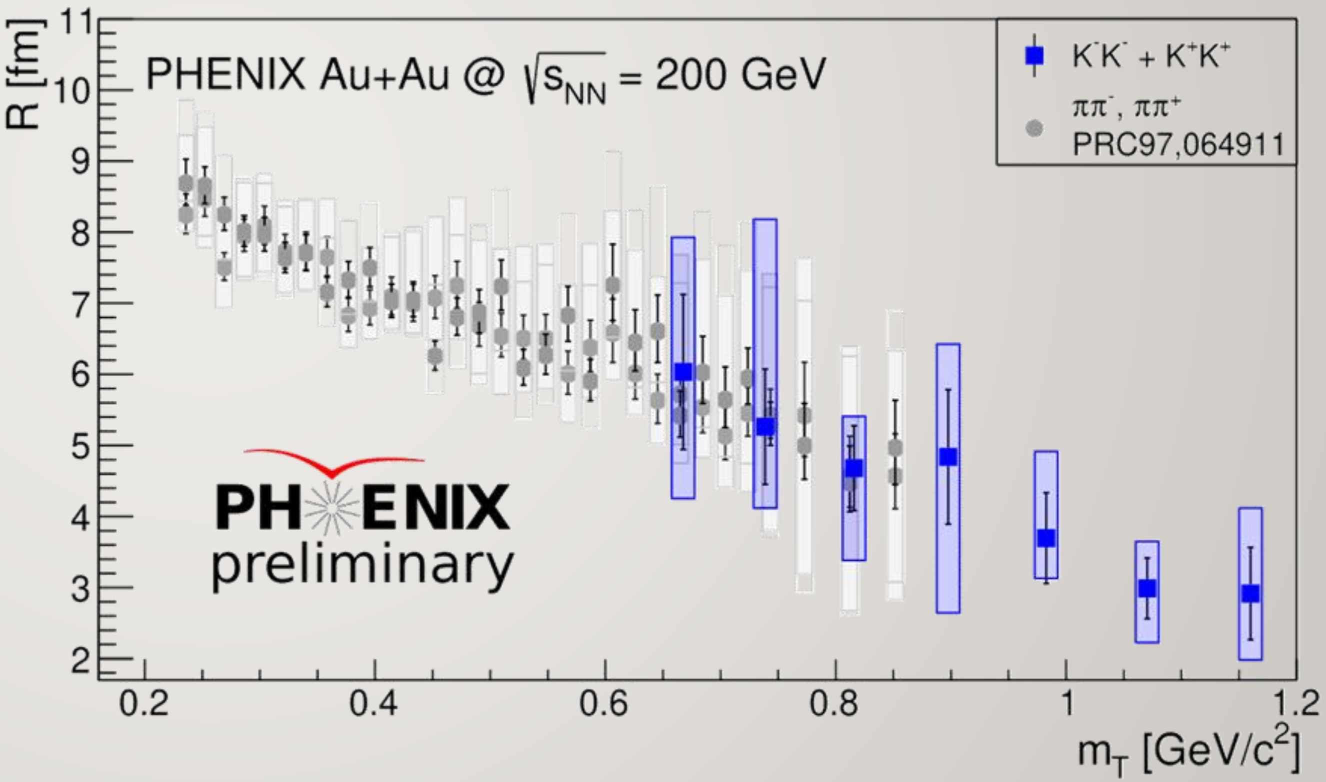
- Good agreement between kaons and pions, experiment and EPOS
  - Slightly surprising: same source shape for kaons and pions!
  - Very different decays and scatterings, how can source shape end up to be the same?



PARTICLE SPECIES COMPARISON, DATA VS EPOS, LÉVY R

- Excellent agreement between kaons and pions, experiment and EPOS
  - Slightly surprising: same source for kaons and pions as expected from hydro
  - Despite role of scattering? Why does it not distort m_T-scaling? Maybe hydro affects shape as well?





CAN HYDRO PRODUCE LÉVY DISTRIBUTED SOURCES?

- Take a simple Maxwell-Jüttner distribution with Cooper-Frye freeze-out

$$S(x,p)d^4x = Nn(x)\exp\left(-\frac{p_\mu u^\mu(x)}{T(x)}\right)p^\mu d^3\Sigma_\mu(x)H(\tau)d\tau$$

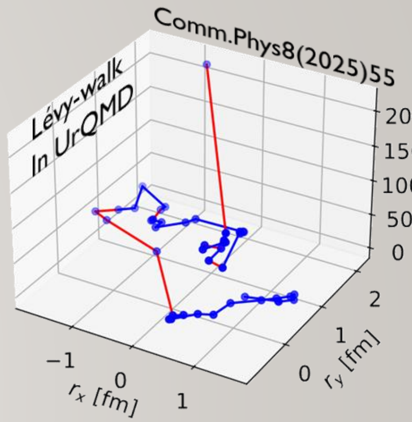
- Can the resulting distribution be Lévy-stable? Probably, if appropriate thermodynamic fields are chosen
- Does hydrodynamics allow that? Surely, for example in a Hubble-flow and $\tau =$ const. freeze-out:
  - $u^\mu(x) = \gamma\left(1,\frac{\dot{R}}{R}\vec{r}\right), n(x) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{L}\left(\frac{r^2}{R^2}\right), T(x) = T_0\left(\frac{\tau_0}{\tau}\right)^{3/\kappa}\frac{1}{\mathcal{L}(r^2/R^2)}$ → Lévy source, unrealistic observables
- Would the observables still be meaningful (compatible with experiment)? That is not so simple!
  - A non-solution final state: $u^\mu(x) = \gamma\left(1,\frac{\vec{r}}{\tau+r}\right), \quad n(x) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{L}\left(\frac{r^2}{R^2}\right), \quad T(x) = T_0\left(\frac{\tau_0}{\tau}\right)^{3/\kappa}$
  - With this, spectra, flow OK, and Lévy-stable source, and HBT-radii decrease with m_T
  - Can be evolved back numerically; possible with full analytic solution as well?
- Is it compatible with realistic initial conditions?

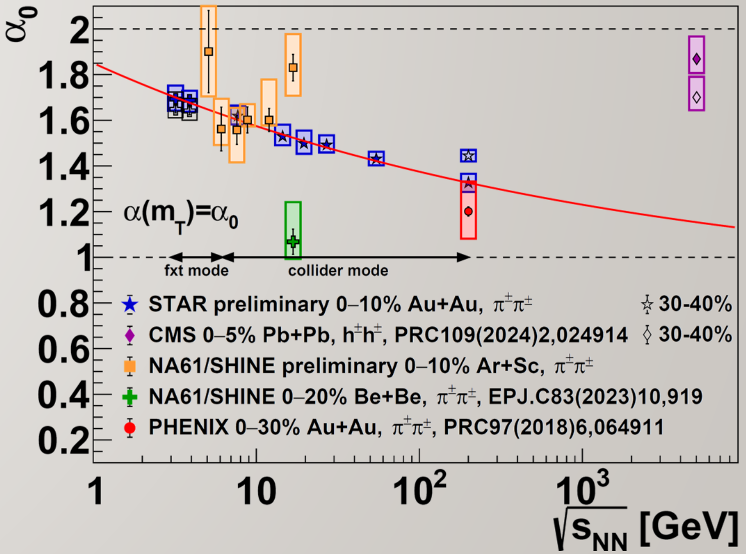
WHAT ABOUT ALTERNATIVES?

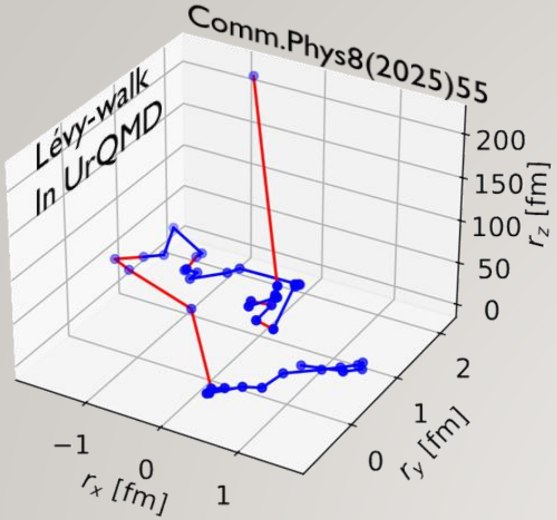
- Usual framework: superdiffusion and subdiffusion, using fractional derivatives
  - Various definitions: Grunwald–Letnikov, Riemann–Liouville, Caputo, Riesz–Feller, …
  - Caputo version, for $p \in \mathbb{R}^+,\ m=[p]$: $f^{(p)}(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-1-p} f^{(m)}(\tau)d\tau$
- Fractional diffusion $\frac{\partial u(x,t)}{\partial t^\gamma}=D\frac{\partial u(x,t)}{\partial |x|^\alpha}$
  - Subdiffusion for $\alpha>2\gamma$, superdiffusion for $\alpha<2\gamma$; leads to Lévy-stable distributions for $\gamma=1,\ 0<\alpha<2$
  - See e.g. Chen et al, Comp. Math. Appl. 59 (2010) 1754 or Metzler, Klafter, Physics Reports 339 (2000) 1-77
- What if superdiffusion happens between hydro (small m.f.p.) and free streaming (infinite m.f.p.)?
- How to connect power-law exponent α to QGP or hadron properties?
  - Jet-dominated correlations: anomalous dimension of QCD (Csörgő, Hegyi, Novák, Zajc, APPolB 36 (2005) 329)
  - At the critical point: critical exponent η (Csörgő, Hegyi, Novák, Zajc, AIP Conf.Proc. 828 (2006) 525)
  - What about the QGP phase, scattering, decays and Lévy-walk?

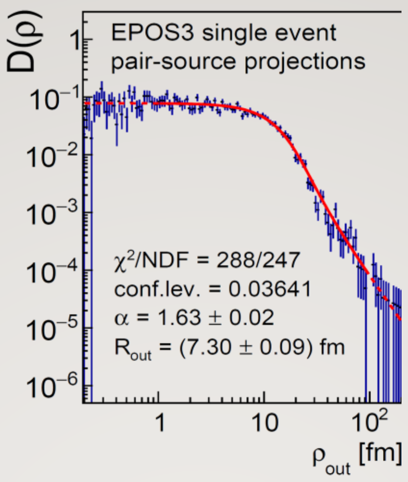
CONCLUSIONS AND OUTLOOK

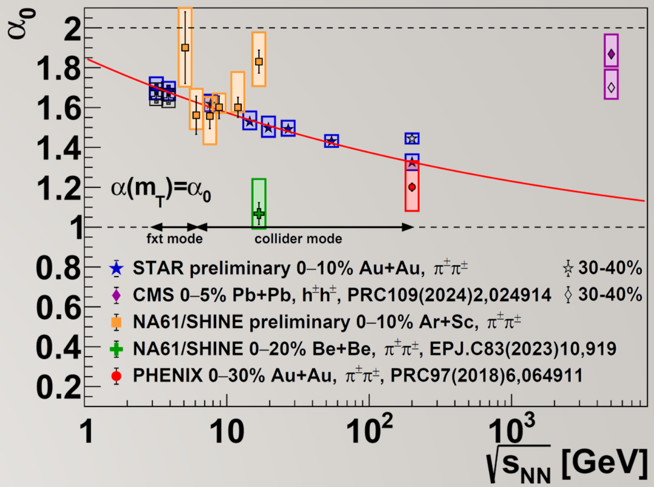
- Lévy parameters for pions measured from 3.2 GeV to 5 TeV from SPS through RHIC to LHC
  - **Lévy α**: between 1 and 2, decrease with $\sqrt{s_{NN}}$ at RHIC, constant with m_T
    - Interesting trends at SPS and towards LHC, incompatibility with simulations
  - **R**: decrease with m_T, similarly to Gaussian radii
    - Relation to Gaussian through HWHM/HWHI
  - **λ**: decrease at low m_T, overall increase with $\sqrt{s_{NN}}$
- Possible reasons for power-law tails and Lévy sources:
  - **Critical phenomena** → **no** non-monotonicity seen in α vs s_{NN}
  - **Resonance decays** → part of the reason, predicts alone larger α
  - **Hadronic scattering, Lévy walk** → plausible explanation
- Questions to be answered:
  - Why are kaon and pion sources similar?
  - Only hadronic phase creates Lévy distributions? Role of hydrodynamics?
  - Origin of Lévy (power-law) exponent?
  - Discrepancy between simulations (UrQMD & EPOS) and data?











THANK YOU FOR YOUR ATTENTION

If you are interested in further developments:

25th Zimányi School Winter Workshop

http://zimanyischool.kfki.hu/25/

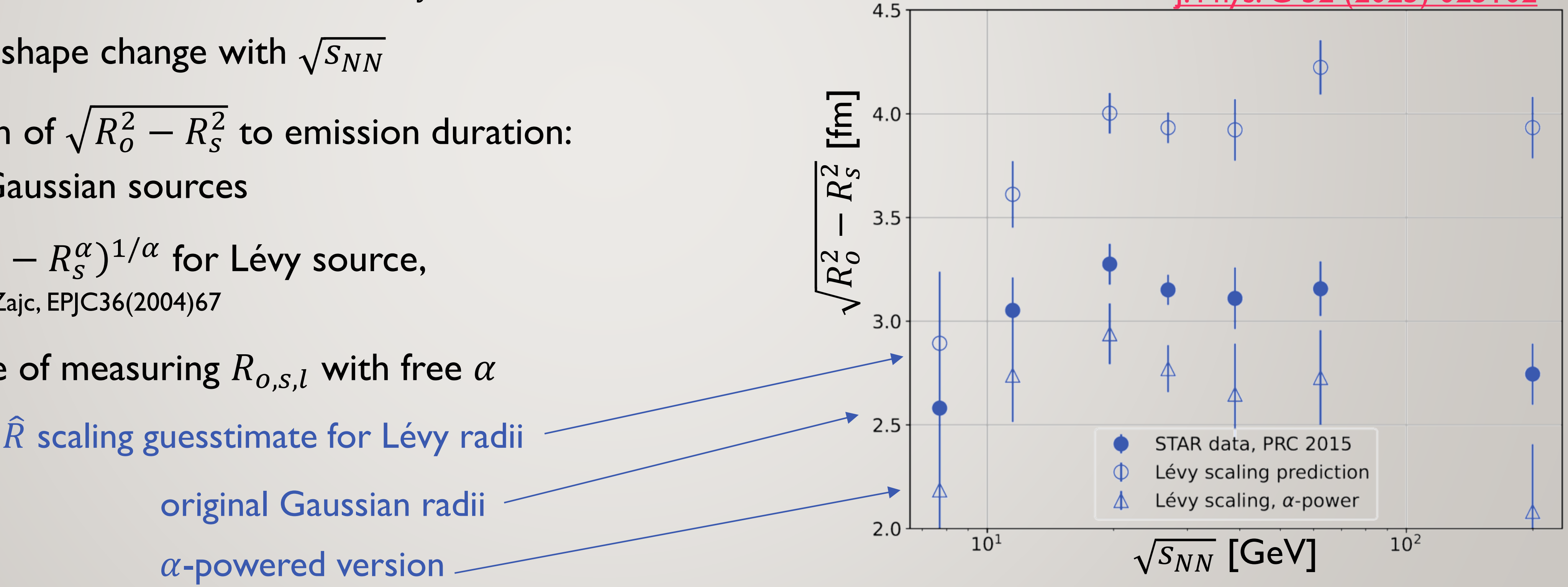
(and also: WPCF 2026 in Budapest)

BACKUP

ENERGY DEPENDENCE OF LÉVY SOURCE SIZE?

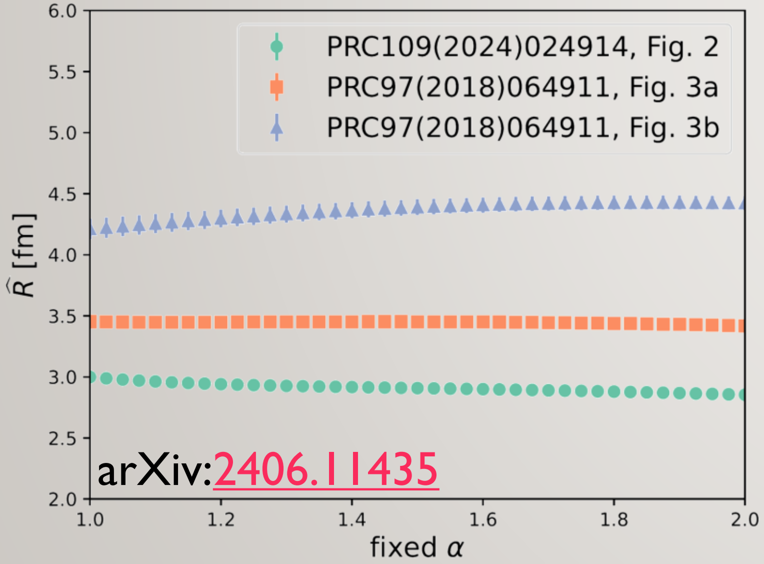
- Experimental observation: $\hat{R} = \frac{R}{\lambda(1+\alpha)}$ doesn't depend on α → can estimate $R_{\text{free }\alpha} = R_{\text{Gauss}} \frac{\lambda_{\text{free }\alpha}(1+\alpha)}{\lambda_{\text{Gauss}}(1+2)}$
  - Assuming trends of α and λ as $A \cdot \sqrt{s_{NN}}^{B}$, with $A_\alpha = 1.85, B_\alpha = -0.06, A_\lambda = 0.6, B_\lambda = 0.06$
- Different trends of guesstimated $R_{\text{Lévy}}$ and R_{Gauss}
- Caused by shape change with $\sqrt{s_{NN}}$
- Connection of $\sqrt{R_o^2 - R_s^2}$ to emission duration: based on Gaussian sources
- Maybe $(R_o^\alpha - R_s^\alpha)^{1/\alpha}$ for Lévy source, Csörgő, Hegyi, Zajc, EPJC36(2004)67
- Importance of measuring $R_{o,s,l}$ with free α

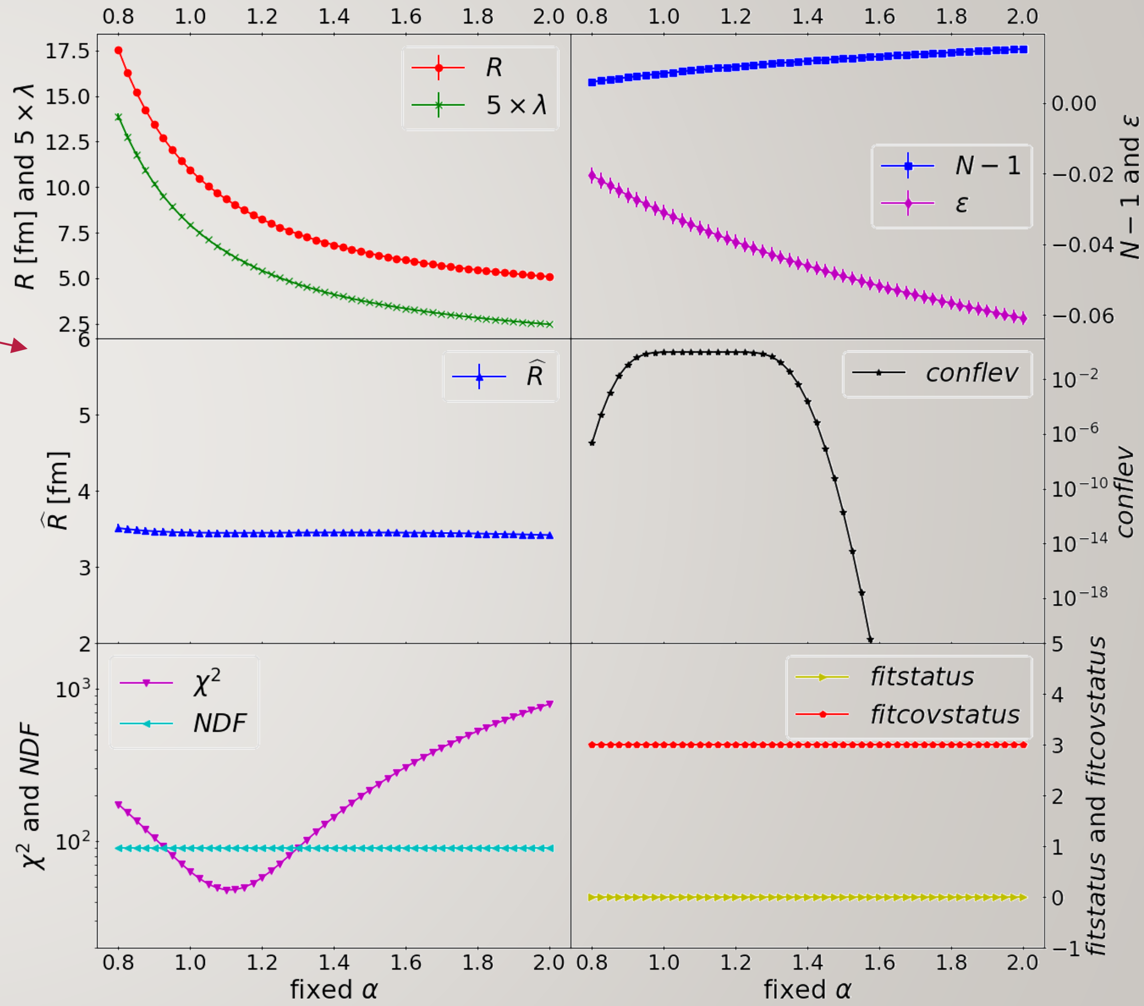
J. Phys. G 52 (2025) 025102



RESCALING HBT RADII FROM GAUSS TO LÉVY

- Source shape and size entangled in Gaussian radii
- Fits possible with many α values
  - Some statistically acceptable, some not
  - Fits to PHENIX HBT paper PRC 2018, Fig 3a
- $\hat{R} = R/[\lambda(1+\alpha)]$ scaling observed generally

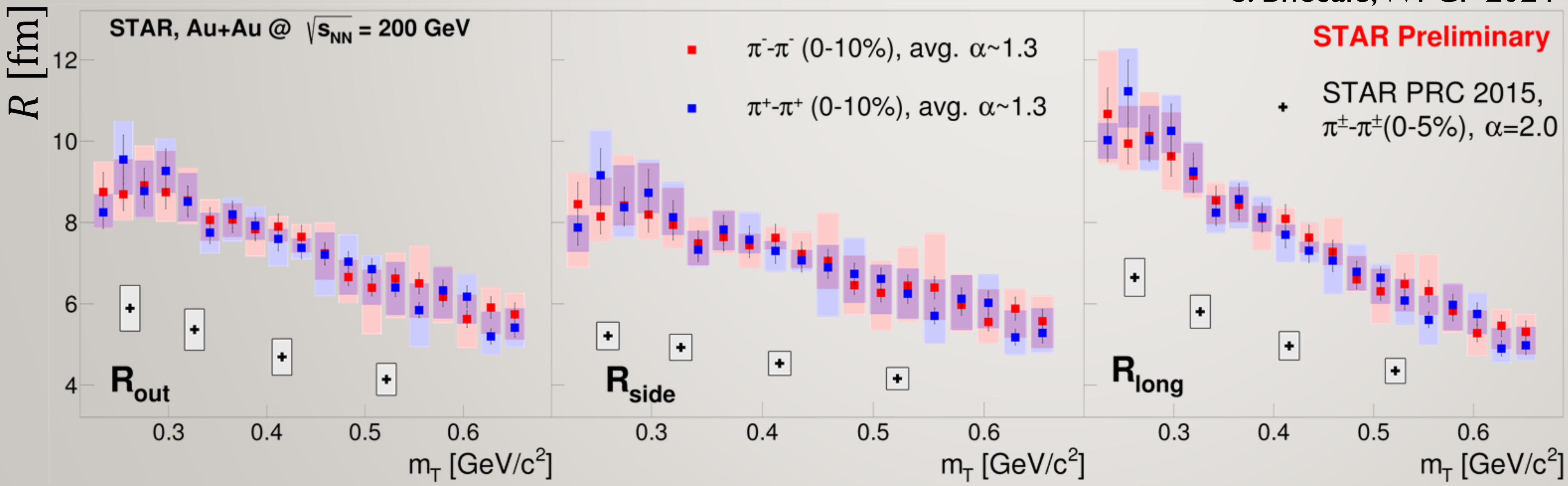




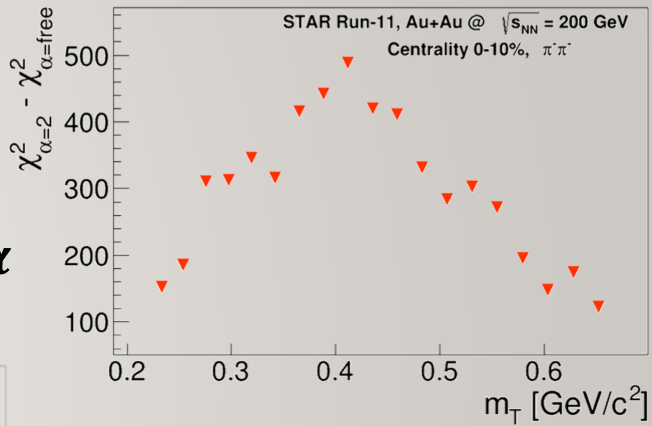
SOURCE RADII: 3D LÉVY MEASUREMENT VS GAUSSIAN

- Lévy-scale R: usual decreasing trend with m_T
- Free α fits reduce χ^2 by 200-500 units compared to Gaussian fits
- χ^2/NDF values within 1-1.04 for all fits
- Confidence levels (p-values) improve by 1-3 orders of magnitude with free $\boldsymbol{\alpha}$

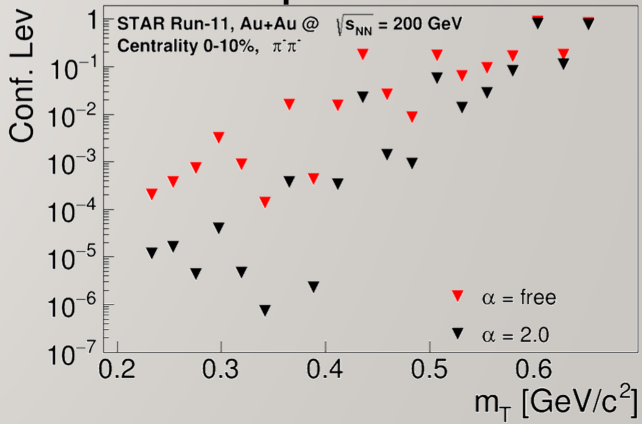
S. Bhosale, WPCF 2024



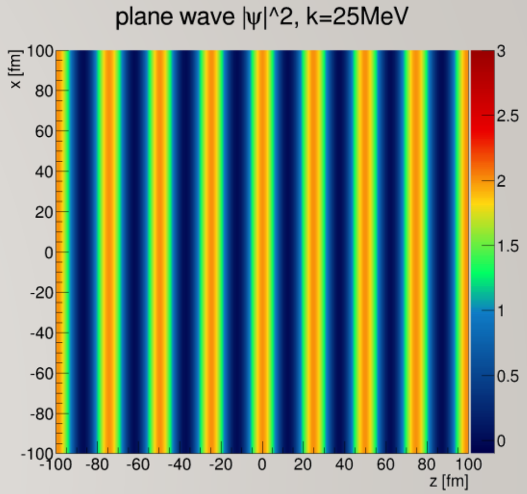
χ^2 difference

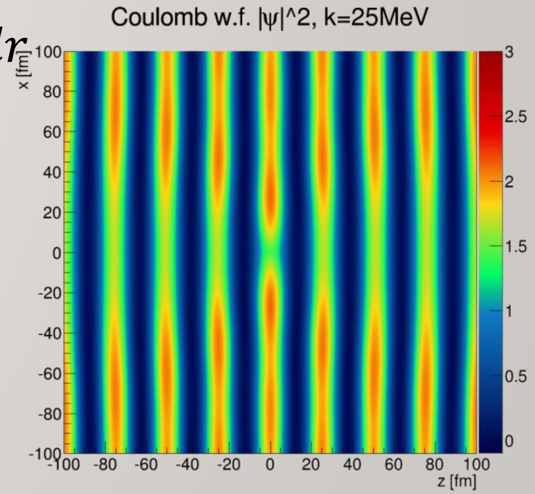


fit p-values



INTERACTIONS





- Plane-wave result, based on $\left|\Psi_{2,q}^{(0)}(r)\right|^2 = 1 + e^{iqr}$, for pair source $D(r)$

$$C_2(q,K) \cong \int D(r,K)\left|\Psi_{2,q}^{(0)}(r)\right|^2 dr = 1 + \int D(r,K)e^{iqr}\,dr$$

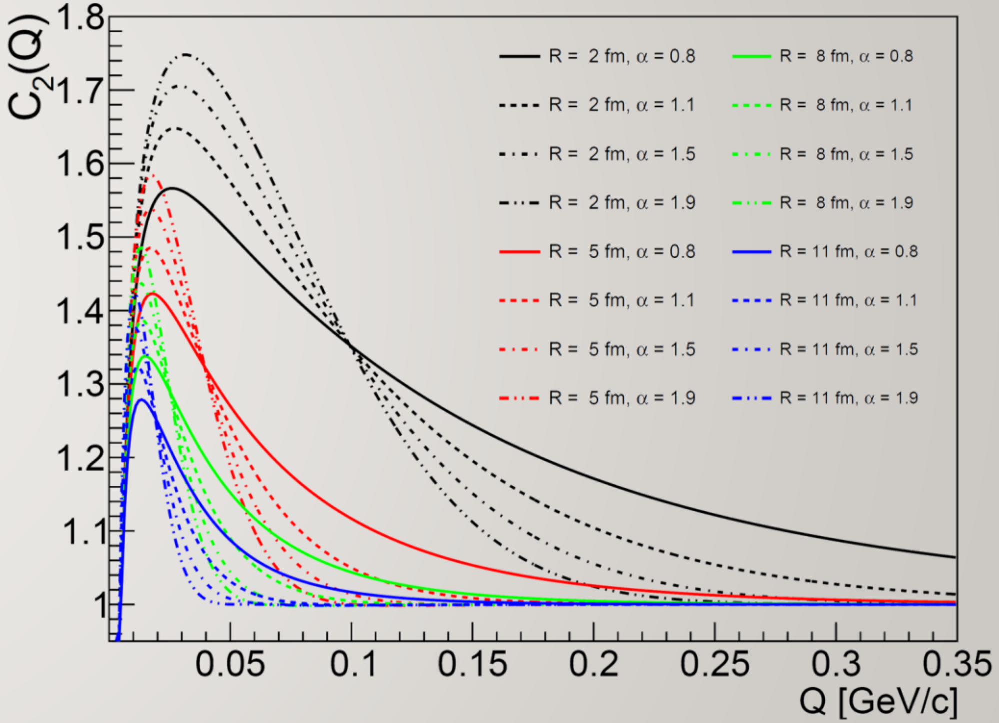
- If there are interactions, solve Schrödinger eq: $\Psi_{2,q}^{(0)}(r) \to \Psi_{2,q}^{(\mathrm{int})}(r_1, r_2)$
- For Coulomb, solution is known: $\left|\Psi_{2,q}^{(C)}(r)\right|^2 = \frac{\pi\eta}{e^{2\pi\eta}-1} \cdot$ (hypergeometric expression)
- Direct fit with this, or the usual iterative Coulomb-correction:

$C_{\mathrm{Bose-Einstein}}(q)K(q)$, where $K(q) = \int D(r,K)\left|\Psi_{2,q}^{(C)}(r)\right|^2 dr / \int D(r,K)\left|\Psi_{2,q}^{(0)}(r)\right|^2 dr$

- Complication: need for integrating power-law tails
  - Precalculated in a tabular form, iterative fitting, e.g., PHENIX, PRC97(2018)064911
  - Interpolating functional form, see Csanád, Lökös, Nagy, Phys.Part.Nucl. 51(2020)238
  - Role of the strong interaction, see Kincses, Nagy, Csanád, PRC102(2020)064912
  - Recent method: EPJC83(2023)1015, code at github.com/csanadm/CoulCorrLevyIntegral
- Many new results, also for the strong interaction: see talk by M. Nagy on Tuesday

HOW TO CALCULATE THE COULOMB EFFECT

- Calculating correlation functions with the Coulomb effect included: time consuming in the past
- Method used in early analyses: Coulomb correction calculated for fixed radius and shape
  - For example, fixing $R = 5$ fm and $\alpha = 2$
- More consistent method: correlation function with Coulomb FSI precalculated in a tabular form
  - Iterative fitting, see e.g., PHENIX, PRC97 (2018) 6, 064911
- Convenient, but somewhat restricted method: interpolating functional form, in a limited R, α range
  - See Csanád, Lökös, Nagy, Phys.Part.Nucl. 51 (2020) 238, used in arXiv:2306.11574 [CMS], arXiv:2302.04593 [NA61]



- Recent method: see talk by Márton Nagy
  - Nagy, Purzsa, Csanád, Kincses Eur. Phys. J. C 83, 1015 (2023), code at github.com/csanadm/CoulCorrLevyIntegral
  - Recent developments: 3D calculation, protons, see talk by M. Nagy on Wednesday

LÉVY INDEX AS A CRITICAL EXPONENT?

- Critical spatial correlation: $\sim r^{-(d-2+\eta)}$; Lévy source: $\sim r^{-(1+\alpha)}$; $\alpha \Leftrightarrow \eta$?

  Csörgő, Hegyi, Zajc, Eur.Phys.J. C36 (2004) 67

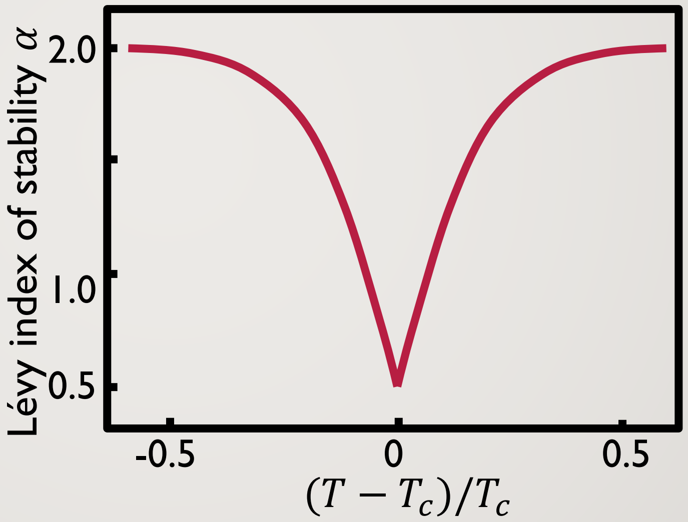
- QCD universality class ↔ 3D Ising

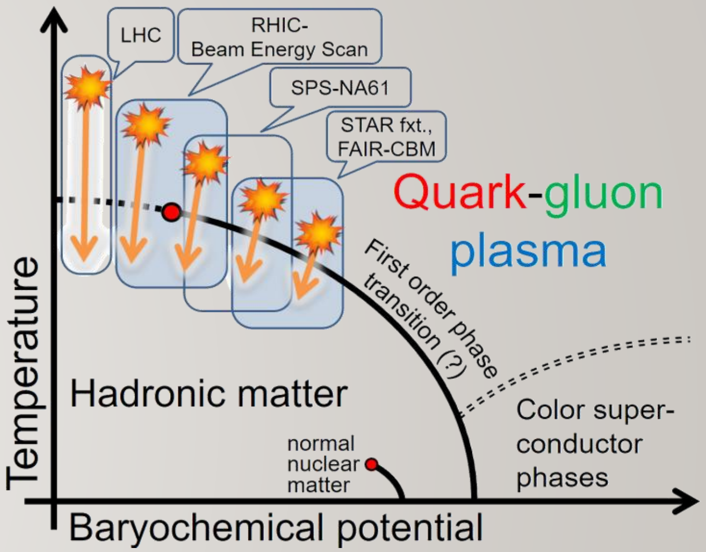
  Halasz et al., Phys.Rev.D58 (1998) 096007
  Stephanov et al., Phys.Rev.Lett.81 (1998) 4816

- At the critical point:
  - Random field 3D Ising: η = 0.50±0.05
    Rieger, Phys.Rev.B52 (1995) 6659
  - 3D Ising: η = 0.03631(3)
    El-Showk et al., J.Stat.Phys.157 (4-5): 869

- Motivation for precise Lévy HBT!

- Change in α_{Levy} proximity of CEP?



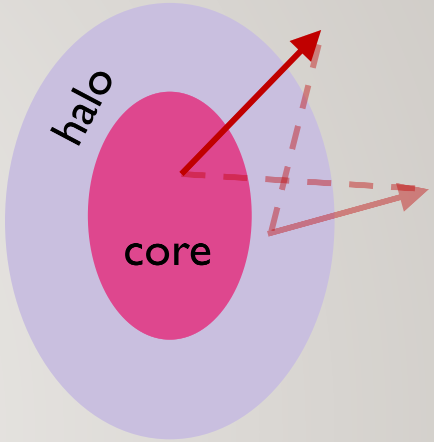


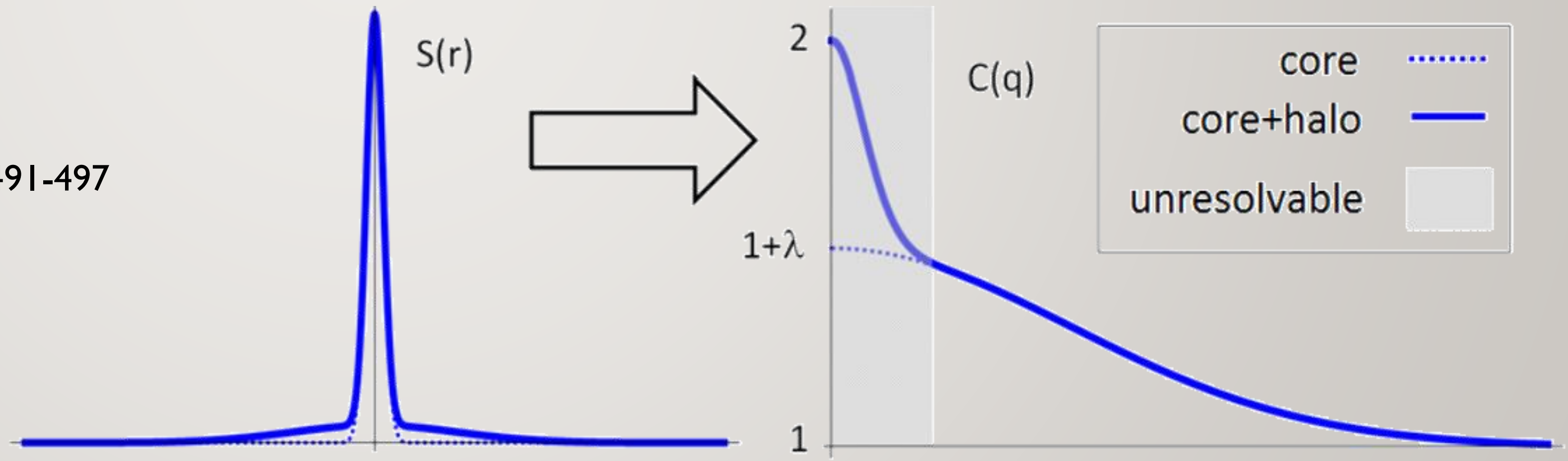
- Finite-size/time & non-equilibrium effects → what does power-law tail mean?
  - Finite-size effects not important? See e.g. Fytas et al, PRE93, 063308 (2016), Ballesteros et al., PLB387 (1996) 125

CORRELATION STRENGTH λ: CORE/HALO

- Two-component core+halo source
  - Core: hydrodynamically expanding, thermal medium
  - Halo: long lived resonances (≳ 10 fm/c, ω,η,η',K_0^S,...)
  - Unresolvable experimentally
  - Define $f_C = N_{\text{core}}/N_{\text{total}}$
- True $q \to 0$ limit: $C(0) = 2$
- Apparently $C(q \to 0) \to 1 + \lambda$
- $\lambda(m_T) = f_C^2(m_T)$

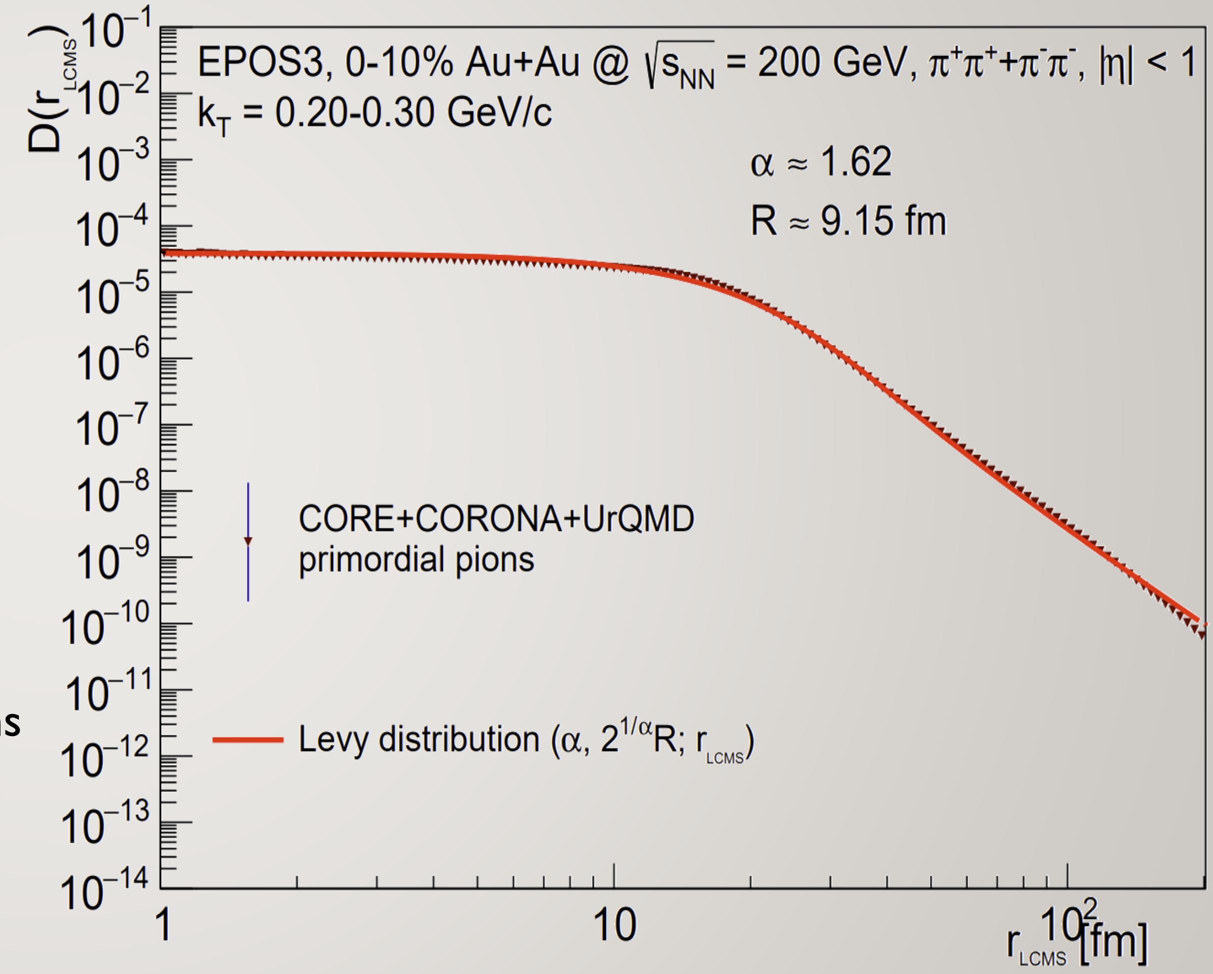
Bolz et al, Phys.Rev. D47 (1993) 3860-3870;
Csörgő, Lörstad, Zimányi, Z.Phys. C71 (1996) 491-497





ROLE OF EVENT AVERAGING?

- Event-averaged source also analyzed
- Not perfectly Lévy shape, very large χ^2
- Nevertheless: similar parameters achieved
  - Event averaged:
    $\alpha \approx 1.62, R \approx 9.15$ fm
  - Event-by-event:
    $\alpha \approx 1.66, R \approx 8.96$ fm
- More reasonable approach for kaons
  - No event-by-event analysis possible for kaons



SOURCE OR PAIR DISTRIBUTION?

- Under some circumstances (thermal emission, no interactions, …):

$$C_2(q,K) = \int S\left(r_1, K+\frac{q}{2}\right) S\left(r_2, K-\frac{q}{2}\right) |\Psi_2(r_1,r_2)|^2 dr_1 dr_2$$
$$\cong 1 + \left|\int S(r,K)e^{iqr}dr\right|^2$$

- Let us introduce the spatial pair distribution:

$$D(r,K) = \int S\left(\rho+\frac{r}{2},K\right) S\left(\rho-\frac{r}{2},K\right) d\rho$$

- Then the Bose-Einstein correlation function becomes:

$$C_2(q,K) \cong \int D(r,K)|\Psi_2(r)|^2 dr = 1 + \int D(r,K)e^{iqr}dr$$

- **Bose-Einstein correlations measure spatial pair distributions!**
- Coulomb and strong Final State Interactions? Under control for Lévy sources

  Csanad, Lökös, Nagy, Phys. Part. Nuclei 51 (2020) 238 [arXiv:1910.02231]

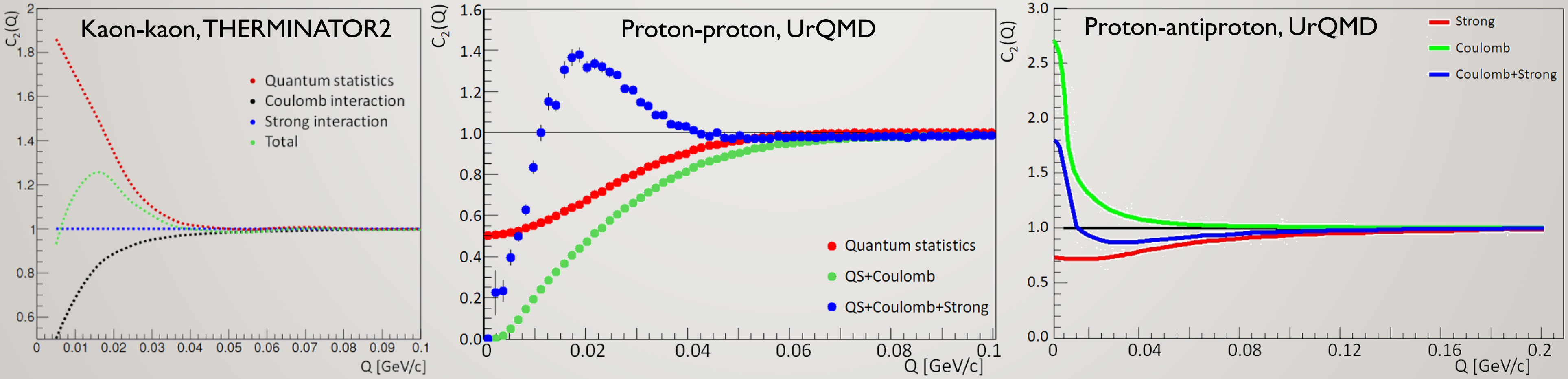
  Kincses, Nagy, Csanad Phys. Rev. C102, 064912 (2020) [arXiv:1912.01381]

ROLE OF THE STRONG INTERACTION

- In case of other interactions or not identical bosons, the formula still works:

$$C_2(q,K) \cong \int D(r,K)|\Psi_2(r)|^2 dr$$

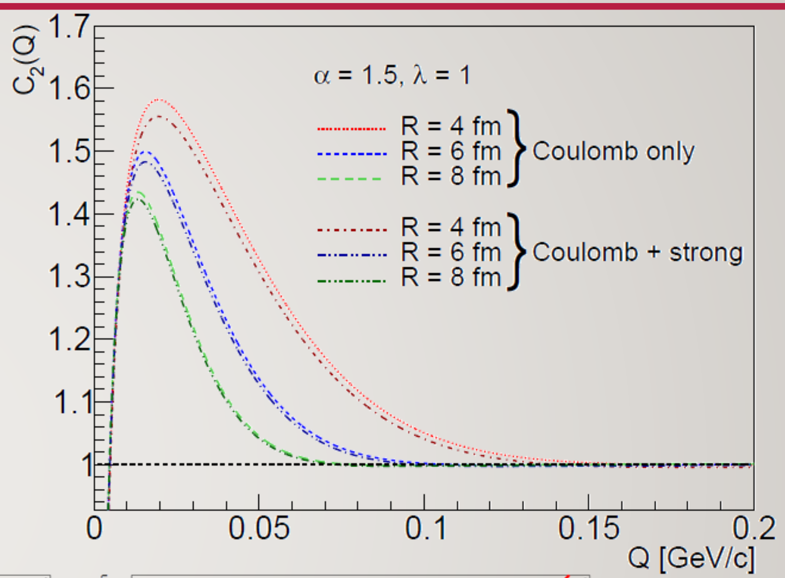
- Pair wave function determines $D \leftrightarrow C_2$ connection
- Mesons, baryons: strong interaction; fermions: anticorrelation
- Non-identical pairs: interaction modifies wave function

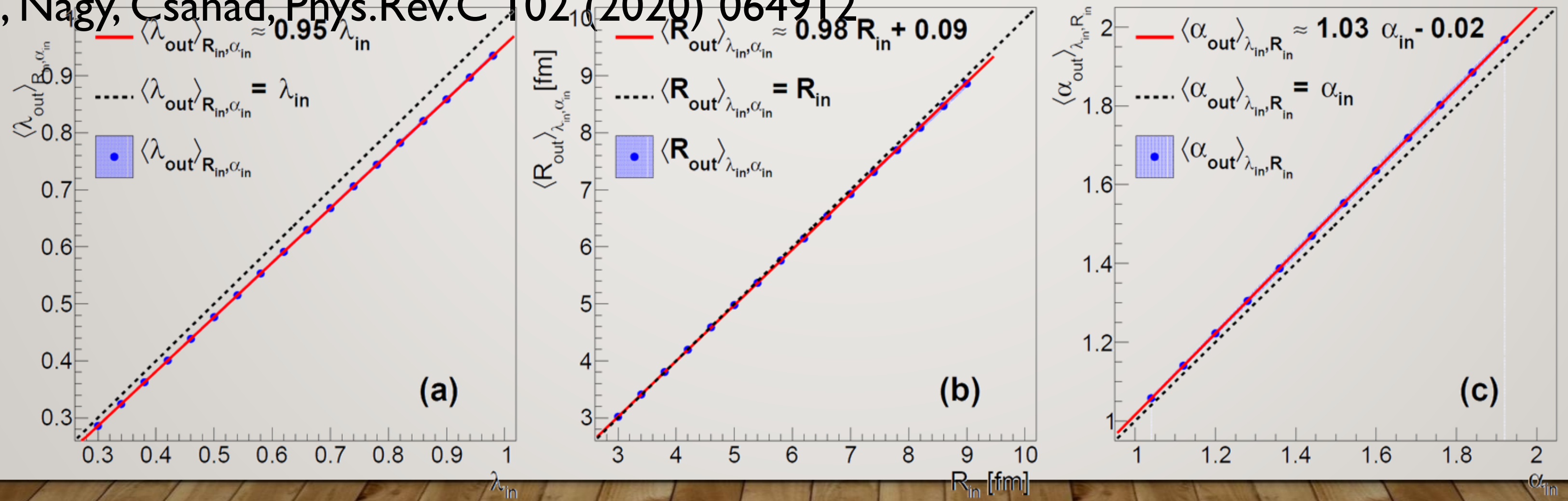


From e.g. H. Zbroszczyk's talk at Zimányi School 2019

STRONG INTERACTION FOR PION PAIRS

- Additional potential appearing
- Possible handling: strong phase shift,
  Modify s-wave component in wave func.
  R. Lednicky, Phys. Part. Nucl.40, 307 (2009)
- Small difference in case of pions
- Few percent modification in λ, α
  Kincses, Nagy, Csanád, Phys.Rev.C 102 (2020) 064912





TWO-PARTICLE SPATIAL CORRELATIONS

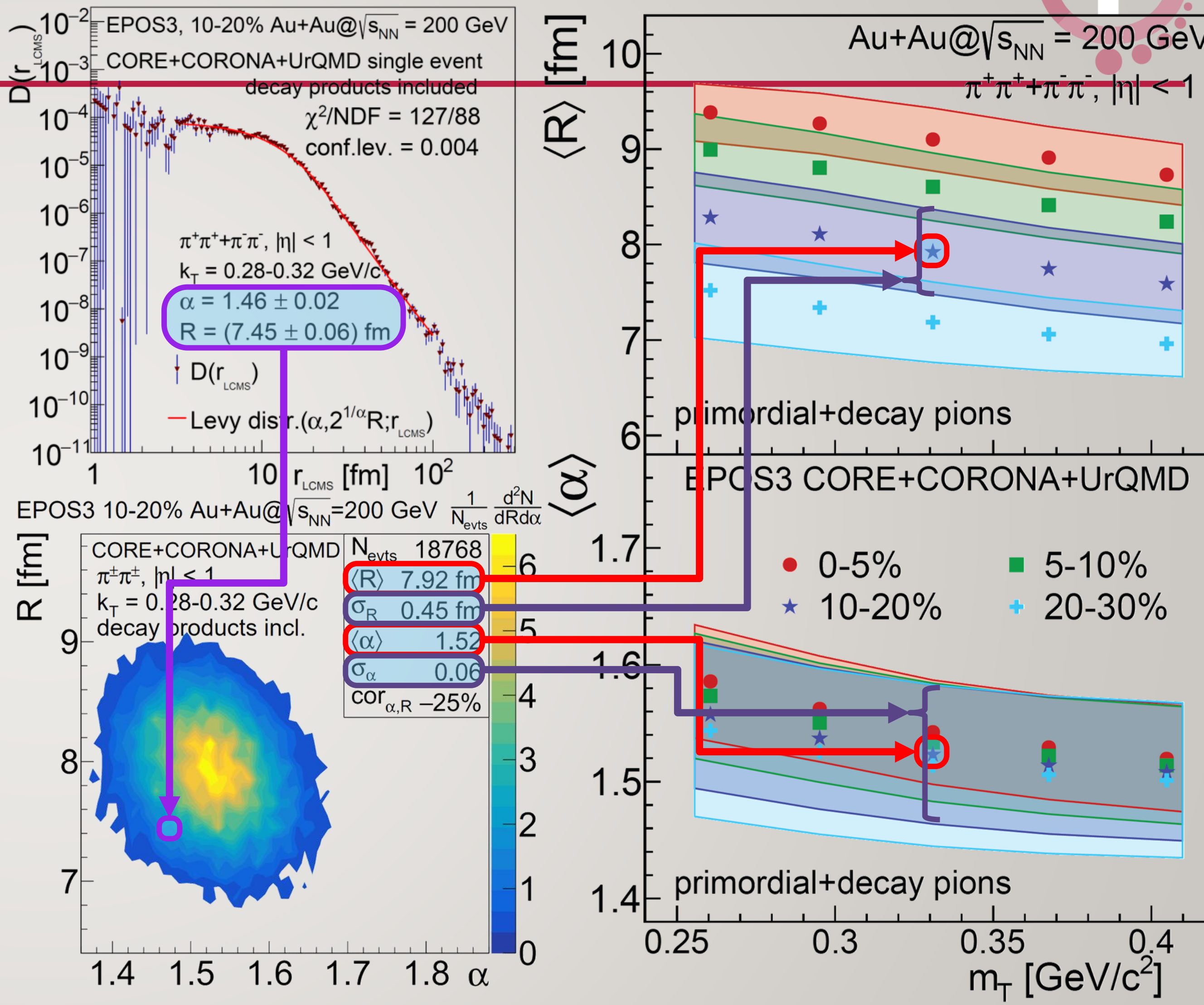
- Object to be investigated: two-particle source

$$D(r,K) = \int d^4\rho S\left(\rho + \frac{r}{2}, K\right) S\left(\rho - \frac{r}{2}, K\right)$$

- Experimental results measure power-law tails, Lévy shapes
  - Measure momentum-space correlations, reconstruct $D(r)$ or fit its parameters
- Why do these Lévy shapes appear?
  - What physics does contribute to it? Rescattering, decays?
  - What role does event averaging have in it?
    Cimerman, Plumberg, Tomasik, Phys.Part.Nucl. 51 (2020) 282, PoS ICHEP2020 538
  - What do specific α values mean?
- Event generator models (like EPOS) – direct access to pair-source!
  - Phenomenological investigations of $D(r)$ possible
  - Effects can be turned off or on, investigated separately

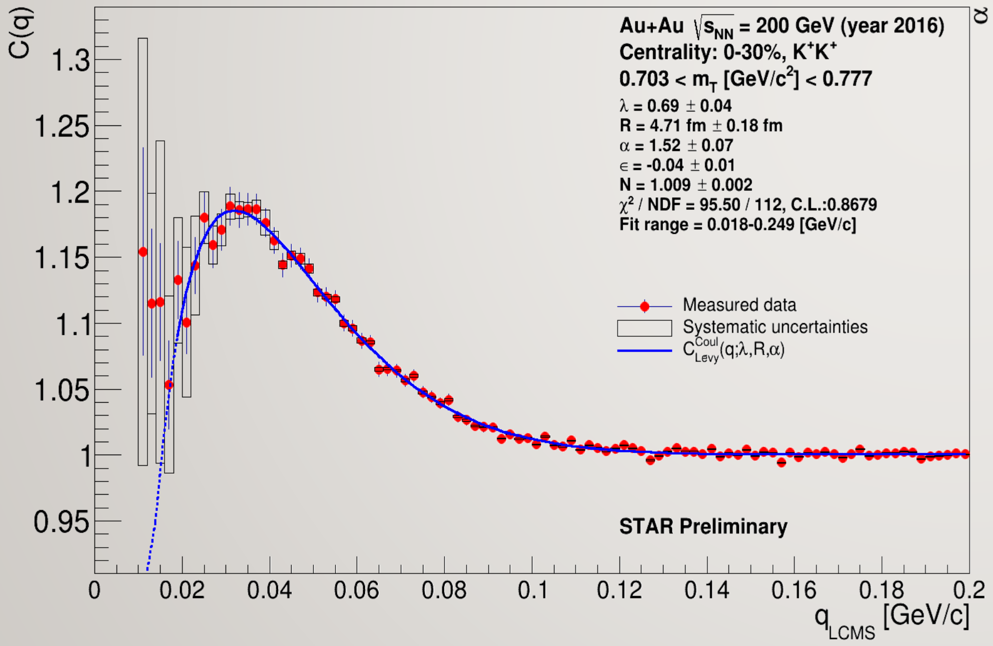
EPOS SUMMARY

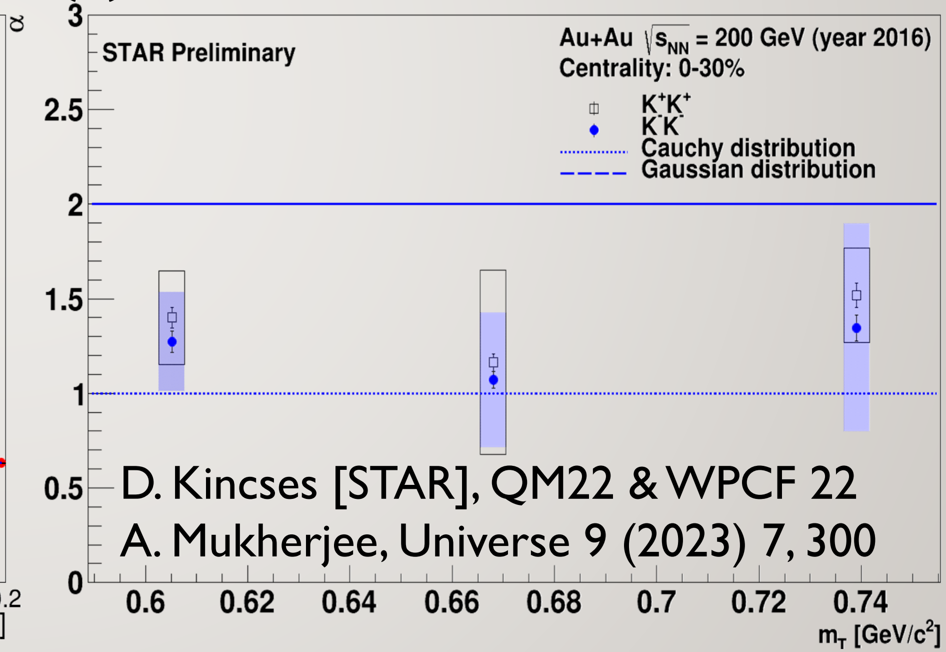
- D(r) calculated in EPOS evt-by-evt
- Lévy fits done evt-by-evt
- Non-Gaussianity in single events
- Extracting mean, & std.dev. of R, α
- m_T & centrality dependence



KAON ANALYSIS AT STAR

- Data successfully described by Lévy fits
- Lévy-stability parameter α between 1 and 2
- Kaon and pion source of same shape at the same m_T?
- Unlike anomalous diffusion expectation of $\alpha(K) < \alpha(\pi)$

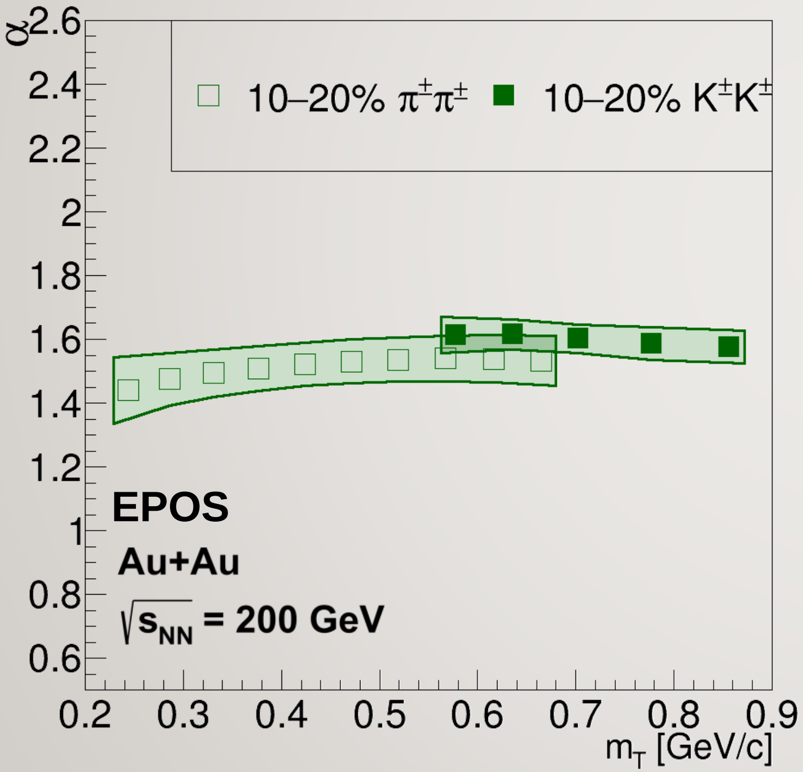




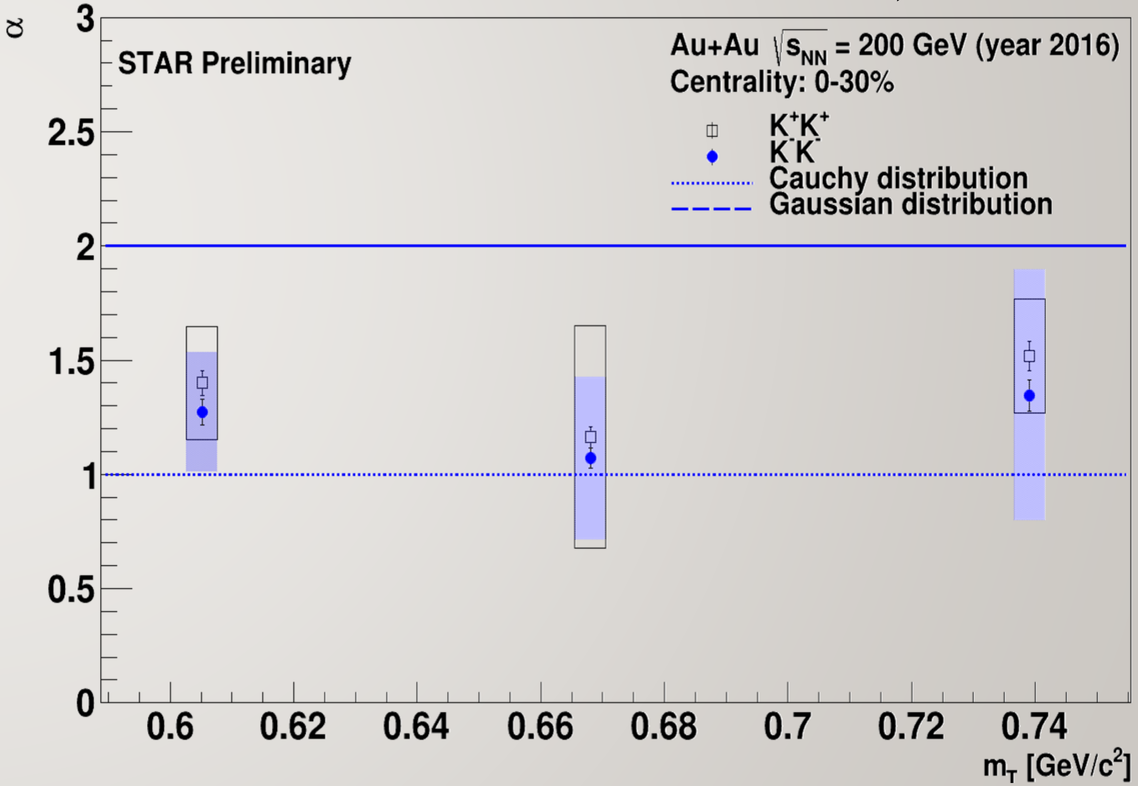
D. Kincses [STAR], QM22 & WPCF 22
A. Mukherjee, Universe 9 (2023) 7, 300

KAON ALPHA

- Good agreement with EPOS

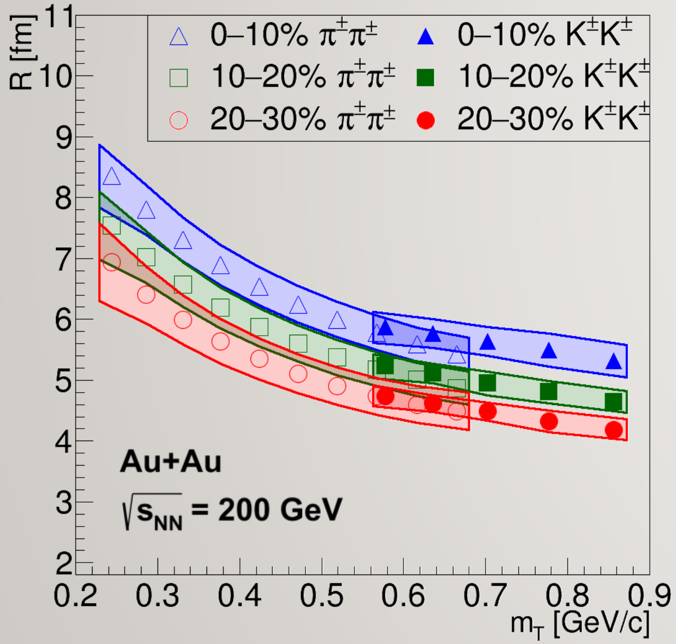


D. Kincses, WPCF 2022



KAON R

- Good agreement with EPOS



D. Kincses, WPCF 2022