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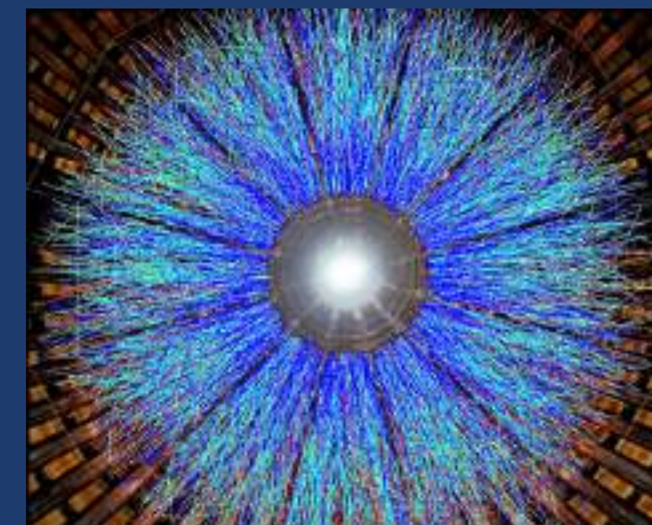
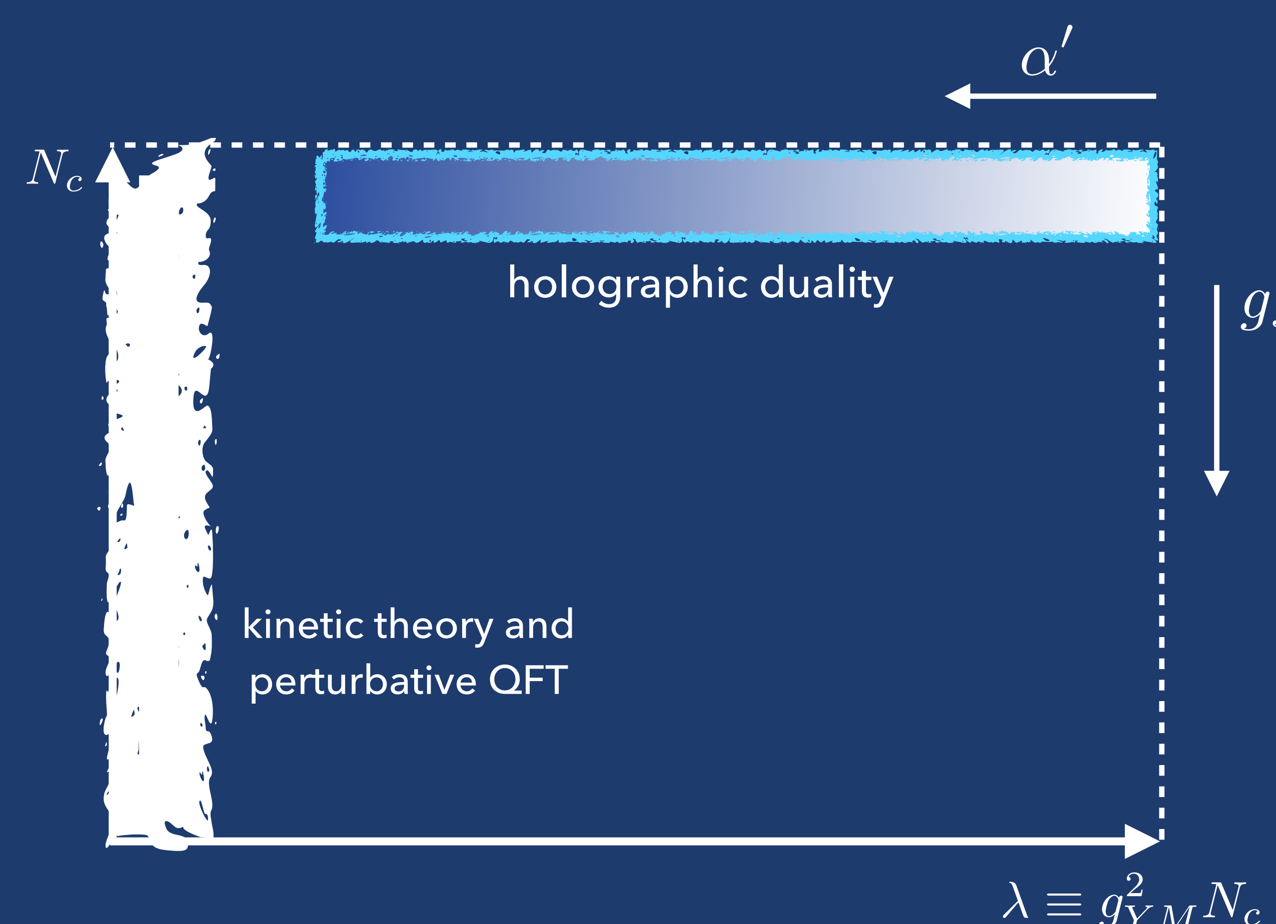
SPECTRAL DUALITY RELATION IN THERMAL FIELD THEORY

GGI, 29.4.2025

THERMAL FIELD THEORY AND BLACK HOLES

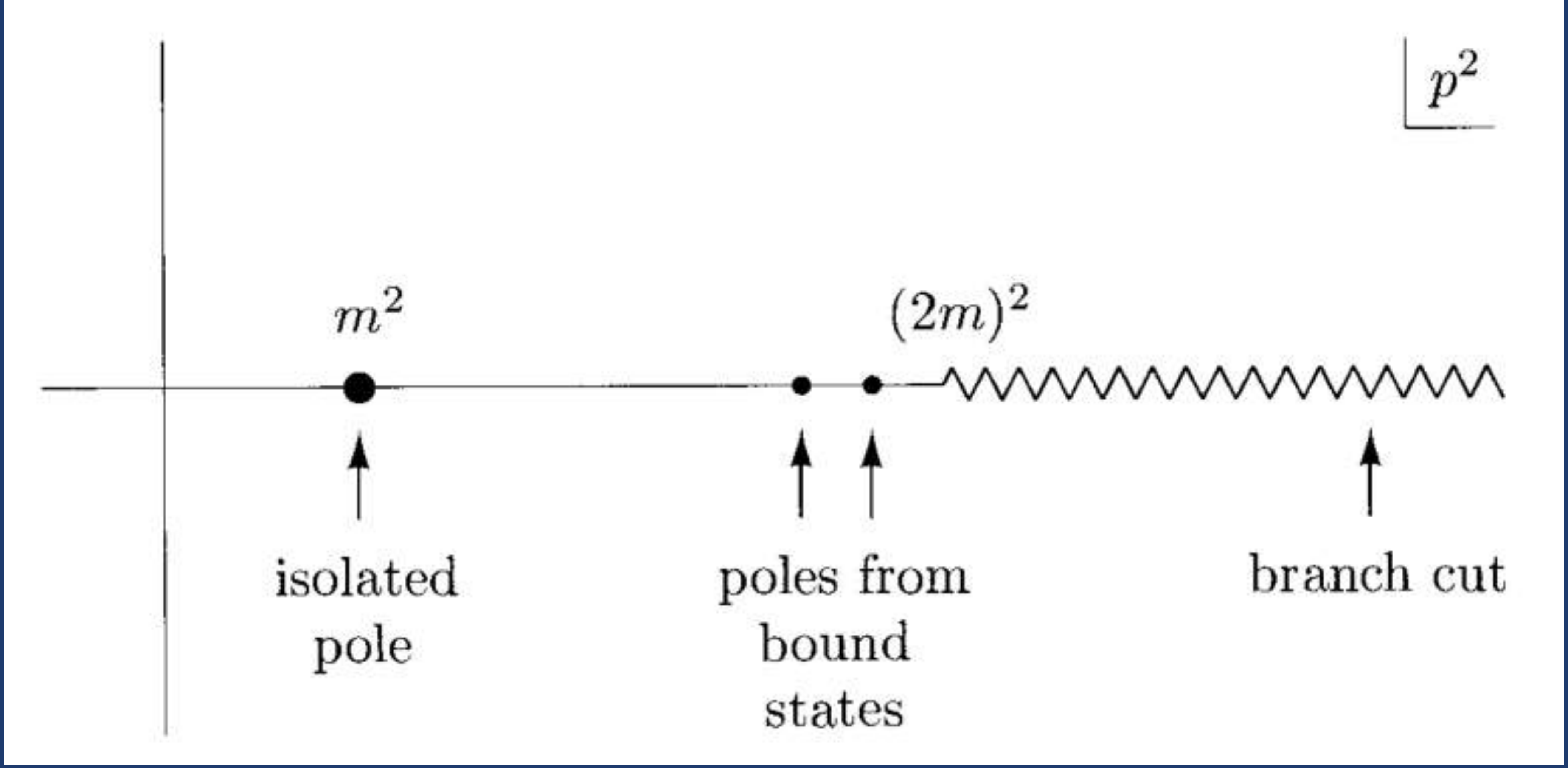


$$Z[\beta = 1/T] = \int \mathcal{D}\Phi \, e^{-\beta H} e^{\frac{i}{\hbar}\int d^d x \, \mathcal{L}(\Phi,\lambda)}$$



SPECTRUM OF A SIMPLE $T = 0$ CORRELATOR

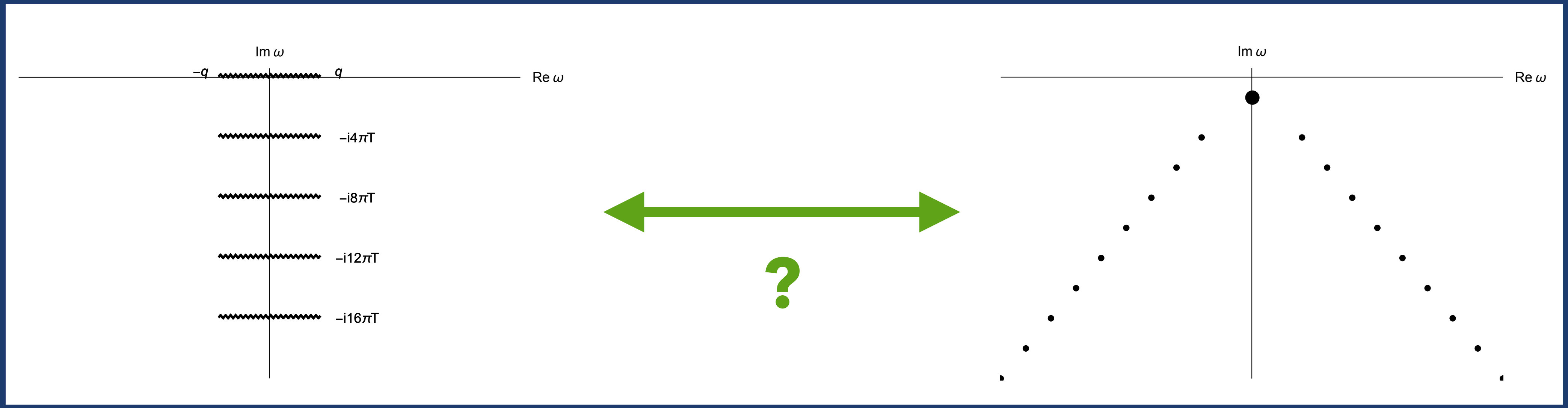
$$\langle \phi(p)\phi(-p) \rangle = \frac{Z(p^2)}{p^2 - m^2 + \Sigma(p^2)}$$



[from Peskin and Schroeder]

ANALYTIC STRUCTURE OF THERMAL CORRELATORS

$$\langle T_{\mu\nu}(-\omega,-q),T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



$\lambda \to 0$

[Hartnoll, Kumar, (2005)]

$\lambda \to \infty$

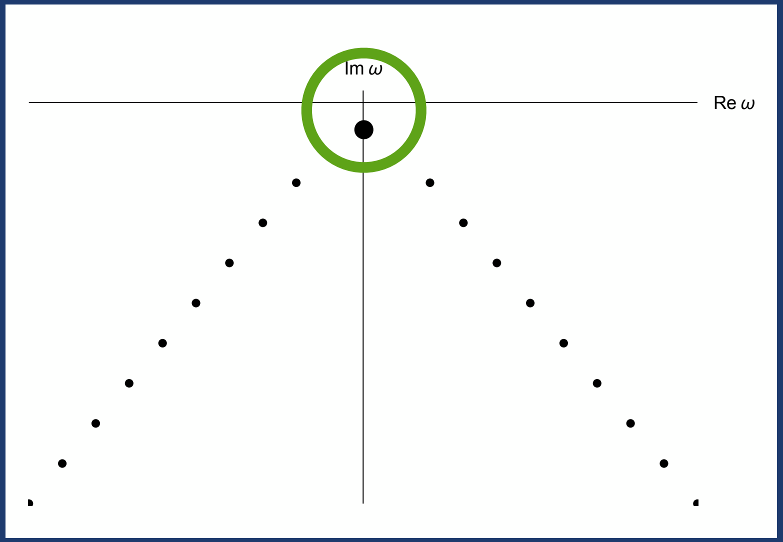
holography (*N=4 SYM-type theories*)

meromorphic momentum space correlator

what is the structure of thermal correlators and QNMs, and what is the minimal information necessary to determine them completely?

- kinetic theory: solving the collision integral
- perturbative QFT: calculation of Feynman diagrams
- holography: solving differential equations but in special theories

LOW-ENERGY SPECTRUM AND HYDRODYNAMICS



- low-energy limit of (some) thermal QFTs is described hydrodynamics
- conservation laws and global conserved operators

$$\nabla_\mu T^{\mu\nu} = 0$$

- tensor structures (symmetries, gradient expansions) and transport coefficients (QFT)

$$T^{\mu\nu} = \sum_{n=0}^{\infty} \left[\sum_{i}^{N} \lambda_i^{(n)} \mathcal{T}_{(n)}^{\mu\nu} \right] \xrightarrow[u^\mu \sim T \sim e^{-i\omega t + iqz}]{\nabla_\mu T^{\mu\nu} = 0} \omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

$$\partial u^\mu \sim \partial T \ll 1 \longrightarrow \omega / T \sim q / T \ll 1$$

equilibrium temperature

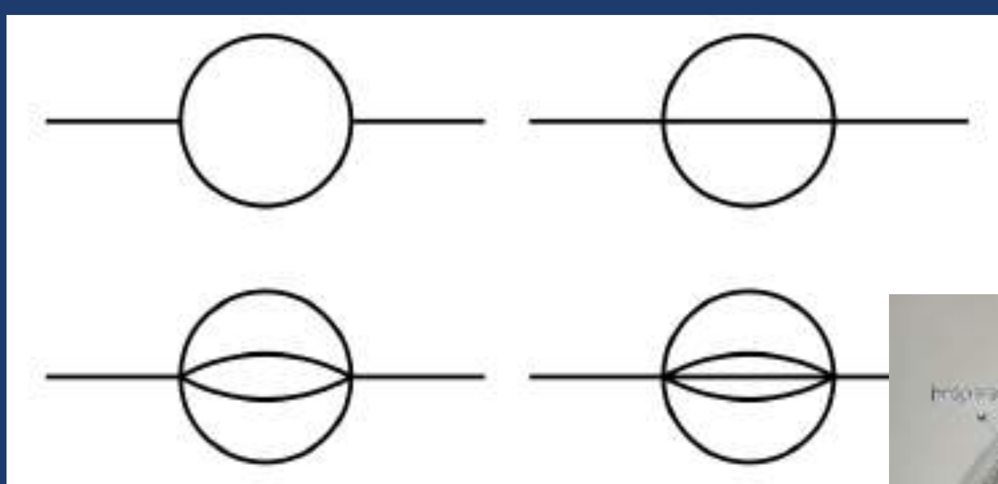
$$q = \sqrt{\mathbf{q}^2}$$

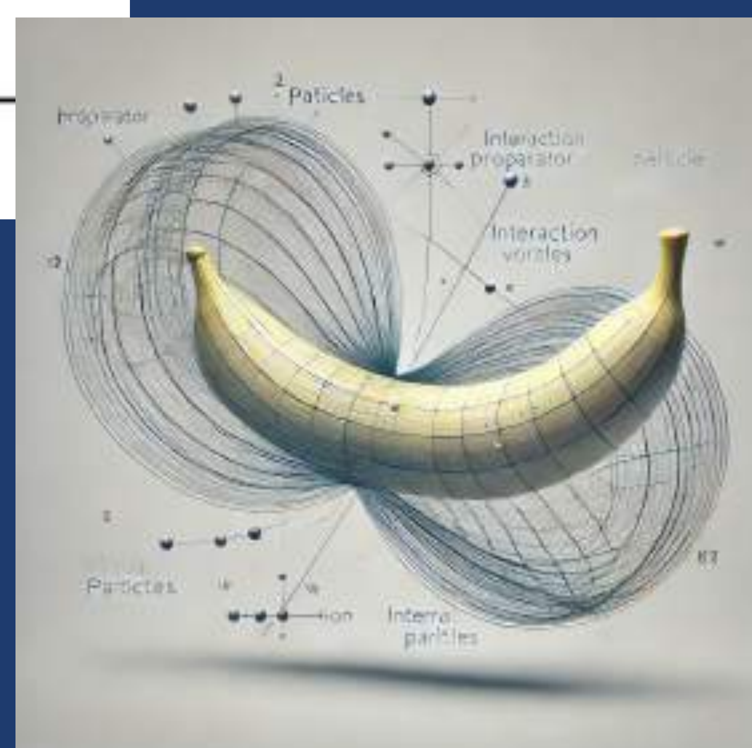
- dispersion relations are poles:

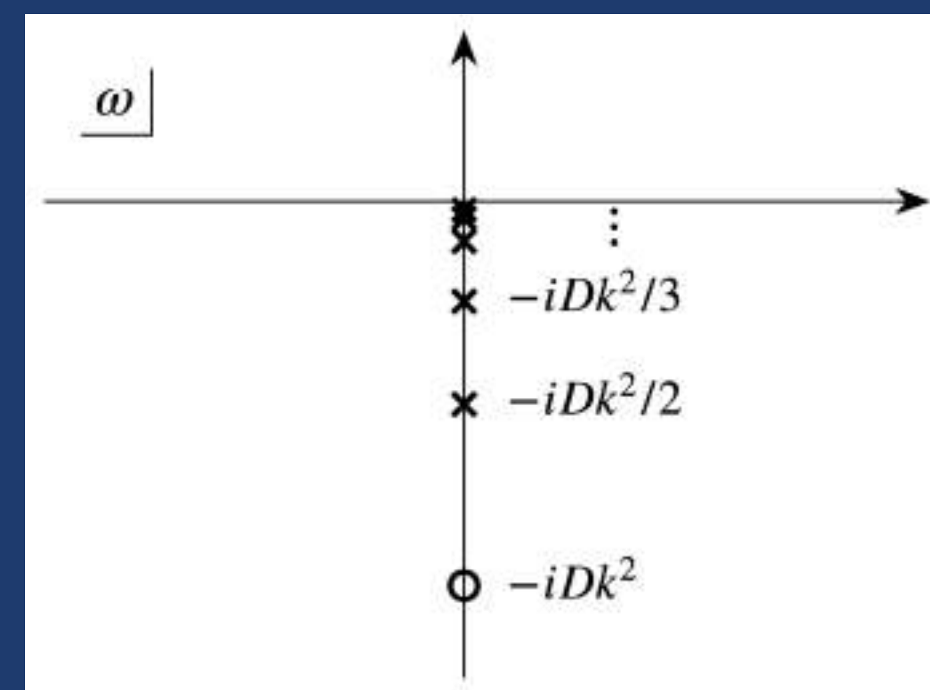
diffusion	sound
$\omega = -iDq^2$	$\omega = \pm v_s q - i\Gamma q^2$

LOW-ENERGY SPECTRUM AND HYDRODYNAMICS

- Schwinger-Keldysh effective field theory of diffusion to all loops: long time tails…
  [Chen-Lin, Delacretaz, Hartnoll (2019) ; Delacretaz (2020); SG, Lemut, Pelaič, Soloviev, PRD (2024)]
- tree-level result (classical hydrodynamics) has one diffusive pole at $\omega = -iDq^2$
- *n*-loop result has a (pair) of diffusive pole(s) and branch point at $\omega = -\frac{iDq^2}{n+1}$
- go to *all* orders with 'bananas':

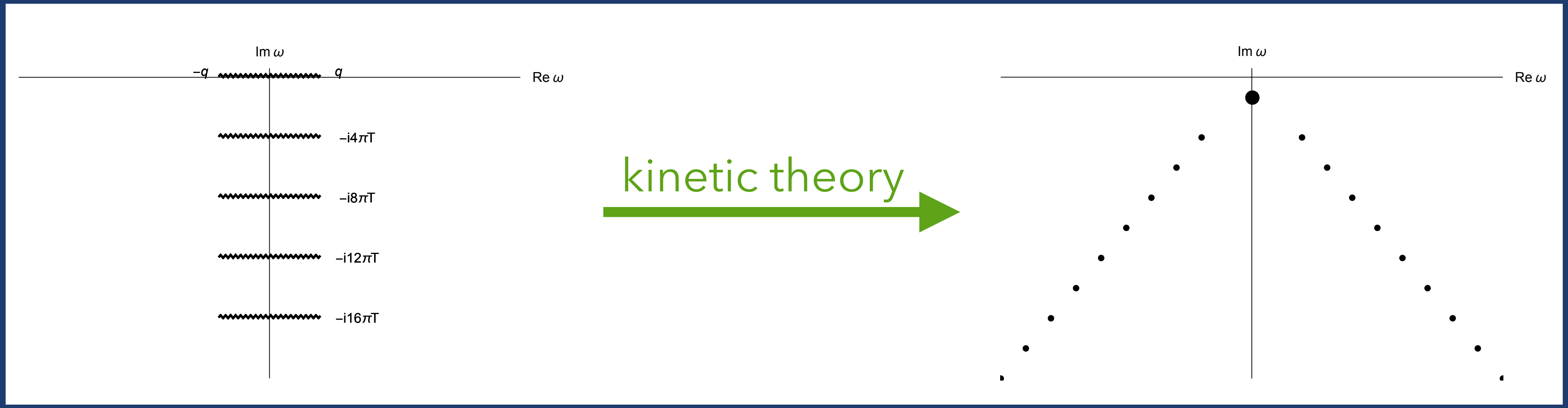






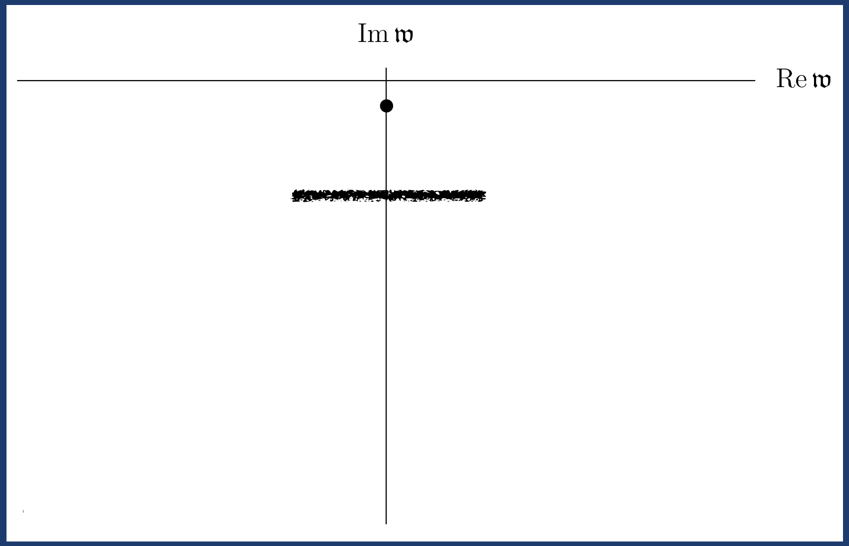
KINETIC THEORY AND THERMAL SPECTRUM

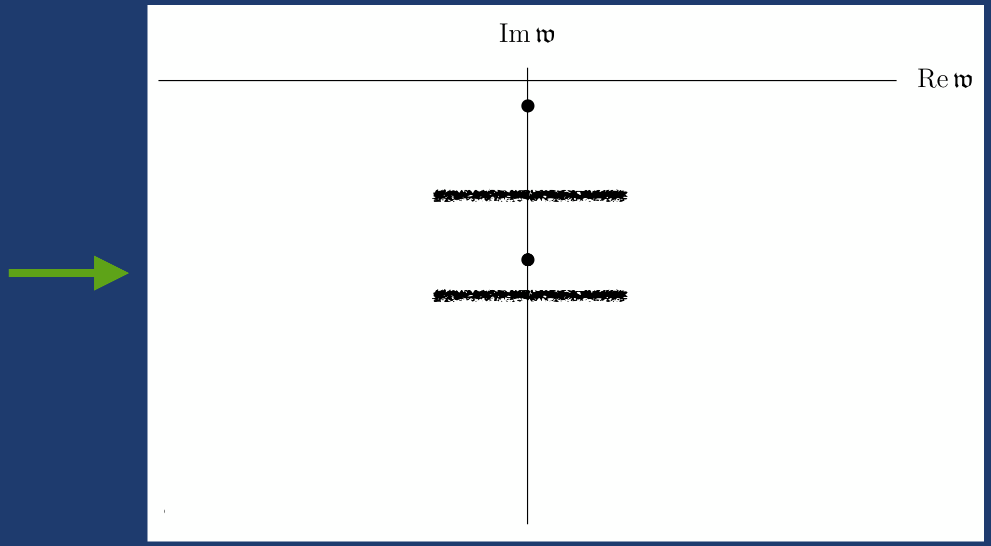
$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



$\lambda \to 0$ $\lambda \to \infty$

poles from a cut?



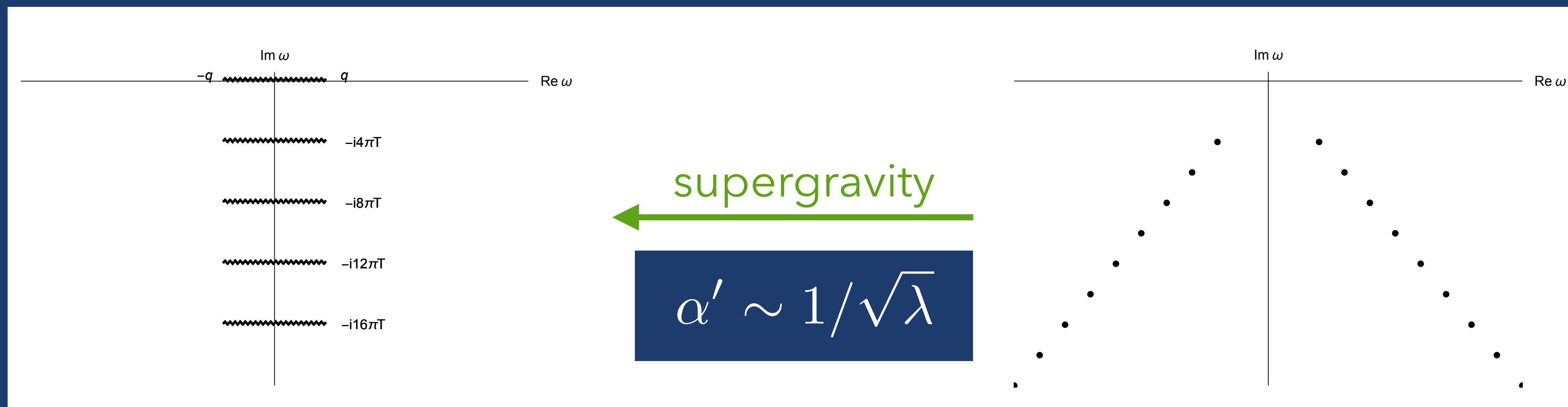


Boltzmann equation – RTA [Romatschke, (2016)]

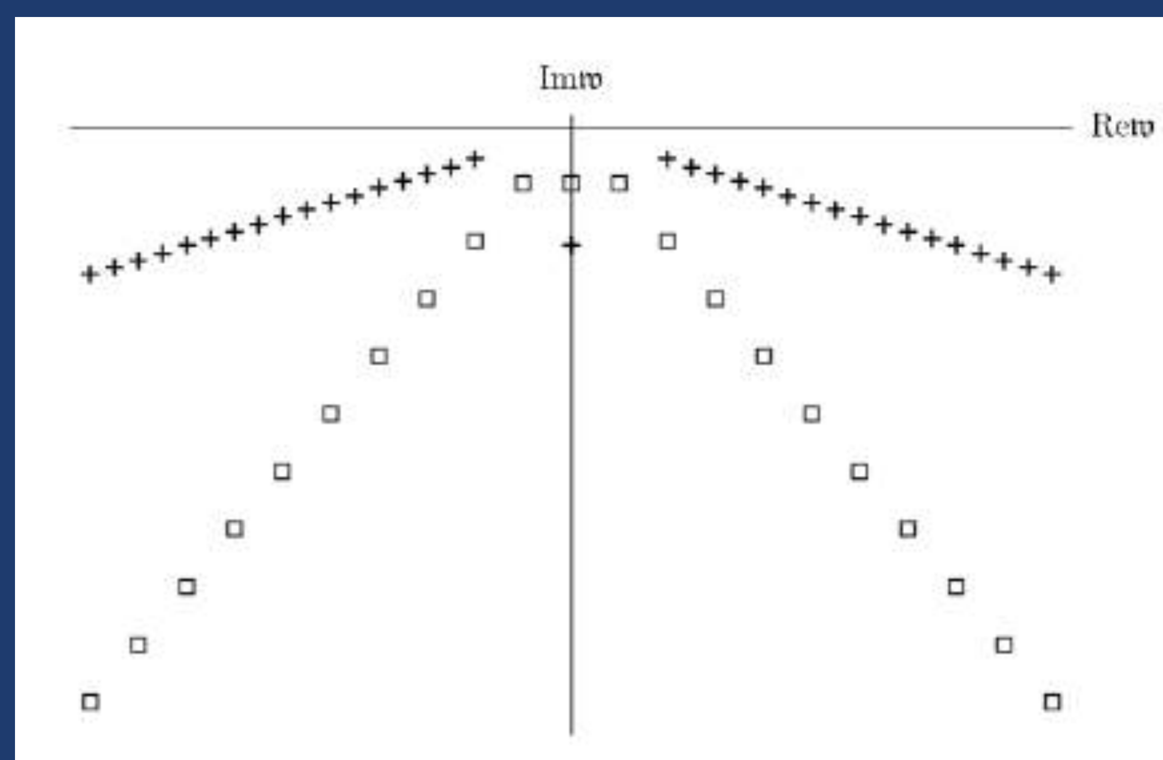
BBGKY hierarchy – RTA-like truncations [SG, Soloviev, *to appear*]

HOLOGRAPHY AND THERMAL SPECTRUM

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$

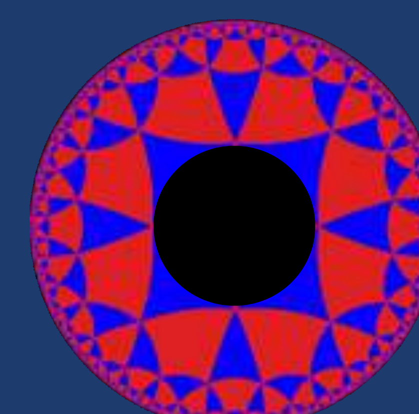


$\lambda \to 0$ | cut from poles | $\lambda \to \infty$



quasinormal modes

of black branes

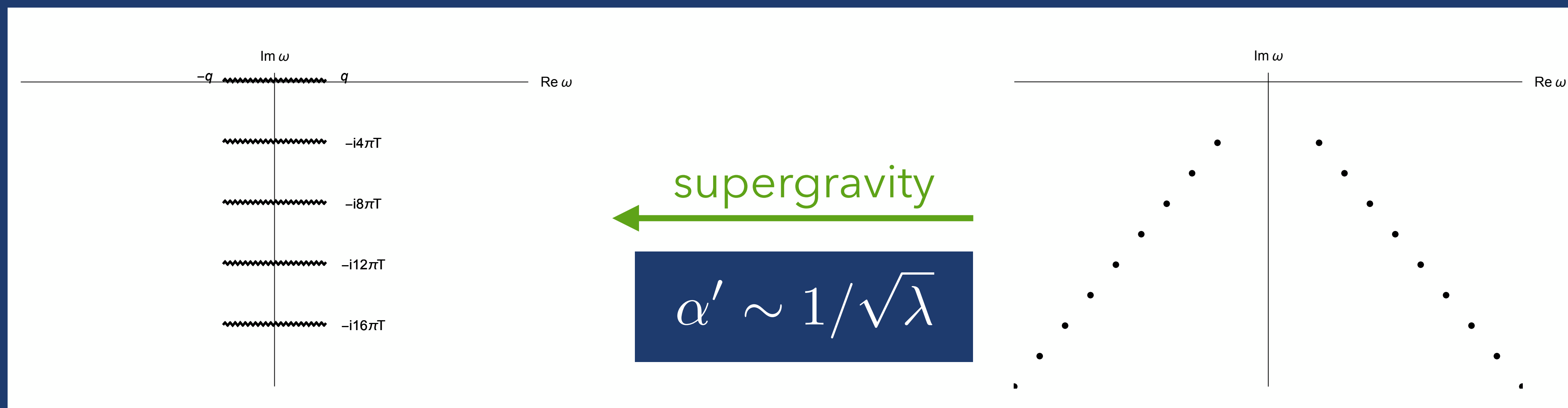


correlators remain meromorphic

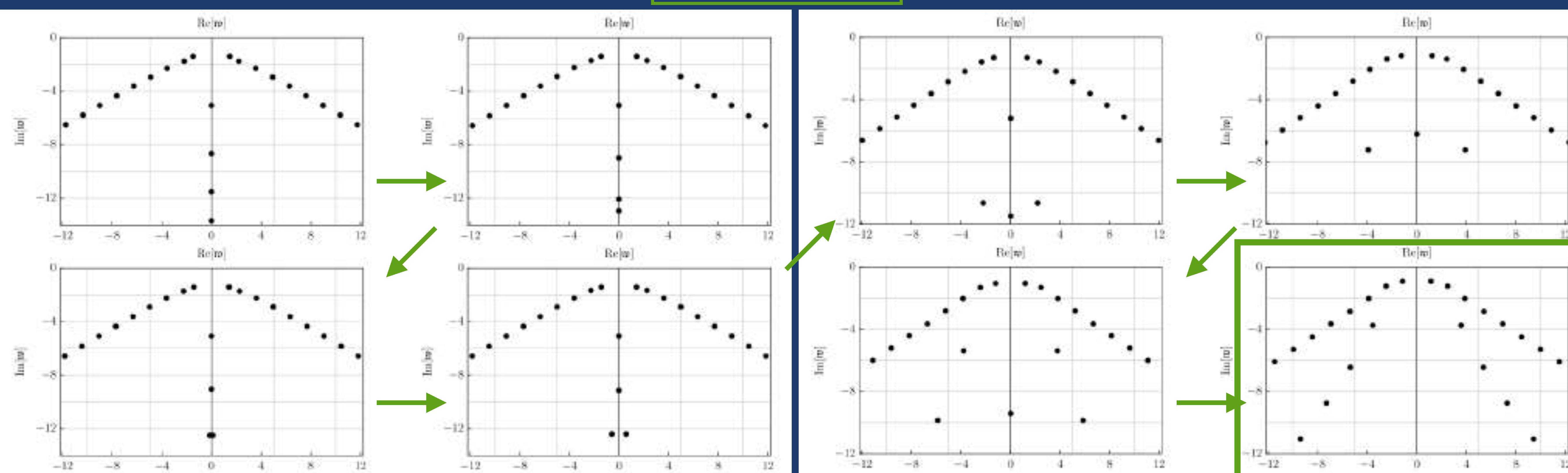
[SG, Starinets ..., several papers]

HOLOGRAPHY AND THERMAL SPECTRUM

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$



$\lambda \to 0$ | cut from poles | $\lambda \to \infty$



[SG, Starinets …, several papers]

OUTLINE

I. spectrum can be reconstructed from a single QNM

II. spectrum can be reconstructed from pole skipping and hydrodynamics

III. spectral duality relation

I. SPECTRUM CAN BE RECONSTRUCTED FROM A SINGLE QNM

[SG, Lemut, JHEP (2022)]

I. SPECTRAL RECONSTRUCTION FROM ONE QNM

- hydrodynamic series are convergent Puiseux series (shear $p=1$, sound $p=2$)
  [SG, Kovtun, Starinets, Tadić, PRL (2019); … ; also Withers; JHEP (2018); Heller, et.al. (2020, …)]

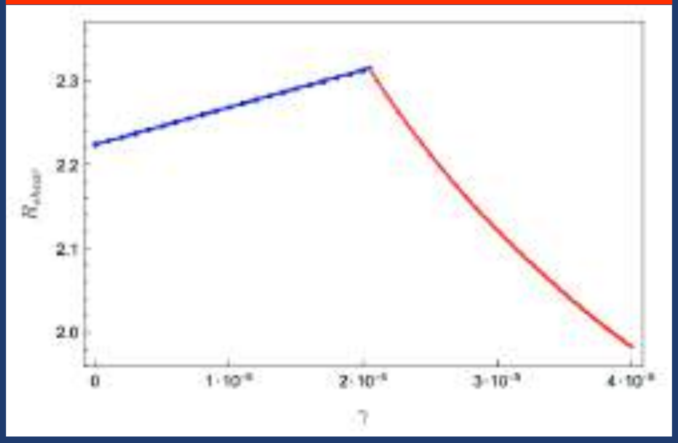
$$\mathfrak{w}_{\text{shear}} = -i\sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots$$

$$\mathfrak{w}_{\text{sound}} = -i\sum_{n=1}^{\infty} a_n e^{\pm\frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} = \pm v_s\mathfrak{q} - \frac{i}{2}\mathfrak{G}\mathfrak{q}^2 + \dots$$

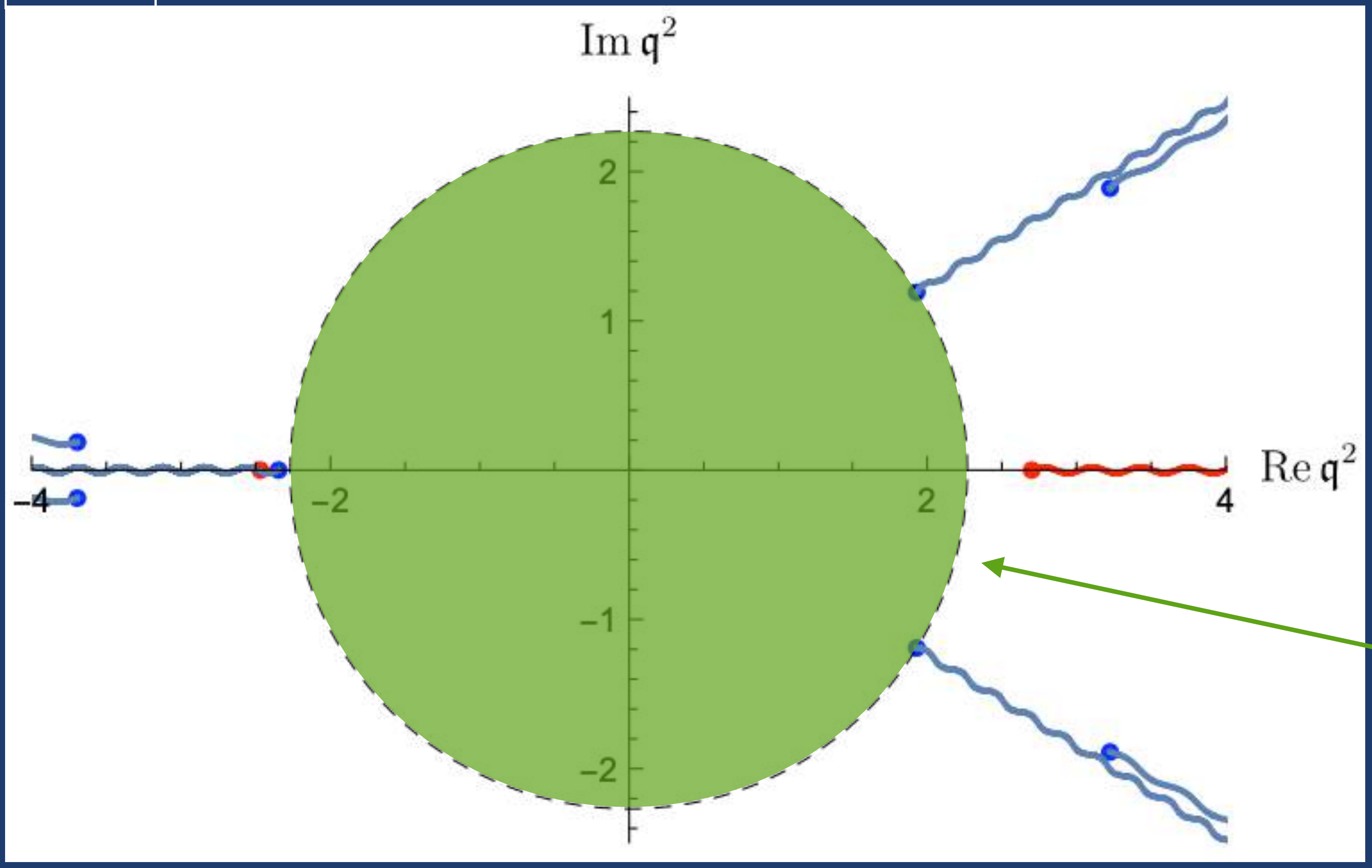
- dispersion relations are holomorphic in a disk

$$R_{\text{shear}}(\lambda) = 2.22\left(1 + 674.15\,\lambda^{-3/2} + \cdots\right)$$
$$R_{\text{sound}}(\lambda) = 2\left(1 + 481.68\,\lambda^{-3/2} + \cdots\right)$$

N=4 SYM radius convergence
[SG, Starinets, Tadić, JHEP (2021)]



$\omega(q^2)$



holomorphic disk

I. SPECTRAL RECONSTRUCTION FROM ONE QNM

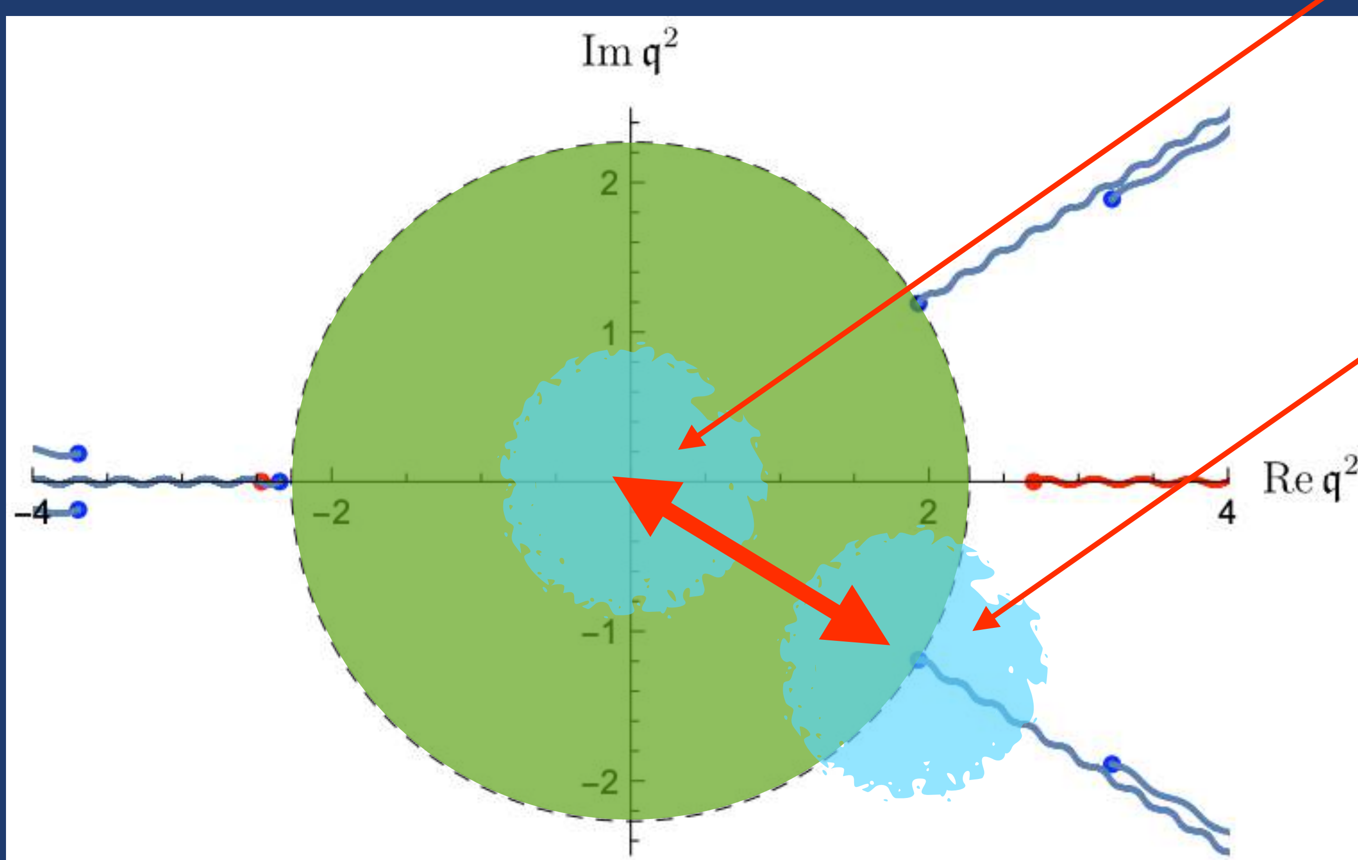
claim: systematic reconstruction of *all* modes connected via *level-crossing* is possible by exploration (analytic continuations) of the Riemann surface connecting physical modes

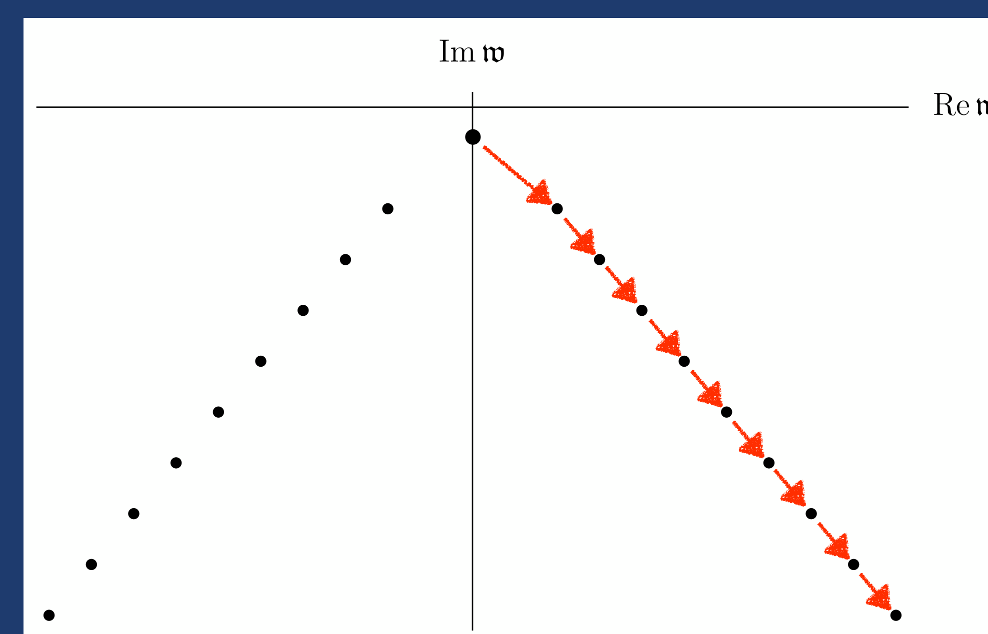
- algorithm combining a theorem by Puiseux and a theorem by Darboux
- statement should hold for spectra that are ‘sufficiently complicated’ like the Heun function

$$\omega_0(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$\omega_0(z) = -i \sum_{n=0}^{\infty} e^{\frac{i\pi n}{2}} b_n (z - z_1)^{n/2}$$

$$\omega_1(z) = -i \sum_{n=0}^{\infty} e^{-\frac{i\pi n}{2}} b_n (z - z_1)^{n/2}$$





all UV modes from one IR mode

EXAMPLE: DIFFUSION OF M2 BRANES (ADS4/CFT3)

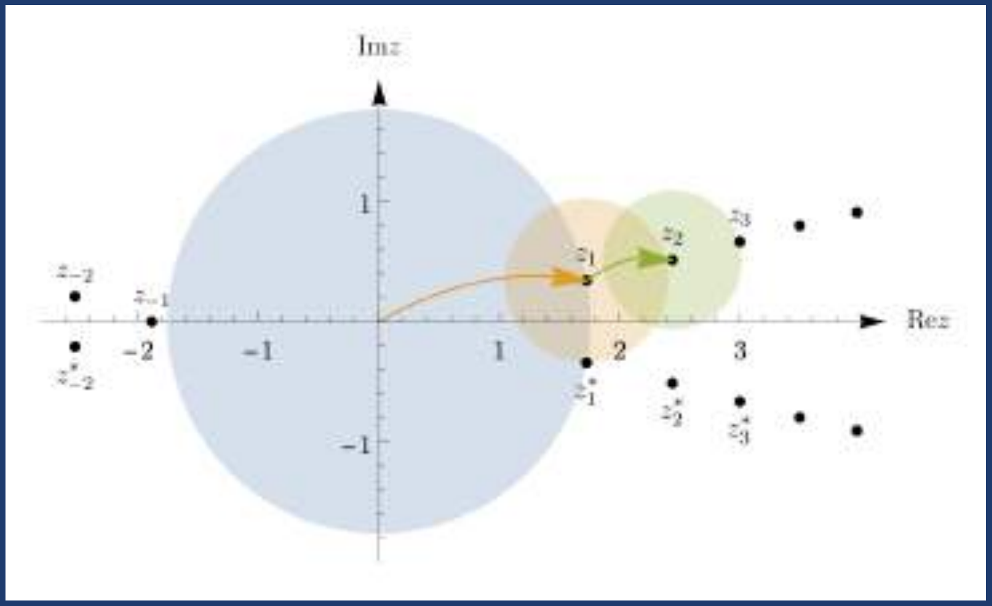
- start from 300 hydrodynamic coefficients $\omega_0(z) = \sum_{n=0}^{\infty} a_n z^n$
- use algorithm with 2 c.c. critical points, 'recover' 12 coefficients and compute the gap with analytic continuation on the same sheet (Padé approximant, …)

$$\mathfrak{w}_1(z) = \sum_{n=0}^{(N_1=12)-1} b_n (z - z_1)^{n/2} \quad \longrightarrow \quad \begin{aligned} \mathfrak{w}_1^{\text{calc}}(0) &= 1.23506 - 1.76338i \\ \mathfrak{w}(0) &= 1.23455 - 1.77586i \end{aligned}$$

- (re)compute the first 300 coefficients, use algorithm with 2 general critical points, 'recover' 12 coefficients and compute the gap

$$\mathfrak{w}_2(z) = \sum_{n=0}^{(N_2=12)-1} c_n (z - z_2)^{n/2} \quad \longrightarrow \quad \begin{aligned} \mathfrak{w}_2^{\text{calc}}(0) &= 2.16275 - 3.25341i \\ \mathfrak{w}_2(0) &= 2.12981 - 3.28100i \end{aligned}$$

- … exploration continues …
- conceptually useful and instructive, practically not (yet)…



II. SPECTRUM CAN BE RECONSTRUCTED FROM POLE SKIPPING

[SG, Lemut, Pedraza, PRD (2023)]

II. SPECTRUM FROM POLE SKIPPING

- pole skipping: ubiquitous feature of thermal correlators and black hole perturbations [SG, Schalm, Scopelliti, PRL (2017); Blake, Lee, Liu, JHEP (2018); Blake, Davison, SG, Liu, JHEP (2018); SG, JHEP (2019)]
- originally: all-order hydrodynamic sound mode $\omega(q) = \sum_{n=1}^{\infty} \alpha_n \left(T, \mu_i, \langle \mathcal{O}_j \rangle, \lambda\right) q^n$

  passes through a 'chaos point' at where the associated *2*-pt function is '0/0':

$$\omega(q = i\lambda_L / v_B) = i\lambda_L = 2\pi T i$$

$$G_R = \frac{0}{0} = N(\delta\omega/\delta q)$$

- relation to maximal quantum chaos as measured by the out-of-time-ordered correlation functions
- triviality of the Einstein equation at the horizon
- infinite number of such '0/0' points at negative Matsubara frequencies for $q \in \mathbb{C}$ [SG, Kovtun, Starinets, Tadić, JHEP (2019); Blake, Davison, Vegh, JHEP (2019)]

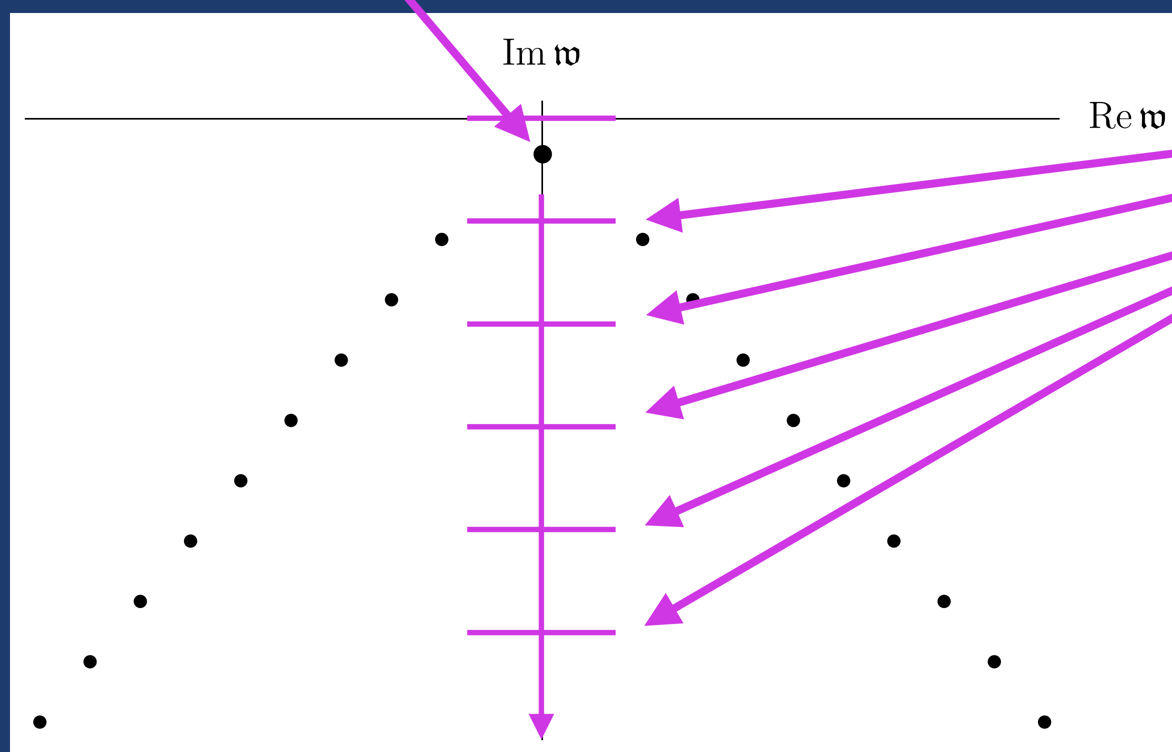
$$\omega_n(q_n) = -2\pi T i n \qquad n \geq 0$$

II. SPECTRUM FROM POLE SKIPPING

- consider diffusion in a neutral CFT dual to AdS-Schwarzschild black brane

$$\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i \mathfrak{D} \mathfrak{q}^2 + \dots$$

for increasing real q



$$\omega_n(q_n) = -2\pi T i n$$

analytic result in AdS4/CFT3
[SG, PRL (2021)]

$$q_n = \frac{4\pi T}{\sqrt{3}} n^{1/4}, \quad n = 0, 1, 2, \dots$$

claim: in holographic theories of the type discussed here (N=4 SYM, M2, M5, …), the entire spectrum can be computed from only a discrete set of pole-skipping points

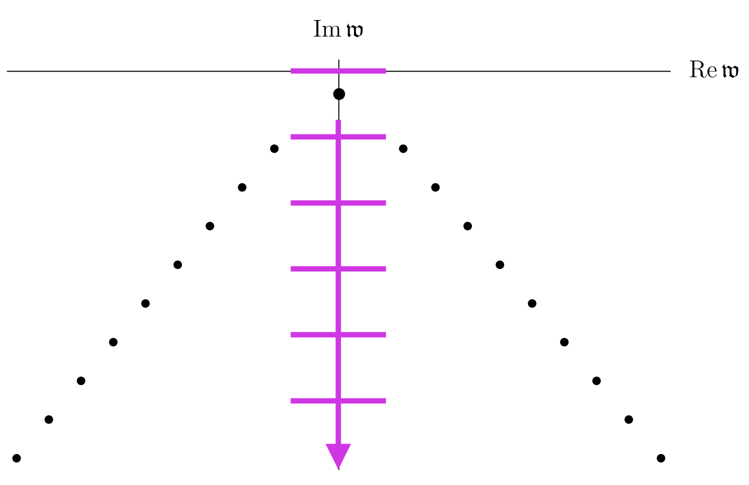
II. SPECTRUM FROM POLE SKIPPING

- interpolation problem:

$$\omega_n(q_n) = -2\pi T i n$$

$$\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots$$

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



for increasing real q

- unique solutions to interpolation problems are extremely hard in the absence of special known properties of the function (e.g. Weierstrass, Hadamard, Nevanlinna-Pick, …)
- trick: ‘analytic continuation’ to *d* spacetime dimensions and expansion around infinite *d*
- general relativity in large *d* drastically simplifies

$$V \sim 1/r^d$$

- recall: large-*d* limit of quantum mechanics is useful in atomic physics (e.g., for Helium)
- convergence of such series depends on the details

II. SPECTRUM FROM POLE SKIPPING

- interpolation: $\omega_n(q_n) = -2\pi T i n$ → $\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots$
- 'analytic continuation' to *d* spacetime dimensions and expansion around infinite *d*

$$\omega_0(q) = -i\left(\frac{q}{\sqrt{d}}\right)^2 - i\sum_{m=2}^{\infty}\frac{1}{d^m}\sum_{j=2}^{m} c_{m,n}\left(\frac{q}{\sqrt{d}}\right)^{2j}$$

$$\frac{q_n}{\sqrt{d}} = \sqrt{\frac{nd}{2}}\left(1 + \sum_{m=1}^{\infty}\frac{b_{n,m}}{d^m}\right)$$

→

$$b_{n,1} = -\sum_{m=2}^{\infty}\frac{n^{m-1}c_{m,m}}{2^m}$$

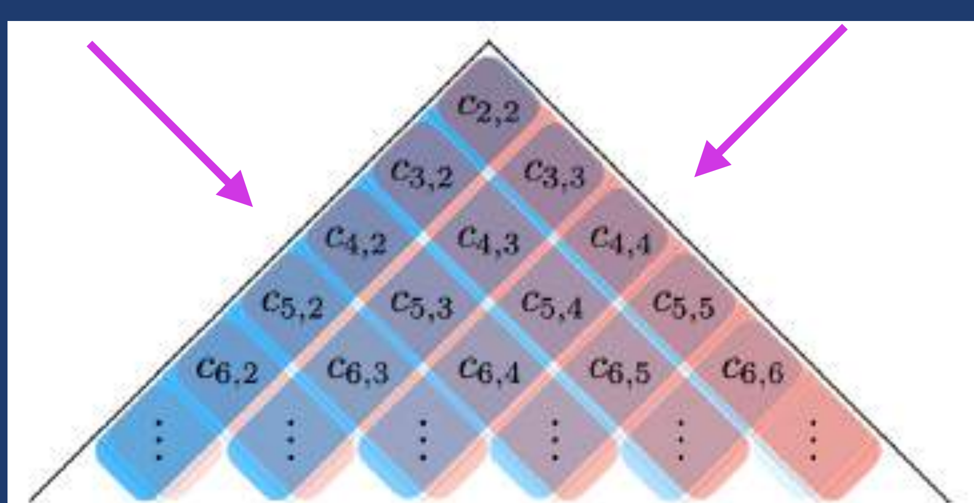
$$b_{n,2} = -\frac{b_{n,1}^2}{2} - \sum_{m=2}^{\infty}\frac{n^{m-1}\left(c_{m+1,m} + 2m b_{n,1} c_{m,m}\right)}{2^m}$$

second analytic continuation

$n \in \mathbb{Z} \to x \in \mathbb{R}$

hydrodynamics

pole skipping



symmetry?

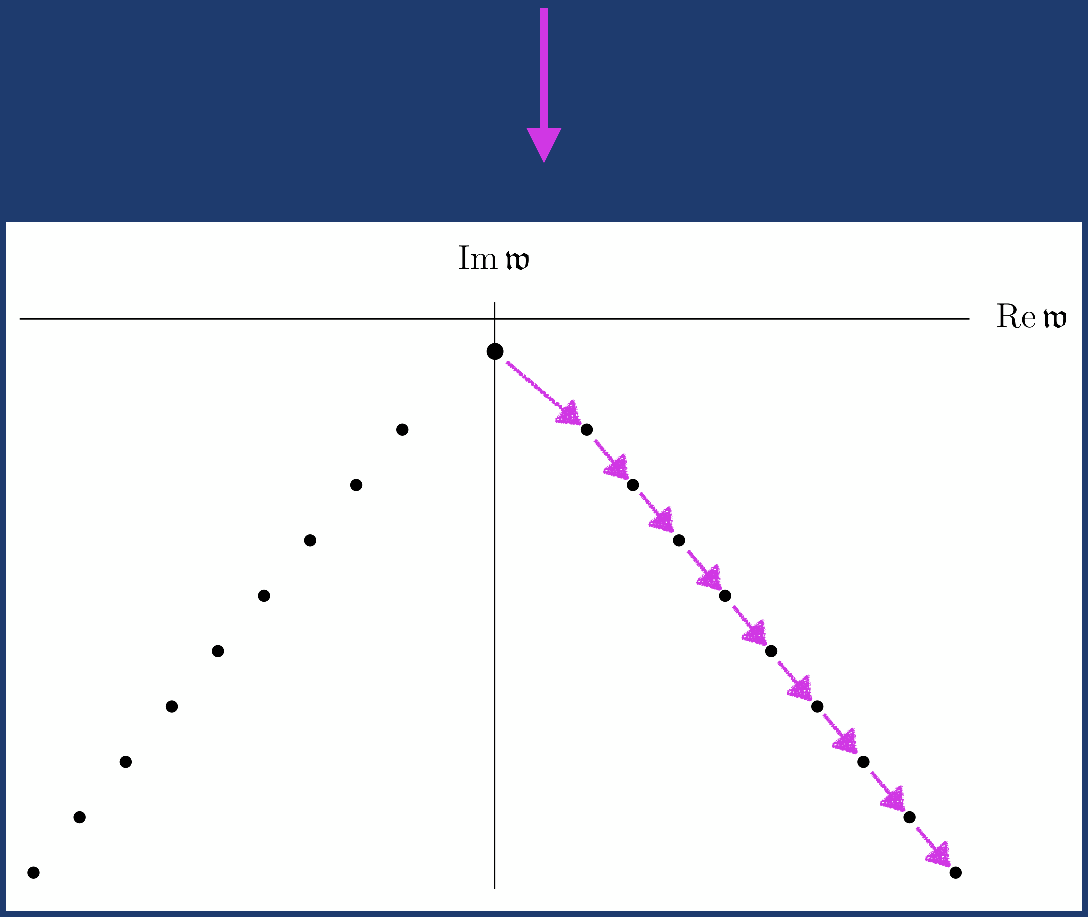
$$c_{m,m} = -\frac{2^m}{(m-1)!}\partial_x^{m-1} b_1(0)$$

$$c_{m+1,m} = -\frac{2^m \partial_x^{m-1} b_2(0)}{(m-1)!} + \sum_{j=2}^{m-1}\left(j - \frac{1}{4}\right) c_{j,j} c_{m-j+1,m-j+1}$$

generating functions

II. SPECTRUM FROM POLE SKIPPING

- the rest of the spectrum follows from a reconstruction in Fact I.



- complete reconstruction of the spectrum using only algebraic 'near-horizon' manipulations (local instead of global (ODE/PDE) analysis)
- thermal product formula: meromorphic correlators follow from poles (QNMs) and a function of q [Dodelson, Iosso, Karlsson, Zhiboedov (2023)]

III. SPECTRAL DUALITY RELATION

[SG, Vrbica, PRL (2024); SG, Vrbica, *2 x May*]

III. SPECTRAL DUALITY RELATION

- two channels of perturbations in AdS4/CFT3 cases: even (or sound) and odd (or shear)
- two meromorphic CFT retarded correlators (e.g. of $T^{\mu\nu}$ or J^{μ}) $G_{\pm}(\omega, q)$ with QNMs $\omega_n^{\pm}(q)$
- CFT3s have S-duality or particle-vortex duality
  gravity in 4*d* has Chandrasekhar duality, Darboux duality, EM duality
- finite-*T* bulk considerations and black holes:

duality:

$$G_+(\omega,q)G_-(\omega,q) = \frac{\omega^2}{\omega_*^2(q)} - 1$$

self-duality: $\omega_*(q) \to \infty$

$$G_+(\omega,q)G_-(\omega,q) = -1$$

algebraically special frequencies
relation to pole skipping [SG, Vrbica, EPJC (2023)]
easy to compute (Robinson-Trautman solution in GR)

III. SPECTRAL DUALITY RELATION

- define infinite convergent product

$$S(\omega, q) \equiv \left(1 + \frac{\omega}{\omega_*(q)}\right) \prod_n \left[1 - \frac{\omega}{\omega_n^+(q)}\right] \left[1 + \frac{\omega}{\omega_n^-(q)}\right]$$

- duality relations, the thermal product formula and details about QNM and Greens function asymptotics give a 'universal' relation:

$$S(\omega, k) - S(-\omega, k) = 2i\lambda(k) \sinh \frac{\beta\omega}{2}$$

$$\lambda(k) = \frac{2}{i\beta} \left[\frac{1}{\omega_*(k)} + \sum_n \left(\frac{1}{\omega_n^-(k)} - \frac{1}{\omega_n^+(k)}\right)\right]$$

III. SPECTRAL DUALITY RELATION

- infinite towers of constraints
- constraints 1:

$$e_{2j+1}(\mathcal{W}) = \frac{i\lambda}{(2j+1)!}\left(\frac{\beta}{2}\right)^{2j+1}$$

elementary symmetric polynomials with $j \in \{0,1,\ldots,n\}$

$$\mathcal{W} = \{1/\omega_*, 1/\omega_1^-, \ldots, 1/\omega_n^-, -1/\omega_1^+, \ldots, -1/\omega_n^+\}$$

- constraints 2:

$$\left(1+\frac{\Omega_\ell}{\omega_*}\right)\prod_n \frac{1-\frac{\Omega_\ell}{\omega_n^+}}{1+\frac{\Omega_\ell}{\omega_n^+}} = \left(1-\frac{\Omega_\ell}{\omega_*}\right)\prod_n \frac{1-\frac{\Omega_\ell}{\omega_n^-}}{1+\frac{\Omega_\ell}{\omega_n^-}}$$

Matsubaras: $\Omega_\ell = 2\pi i\ell/\beta$

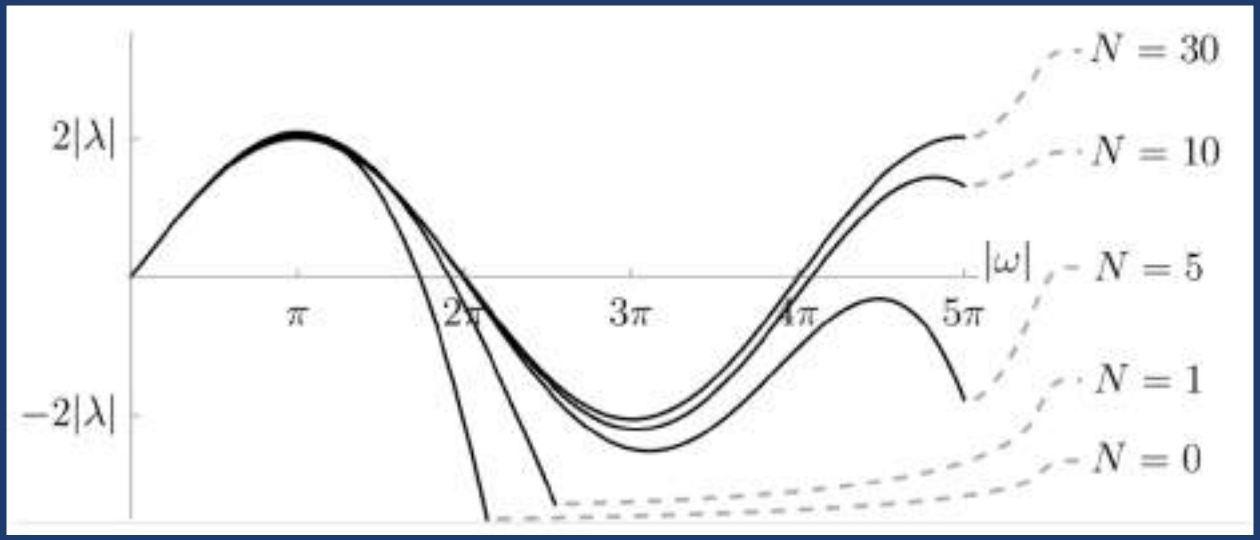
III. SPECTRAL DUALITY RELATION

- AdS-Schwarzschild black brane
  - $\langle J, J \rangle_R$ is self-dual

    poles must converge to Matsubara frequencies $\frac{iD_c\beta}{2} \lim_{q\to 0} k^2 S(\omega) = \sinh \frac{\beta\omega}{2}$

    ...
  - $\langle T, T \rangle_R$ has $\omega_* = i\frac{\gamma q^4}{6\bar{\epsilon}}$

    various hydro constraints follow, e.g.: $D/\Gamma = 2$



- AdS-Reissner-Nordström black brane
  - channels are coupled with $\omega_* = i\frac{\gamma q^4}{6\bar{\epsilon}}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \left(\frac{2Q\gamma}{3\bar{\epsilon}}\right)^2 q^2}\right)^{-1}$

III. SPECTRAL DUALITY RELATION

- duality relation for any pair of meromorphic correlators:

$$G_+(\omega)G_-(\omega) = -1$$

$$S(\omega) = \prod_n \left(1 - \frac{\omega}{\omega_n^+}\right)\left(1 + \frac{\omega}{\omega_n^-}\right)$$

$$S(\omega) - S(-\omega) = 2i\lambda \sinh \frac{\beta\omega}{2}$$

- knowing one spectrum is sufficient for determining the other spectrum!

- we also prove various statements about spectra (poles and zeros)
- zeros and poles of any correlator:

$$-G(\omega)G^{-1}(\omega) = -1$$

$$S(\omega) = \prod_n \left(1 - \frac{\omega}{\omega_n}\right)\left(1 + \frac{\omega}{z_n}\right)$$

thermal product formula →

$$r_n = \lambda G(0) \frac{\omega_n \sinh \frac{\beta\omega_n}{2}}{2\prod_{\substack{m \\ m\neq n}} \left(1 - \frac{\omega_n^2}{\omega_m^2}\right)}$$

III. SPECTRAL DUALITY RELATION

- large-N field theory and double-trace deformed RG flow

$$Z_f[J] = e^{\Gamma[J,f]} = \left\langle e^{\int \mathcal{O} J - \frac{f}{2}\int \mathcal{O}^2} \right\rangle \qquad G(f) = \frac{G_0}{1 + f G_0}$$

$$[G(f_1)(f_2 - f_1) + 1]\,[G(f_2)(f_2 - f_1) - 1] = -1$$

- using duality relations between UV and IR CFTs

$$Z_{f_-}[J_-] = e^{\Gamma_-[J_-, f_-]} = \left\langle e^{\int \mathcal{O}_- J_- - \frac{f_-}{2}\int \mathcal{O}_-^2} \right\rangle_-$$

$$Z_{f_+}[J_+] = e^{\Gamma_+[J_+, f_+]} = \left\langle e^{\int \mathcal{O}_+ J_+ - \frac{f_+}{2}\int \mathcal{O}_+^2} \right\rangle_+$$

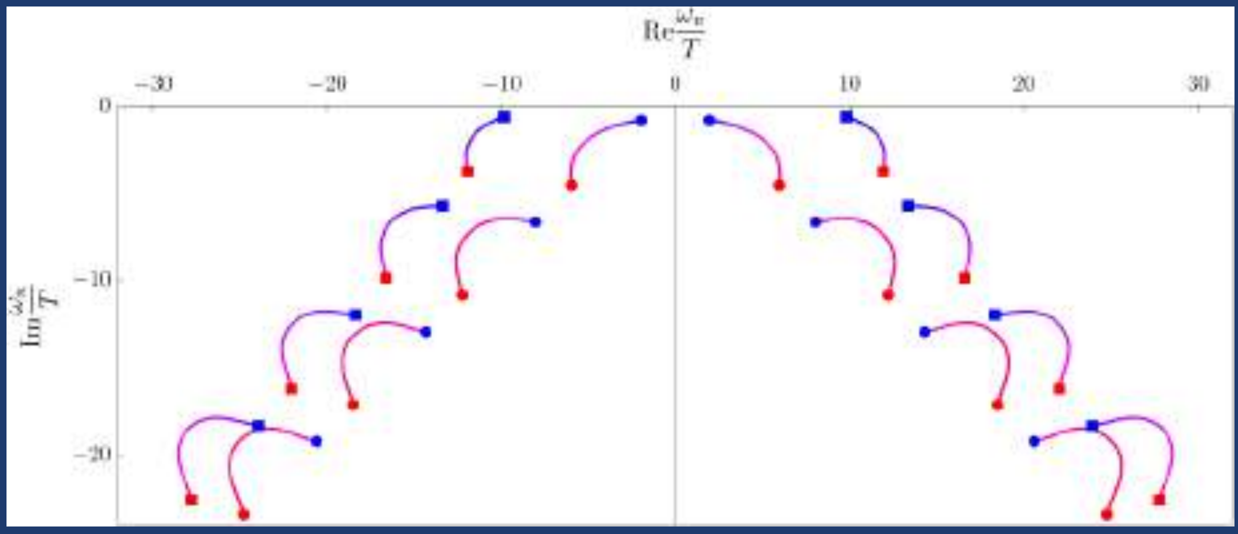
$$\Delta_+ + \Delta_- = d$$

$$f_+ f_- = -1$$

$$G_-(f_-) = f_+\,[f_+ G_+(f_+) - 1]$$

$$G_0^- G_0^+ = -1$$

- same statement applies to correlators in any pair of theories related by Legendre transform



SUMMARY AND FUTURE DIRECTIONS

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- QFTs and gravity exhibit extremely interesting and powerful (mathematical) structures
- large classes of large-N QFTs exhibit stringent constraints on spectra
- results apply to gravity
- keep exploring…
  … ideally in realistic QFTs

THANK YOU!

STUFF

MORE DETAILS ON II. SPECTRUM FROM POLE SKIPPING

$$\omega_0(q) = -i\left(\frac{q}{\sqrt{d}}\right)^2 - i\sum_{m=2}^{\infty}\frac{1}{d^m}\sum_{j=2}^{m} c_{m,n}\left(\frac{q}{\sqrt{d}}\right)^{2j}$$

$$\frac{q_n}{\sqrt{d}} = \sqrt{\frac{nd}{2}}\left(1+\sum_{m=1}^{\infty}\frac{b_{n,m}}{d^m}\right)$$

- first level: $b_{n,1} = -\frac{1}{2}H_n = -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$ → $H_n \to H(x) = \sum_{k=1}^{\infty}\frac{x}{k(x+k)}$

→ $c_{m,2} = c_{m,m} = (-1)^m 2^{m-1}\zeta(m)$ $\quad$ $\omega_0(q) = -i\bar{q}^2 - i\frac{\bar{q}^2}{d}H_{2\bar{q}^2/d} + \dots$

- second level, …
- results: $b_{n,1}$ → $c_{m,2} = c_{m,m} = (-1)^m 2^{m-1}\zeta(m)$

$b_{n,3}$ → $c_{4,2} \approx 1.000 \times 8\zeta(4)$, $c_{5,3} \approx -15.502 \times 16\zeta(5)$

$b_{n,2}$ → $c_{3,2} \approx -1.000 \times 4\zeta(3)$, $c_{4,3} \approx 7.001 \times 8\zeta(4)$, $c_{5,4} \approx -15.548 \times 16\zeta(5)$, $c_{6,5} \approx 27.546 \times 32\zeta(6)$

- hidden symmetry?