Is Schwinger-Keldysh Hydrodynamics Compatible with Thermodynamic Equilibrium?

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Foundations and Applications of Relativistic Hydrodynamics
May 05, 2025 • GGI Florence

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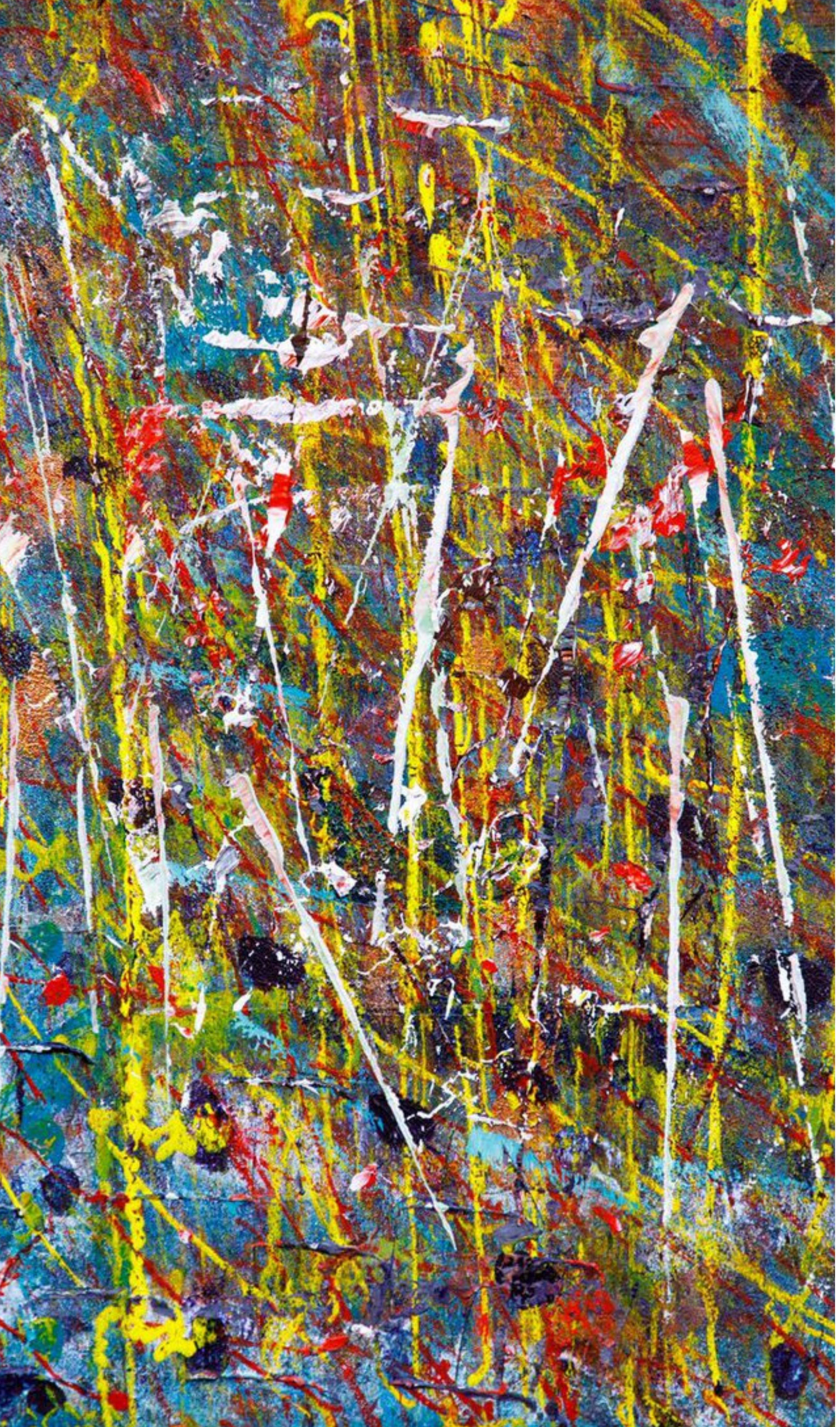
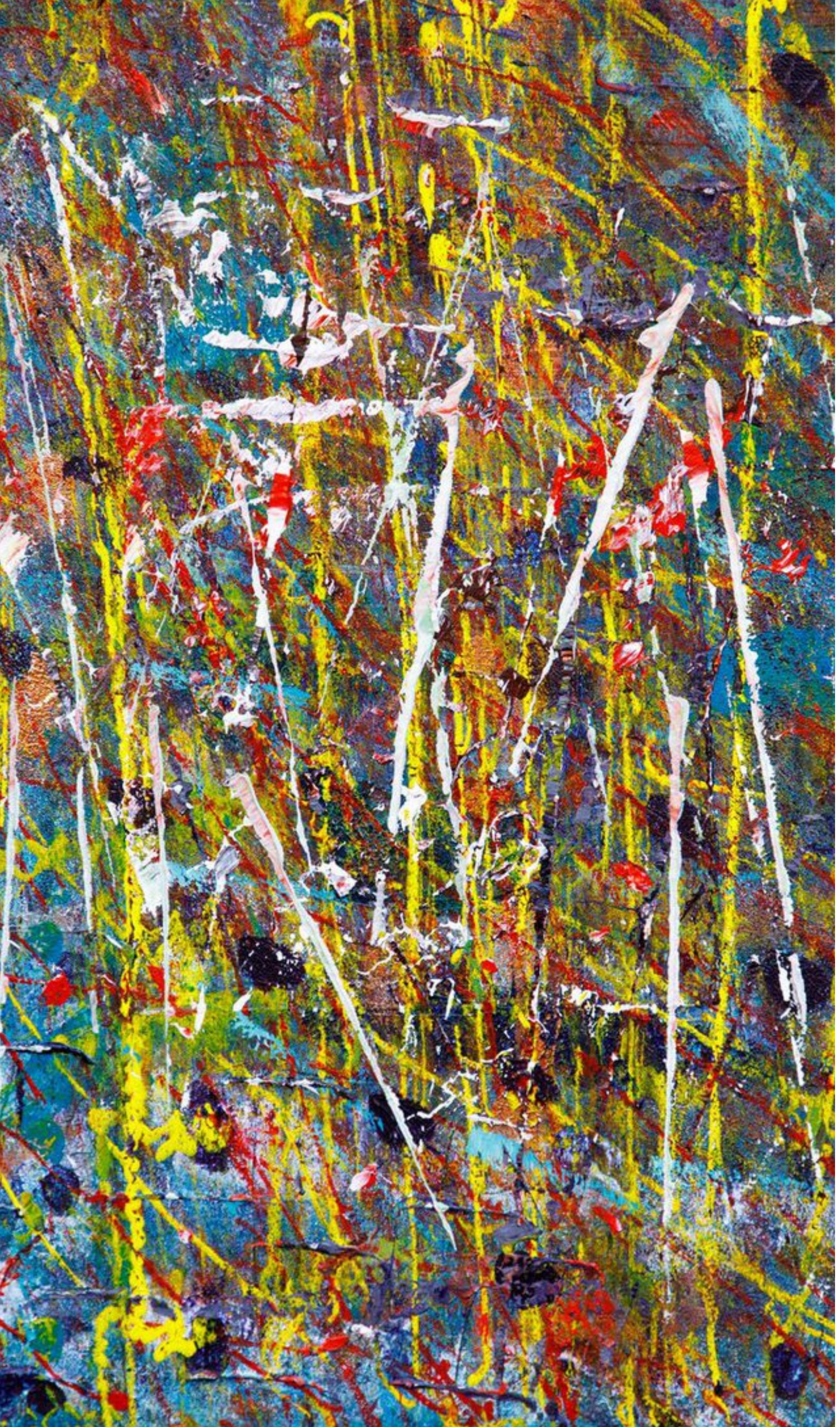


The big picture

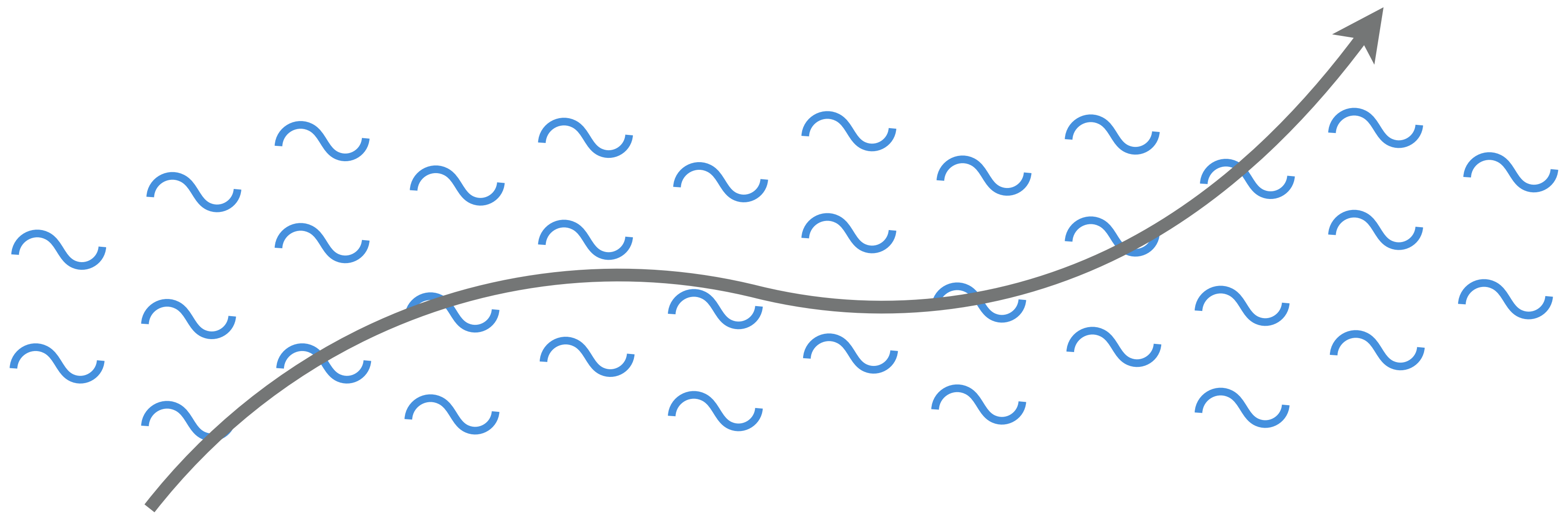
- Effective field theory (EFT) is a powerful tool to construct low-energy effective models for complex many-body systems, without the detailed knowledge of their microscopic constituents.
- The conventional EFT framework is only suited to systems at zero temperature or systems in thermal equilibrium (statistical field theory).
- Hydrodynamics provides a universal framework to describe low-energy effective dynamics of complex many-body systems operating *close to thermal equilibrium*.
- Classical hydrodynamics is *"like"* an EFT. It involves *effective fields* for local fluid velocity, temperature, density, etc, whose dynamics is dictated by universal symmetry principles.
  However, it is *not* based on an effective action principle.

This has changed recently!



The big picture

- Classical hydrodynamics has limited applicability due to the presence of stochastic thermal fluctuations.



- Classical hydrodynamics *includes dissipation but not fluctuations*, which are neccesiated by the fluctuation-dissipation theorem (FDT).
- Schwinger-Keldysh (SK) hydrodynamics is an EFT extension of classical hydrodynamics, which systematically incorporates the effects of stochastic thermal fluctuations.

[Kovtun, Moore, Romatschke 2014]
[Harder, Kovtun, Ritz 2015]
[Crossley, Glorioso, Liu 2015]
[Haehl, Loganayagam, Rangamani 2015]
[Jensen, Pinzani-Fokeeva, Yarom 2017]

The hydrodynamic landscape

Hydrodynamics

- ✓ Near-equilibrium many-body dynamics
- ✓ Based on universal conservation laws
- ✘ Includes dissipation but not fluctuations
- ✘ Incompatible with FDT

Stochastic Hydrodynamics

- ✓ Includes fluctuations
- ▲ Might violate FDT at loop levels
- ✘ Does not include non-linear stochastic noise
- ✘ No effective action principle

Statistical Field Theory

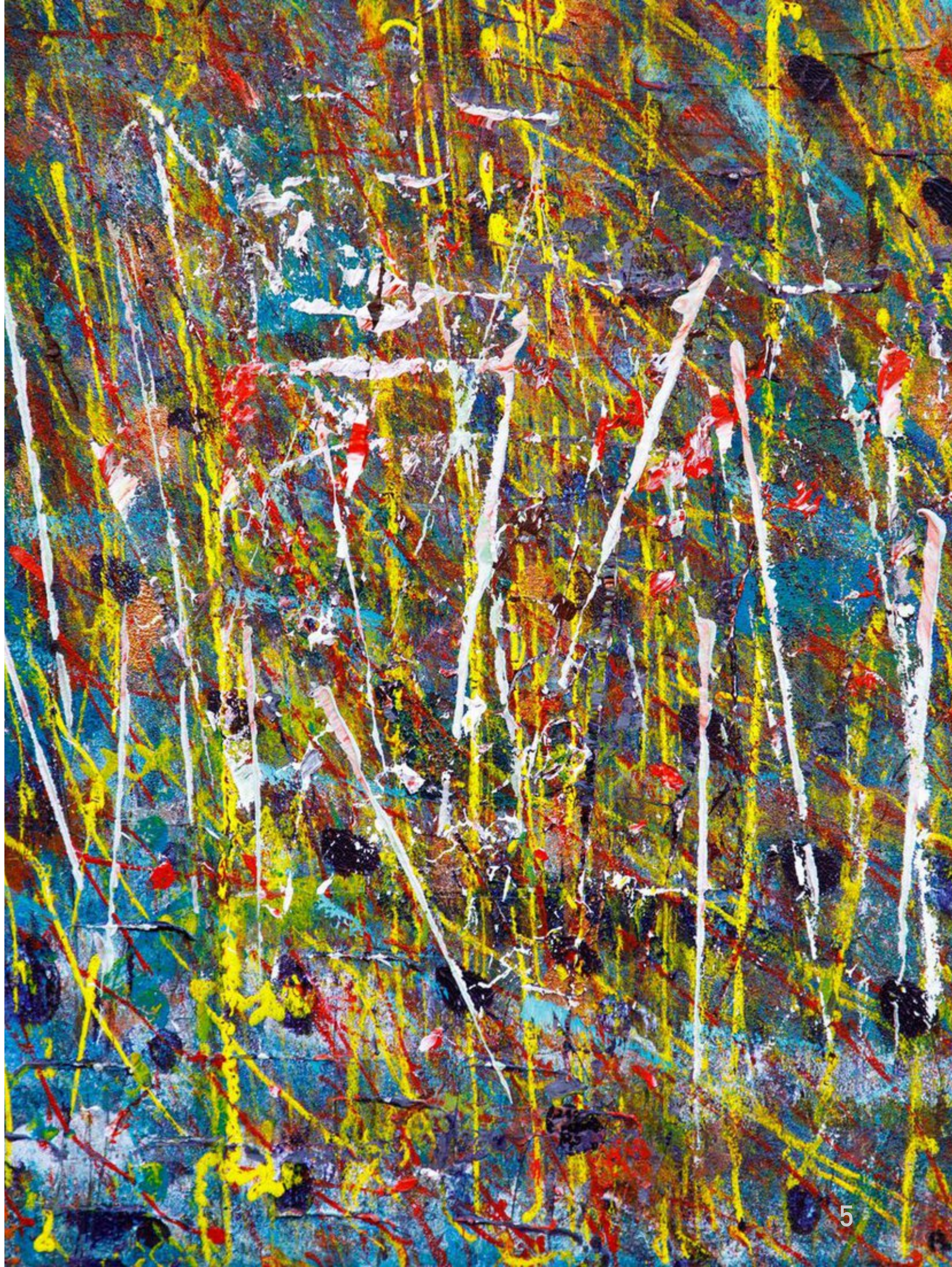
- ✓ Governs equilibrium configurations
- ✓ Equilibrium effective action (free energy)
- ✓ Includes (equilibrium) fluctuations
- ✘ Cannot be used to obtain dynamics

Schwinger-Keldysh Hydrodynamics

- ✓ Effection principle for (stochastic) hydrodynamics
- ✓ Includes general stochastic noise
- ✓ Features fully non-linear and loop-level FDT → KMS symmetry
- ✓ Symmetry based understanding of the second law
- ▲ Fairly new and unexplored!

Outline

- **Hydrostatic compatibility**
- Schwinger-Keldysh hydrodynamics
- Hydrostatic compatibility at one-loop
- BRST ghosts
- Lessons and outlook

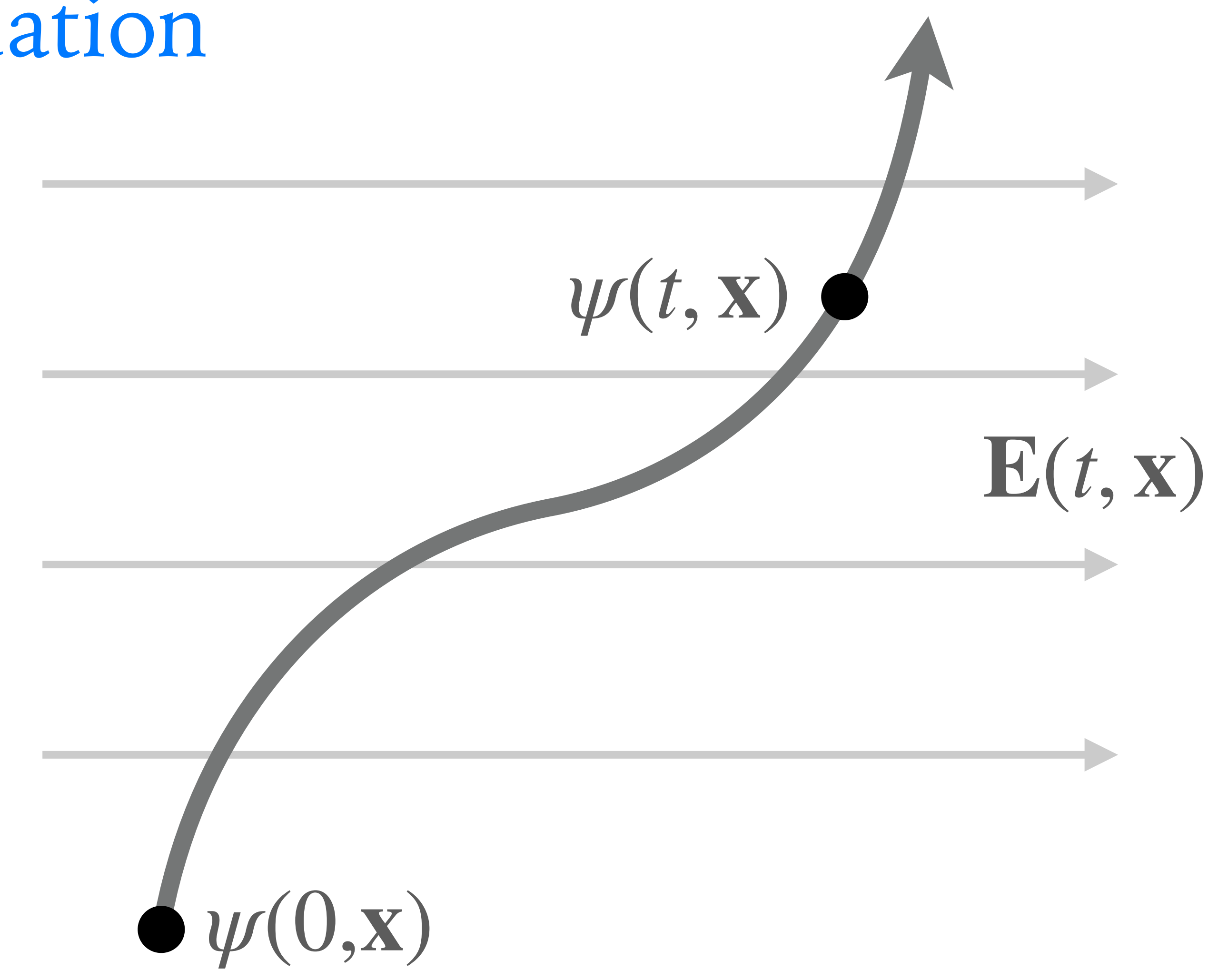


Hydrodynamics 101

- Consider hydrodynamics of a conserved scalar variable ψ, with conservation equation

$$\partial_t \psi + \nabla \cdot \mathbf{J} = 0$$

Given $\mathbf{J} = \mathbf{J}(\psi)$, this equation governs the classical evolution of ψ.



$\mathbf{E} = \nabla h$ is applied electric field

- The constitutive relations are given as

$$\mathbf{J} = -D(\psi)\,\nabla\psi \; + \; \sigma(\psi)\,\mathbf{E} \; + \; \mathcal{O}(\nabla^2)$$

$D(\psi)$: diffusion; $\sigma(\psi)$: electric conductivity

- The diffusion and conductivity are related via the chemical potential $\mu(\psi) = \delta F[\psi]/\delta\psi$,

$$D(\psi) = \mu'(\psi)\,\sigma(\psi) \quad \Longrightarrow \quad \mathbf{J} = -\sigma(\psi)\Big(\nabla\mu(\psi) - \mathbf{E}\Big) + \mathcal{O}(\nabla^2)$$

Arises from the local second law of thermodynamics, together with $\sigma \geq 0$.

Hydrostatic compatibility

- The second law *equality* constraints, such as the diffusion-conductivity constraint, ensure the hydrostatic compatibility of classical hydrodynamics:

> In the presence of time-independent background sources,
> hydrodynamic equations must admit a time-independent equilibrium solution,
> governed by an equilibrium generating functional.

[Banerjee et al. 2012]
[Jensen et al. 2012]

$$\mathcal{Z}_{\mathrm{eqb}}[h] = \int_{\mathrm{eqb}} \mathcal{D}\psi \, \exp\left(-F[\psi] + \int_{\mathbf{x}} h\,\psi \right)$$

- Classically, the generating functional is dominated by the saddle point $\mu(\psi_{\mathrm{eqb}}) = h$.
  This solves the hydrodynamic equation, provided that the second law equality constraints are obeyed

$$\partial_t \psi_{\mathrm{eqb}} + \nabla \cdot \mathbf{J}_{\mathrm{eqb}} = -\nabla \cdot \left(D(\psi_{\mathrm{eqb}})\, \nabla \psi_{\mathrm{eqb}} - \sigma(\psi_{\mathrm{eqb}})\, \mathbf{E} \right) + \mathcal{O}(\nabla^3)$$

$$= -\nabla \cdot \left(\left(\frac{D(\psi_{\mathrm{eqb}})}{\mu'(\psi_{\mathrm{eqb}})} - \sigma(\psi_{\mathrm{eqb}}) \right) \nabla h \right) + \mathcal{O}(\nabla^3) = 0$$

We will take $\beta = 1/(k_B T) = 1$.

Model A and Model B

- We will consider a slight generalisation of our scalar hydrodynamic model [Hohenberg, Halperin 1977]

$$\partial_t \psi + \Gamma\left(\mu(\psi) - h\right) = 0$$

$\Gamma = \Gamma(\psi)$ if ψ is not conserved (Model A)

$\Gamma = -\nabla\cdot\left(\sigma(\psi)\nabla\circ\right)$ if ψ is conserved (Model B)

Technically, only Model B is hydrodynamic.

- For concreteness, we use

$$F[\psi] = \frac{1}{2}\int_{\mathbf{x}}\left(\gamma(\nabla\psi)^2 + a_2\psi^2 + \frac{1}{3}a_3\psi^3 + \frac{1}{12}a_4\psi^4\right)$$

$$\mu(\psi) = \left(a_2 - \gamma\nabla^2\right)\psi + \frac{1}{2}a_3\psi^2 + \frac{1}{6}a_4\psi^3$$

$$\Gamma(\psi) = \Gamma_0 + \Gamma_1\psi + \frac{1}{2}\Gamma_2\psi^2 \qquad \text{(Model A)}$$

$$\sigma(\psi) = \sigma_0 + \sigma_1\psi + \frac{1}{2}\sigma_2\psi^2 \qquad \text{(Model B)}$$

Hydrostatic compatibility of correlation functions

- Hydrodynamics can be used to compute *classical* retarded correlation functions

[Kadanoff, Martin 1963]

$$\partial_t \psi + \Gamma\left(\mu - h\right) = 0$$

$$\mu = -\gamma \nabla^2 \psi + a_2 \psi + \dots$$

$$\Gamma_{(A)} = \Gamma_0 + \dots$$

$$\Gamma_{(B)} = -\nabla\left(\left(\sigma_0 + \dots\right)\nabla \circ\right)$$

$$G^{\text{ret,cl}} \equiv \frac{\delta}{\delta h}\psi_{\text{onshell}}[h]$$

$$G^{\text{ret,cl}}(\omega, \mathbf{k})_{(A)} = \frac{\Gamma_0}{(a_2 + \gamma\,\mathbf{k}^2)\,\Gamma_0 - i\omega} \qquad G^{\text{ret,cl}}(\omega, \mathbf{k})_{(B)} = \frac{\sigma_0\,\mathbf{k}^2}{(a_2 + \gamma\,\mathbf{k}^2)\,\sigma_0\,\mathbf{k}^2 - i\omega}$$

- The equilibrium correlation functions are governed by the generating functional

$$G^{\text{eqb}} = \frac{\delta}{\delta h}\frac{\delta}{\delta h}\ln \mathcal{Z}_{\text{eqb}}[h] \quad \longrightarrow \quad G^{\text{eqb,cl}} = \frac{\delta}{\delta h}\psi_{\text{eqb}}[h]$$

$$\mathcal{Z}_{\text{eqb}}[h] = \int_{\text{eqb}} \mathcal{D}\psi\, \exp\left(-F[\psi] + \int_{\mathbf{x}} h\,\psi\right)$$

$$G^{\text{eqb,cl}}(\mathbf{k}) = \frac{1}{a_2 + \gamma\,\mathbf{k}^2}$$

$$F[\psi] = \frac{1}{2}\int_{\mathbf{x}}\left(\gamma\,(\nabla\psi)^2 + a_2\,\psi^2 + \dots\right)$$

- Hydrostatic compatibility implies

$$G^{\text{ret,cl}}(\omega = 0, \mathbf{k}) = G^{\text{eqb,cl}}(\mathbf{k})$$

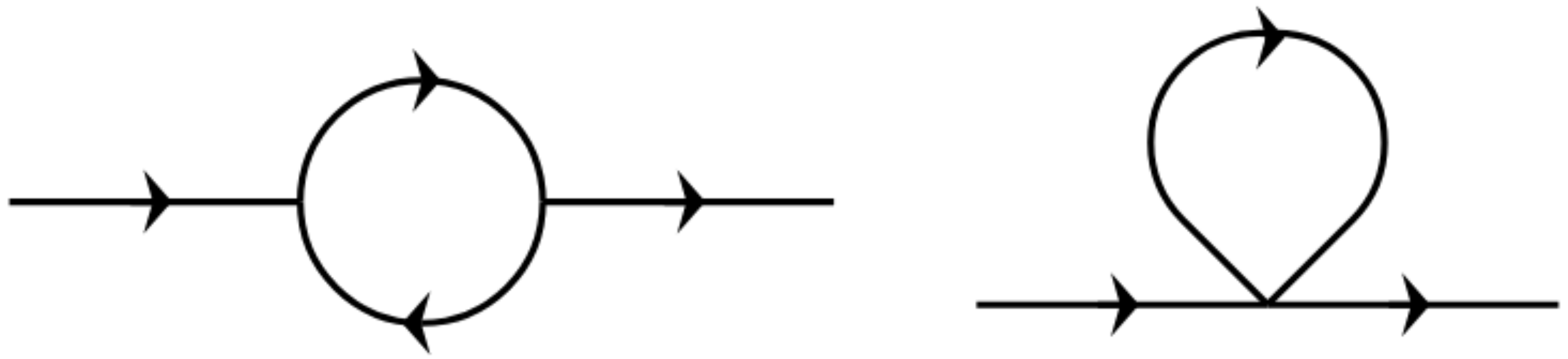
Similar compatibility conditions exist for higher-point correlation functions.

Stochastic hydrostatic compatibility

- Hydrostatic compatibility must remain intact in the presence of interactions

$$G^{\text{ret,cl}}(\omega = 0, \mathbf{k}) = G^{\text{eqb,cl}}(\mathbf{k}) \longrightarrow G^{\text{ret}}(\omega = 0, \mathbf{k}) \overset{?}{=} G^{\text{eqb}}(\mathbf{k})$$

- $G^{\text{eqb}}(\mathbf{k})$ can be obtained using the equilibrium generating functional.
  One-loop corrections:



$$F[\psi] = \frac{1}{2}\int_{\mathbf{x}} \left(\gamma\,(\nabla\psi)^2 + a_2\,\psi^2 + \frac{1}{3}\,a_3\,\psi^3 + \frac{1}{12}\,a_4\,\psi^4\right)$$

- To compute $G^{\text{ret}}(\mathbf{k})$, we need the framework of Schwinger-Keldysh (stochastic) hydrodynamics.

Is Schwinger-Keldysh hydrodynamics
compatible with hydrostatics?

Stochastic hydrostatic compatibility

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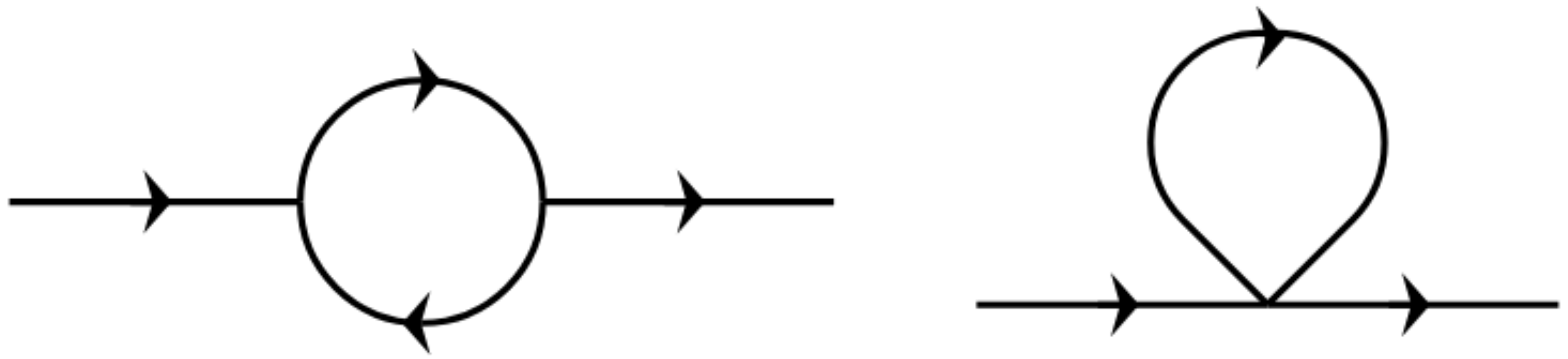

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- To compute $G^{\text{ret}}(\mathbf{k})$, we need the framework of Schwinger-Keldysh (stochastic) hydrodynamics.

Is Schwinger-Keldysh hydrodynamics compatible with hydrostatics?

Depends…

Outline

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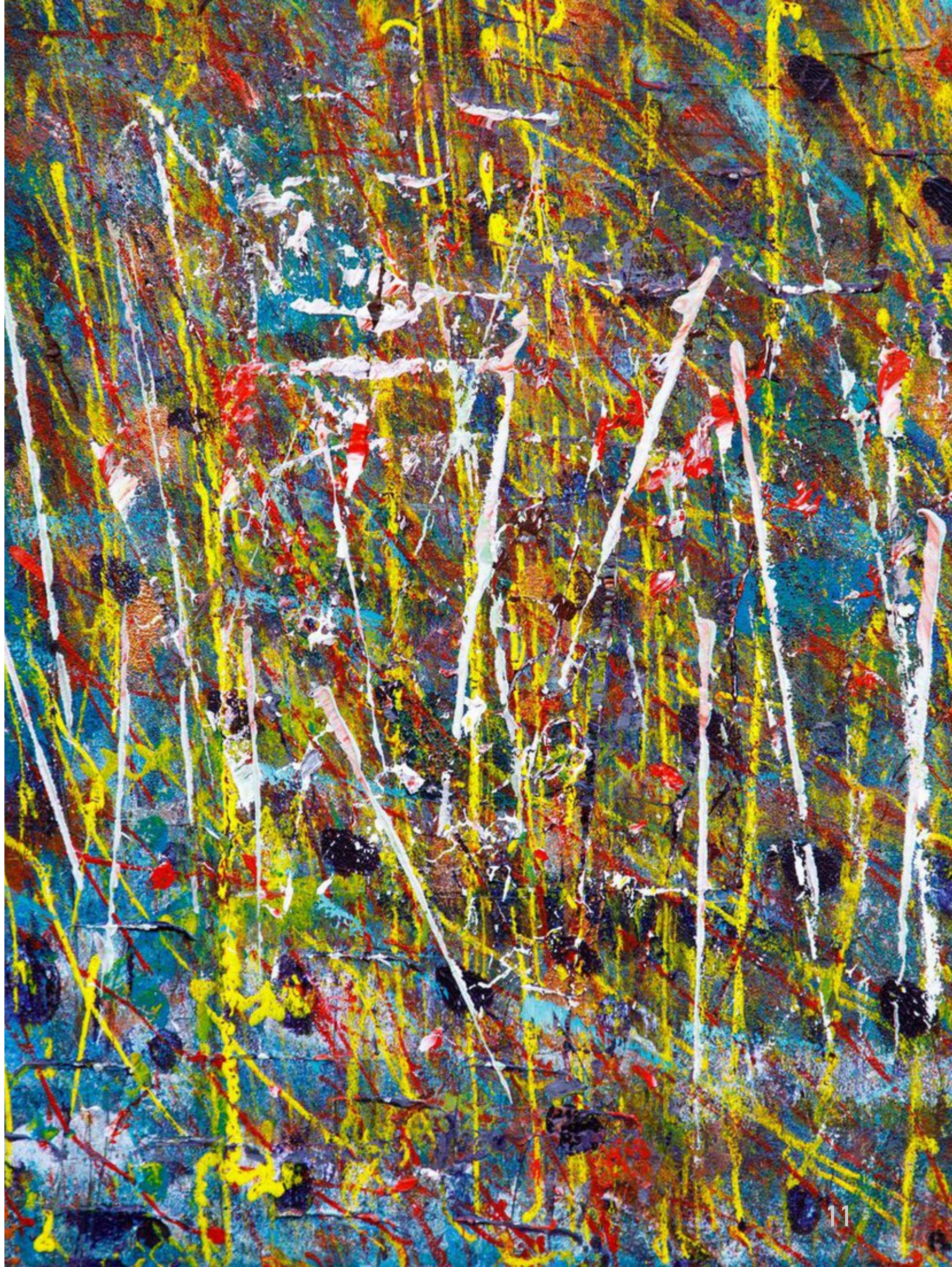
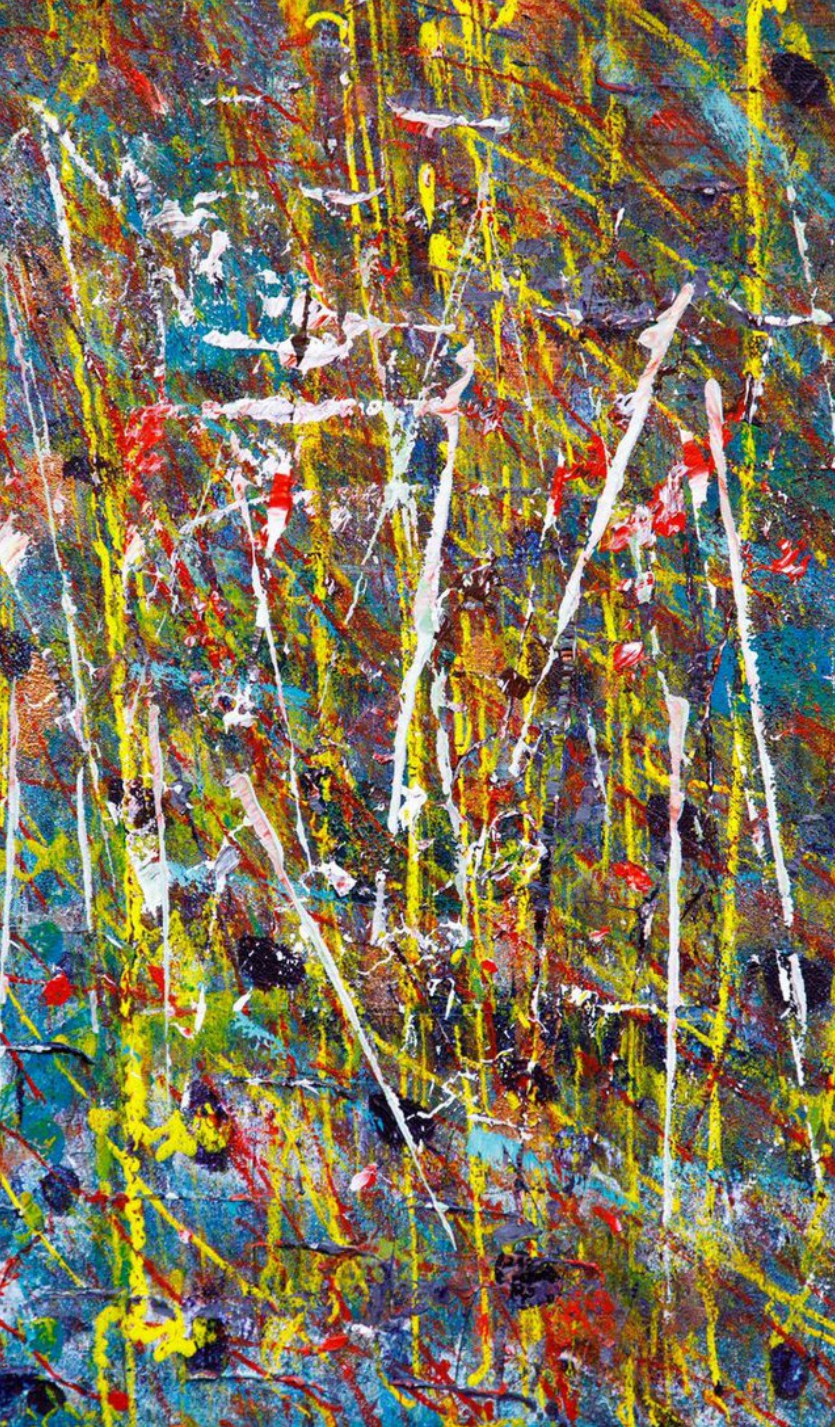
- **Schwinger-Keldysh hydrodynamics**
- Hydrostatic compatibility at one-loop
- BRST ghosts
- Lessons and outlook





Stochastic fluctuations

➤ Fluctuation-dissipation theorem relates retarded correlation functions to symmetric correlation functions

$$G^{\text{sym}}(\omega, \mathbf{k}) = \frac{2}{\omega} \operatorname{Im} G^{\text{ret}}(\omega, \mathbf{k})$$

$\langle \psi^2 \rangle - \langle \psi \rangle^2$

$\psi(t) \sim \mathrm{e}^{-\# t}$

➤ Symmetric functions compute stochastic fluctuations of a variable, which are absent in classical hydrodynamics.

➤ Fluctuations *backreact* on hydrodynamics via interactions and induce stochastic-loop corrections in retarded correlation functions.

➤ Schwinger-Keldysh (SK) hydrodynamics provides a systematic symmetry-based EFT framework that incorporates the effects of stochastic thermal fluctuations.

*We will take $\beta = 1/(k_B T) = 1$.

MSR path integral

- Stochastic hydrodynamics with Gaussian noise:

$$\underbrace{\partial_t \psi_r + \Gamma\left(\mu(\psi_r) - h_r\right)}_{E} + f(\psi_r)\,\theta = 0$$

$f^2 = \Gamma$ due to FDT

$$\langle \theta(t, \mathbf{x}) \rangle = 0$$

$$\langle \theta(t, \mathbf{x})\, \theta(t', \mathbf{x}') \rangle = 2\,\delta(t - t', \mathbf{x} - \mathbf{x}')$$

$$\langle \dots \rangle = \int \mathscr{D}\theta\, (\dots)_{\text{onshell}} \exp\left(-\frac{1}{4}\int_x \theta^2\right)$$

$\psi \equiv \langle \psi_r \rangle$

$h \equiv h_r$

- Martin-Siggia-Rose (MSR) path integral [Martin, Siggia, Rose 1973]

$$\mathscr{Z}[h_r, h_a] = \int \mathscr{D}\theta\, \mathscr{D}\psi_r\ \delta\big[E + f\,\theta\big]\ \exp\left(i \int_x \frac{i}{4}\theta^2 + h_a \psi_r\right)$$

$$G^{\text{ret}} = \frac{\delta}{\delta h_r}\frac{-i\delta}{\delta h_a} \ln \mathscr{Z}[h_r, h_a] = \frac{\delta}{\delta h_r}\langle \psi_r \rangle$$

$$G^{\text{sym}} = \frac{-i\delta}{\delta h_a}\frac{-i\delta}{\delta h_a} \ln \mathscr{Z}[h_r, h_a] = \langle \psi_r^2 \rangle - \langle \psi_r \rangle^2$$

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$$\mathscr{Z}[h_r,h_a] = \int \mathscr{D}\theta\,\mathscr{D}\psi_r\;\delta\!\left[E + f\theta\right]\,\exp\left(i\int_x \frac{i}{4}\theta^2 + h_a\psi_r\right) = \int \mathscr{D}\theta\,\mathscr{D}\psi_r\,\mathscr{D}\phi_a \exp\left(i\int_x \frac{i}{4}\theta^2 - \phi_a\left(E + f\theta\right) + h_a\psi_r\right)$$

$$= \int \mathscr{D}\psi_r\,\mathscr{D}\phi_a \exp\Bigg(\underbrace{i\int_x i\Gamma\,\phi_a^2 - \phi_a E + h_a\psi_r}_{S[\psi_r,\phi_a\,;\,h_r,h_a]}\Bigg)$$

$$G^{\text{ret}} = \frac{\delta}{\delta h_r}\frac{-i\delta}{\delta h_a}\ln\mathscr{Z}[h_r,h_a] = \frac{\delta}{\delta h_r}\langle\psi_r\rangle$$

$$G^{\text{sym}} = \frac{-i\delta}{\delta h_a}\frac{-i\delta}{\delta h_a}\ln\mathscr{Z}[h_r,h_a] = \langle\psi_r^2\rangle - \langle\psi_r\rangle^2$$

Schwinger-Keldysh generating functional

- Schwinger-Keldysh generating functional $\mathcal{Z}[h_r, h_a]$

h_r : physical background field
h_a : noise background field
(same as the classical background field h)

$$G^{\text{ret}} = \frac{\delta}{\delta h_r}\frac{-i\delta}{\delta h_a}\ln \mathcal{Z}[h_r, h_a] \qquad G^{\text{sym}} = \frac{-i\delta}{\delta h_a}\frac{-i\delta}{\delta h_a}\ln \mathcal{Z}[h_r, h_a]$$

- Microscopic unitarity constraints:

$$\mathcal{Z}\Big|_{h_a\to 0} = 1 \qquad \mathcal{Z}\Big|_{h_a\to -h_a} = \mathcal{Z}^* \qquad \text{Re}\,\mathcal{Z} \le 0 \qquad \Longrightarrow$$

Poles of $G^{\text{ret}}(\omega, \mathbf{k})$ lie at $\text{Im}\,\omega < 0$

$$G^{\text{ret}}(t - t', \mathbf{x} - \mathbf{x}') \propto \Theta(t - t')$$

$$G^{\text{sym}}(\omega, \mathbf{k}) \ge 0$$

- Kubo-Martin-Schwinger (KMS) symmetry:

$$\text{K}\, h_r(t) = h_r(-t)$$

$$\text{K}\, h_a(t) = \left(h_a + i\partial_t h_r\right)(-t) \qquad \Longrightarrow$$

$$G^{\text{sym}}(\omega, \mathbf{k}) = \frac{2}{\omega}\text{Im}\, G^{\text{ret}}(\omega, \mathbf{k})$$

Fluctuation-Dissipation Theorem

Schwinger-Keldysh effective field theory

- Schwinger-Keldysh effective field theory (SK-EFT)

$$\mathcal{Z}[h_r, h_a] = \int \mathcal{D}\Psi_r \mathcal{D}\Psi_a \exp\Big(i\, S[\Psi_r, \Psi_a ; h_r, h_a] \Big)$$

Ψ_r : physical dynamical field(s)

Ψ_a : noise dynamical field(s)

(not necessarily the classical field ψ)

- The SK-EFT must correctly reproduce the classical hydrodynamic equations.
- The SK-EFT must satisfy:
  - ✓ Unitarity constraints:

$$\mathcal{Z}\Big|_{h_a\to 0} = 1 \qquad \mathcal{Z}\Big|_{h_a\to -h_a} = \mathcal{Z}^* \qquad \mathrm{Re}\,\mathcal{Z} \leq 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$S\Big|_{h_a, \Psi_a\to 0} = 0 \qquad S\Big|_{h_a\to -h_a, \Psi_a\to -\Psi_a} = -S^* \qquad \mathrm{Im}\,S \geq 0$$

  - ✓ KMS symmetry:

$$\mathrm{K}\, h_r(t) = h_r(-t) \qquad \mathrm{K}\,\Psi_r(t) = \pm\,\Psi_r(-t)$$

$$\mathrm{K}\, h_a(t) = \Big(h_a + i\partial_t h_r\Big)(-t) \qquad \mathrm{K}\,\Psi_a(t) = \pm\,\Big(\Psi_a + i\partial_t \Psi_r\Big)(-t)$$

time-reversal eigenvalue of $\Psi_{r,a}$

Schwinger-Keldysh effective field theory

- Schwinger-Keldysh effective field theory (SK-EFT)

Ψ_r : physical dynamical field(s)

Ψ_a : noise dynamical field(s)

(*not necessarily the classical field* ψ)

$$\mathcal{Z}[h_r, h_a] = \int \mathcal{D}\Psi_r \mathcal{D}\Psi_a \exp\left(i\, S[\Psi_r, \Psi_a; h_r, h_a] \right)$$

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- The SK-EFT must satisfy:
  - ✓ Unitarity constraints:

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$$\mathcal{Z}\Big|_{h_a \to -h_a} = \mathcal{Z}^* \quad \downarrow \quad S\Big|_{h_a \to -h_a, \Psi_a \to -\Psi_a} = -S^*$$

$$\mathrm{Re}\,\mathcal{Z} \le 0 \quad \downarrow \quad \mathrm{Im}\, S \ge 0$$

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time-reversal eigenvalue of $\Psi_{r,a}$

Schwinger-Keldysh hydrodynamics 101

- There may be multiple SK-EFTs for a hydrodynamic model. Consider ($S = \int \mathrm{d}t\, \mathrm{d}x\, \mathscr{L}$) $\quad \partial_t \psi + \Gamma\left(\mu - h\right) = 0$

Canonical Scheme (Can)

$$\mathscr{L} = -\psi_a\left(\mu(\psi_r) - h_r\right) + \psi_a \frac{i}{\Gamma}\left(\psi_a + i\partial_t \psi_r\right) + h_a \psi_r$$

$$\mathrm{K}\,\psi_r(t) = \psi_r(-t)$$

$$\mathrm{K}\,\psi_a(t) = \left(\psi_a + i\partial_t \psi_r\right)(-t)$$

Not suitable for Model B

$\psi \equiv \psi_r$

$\psi_a \leftrightarrow \Gamma\phi_a$

$\mu(\psi_r) \leftrightarrow \partial_t \phi_r + h_r$

Grand Canonical Scheme (GCan)

$$\mathscr{L} = \psi(\partial_t\phi_r + h_r)\left(\partial_t \phi_a + h_a\right) + i\phi_a \Gamma\left(\phi_a + i\partial_t \phi_r\right)$$

$$\mathrm{K}\,\phi_r(t) = -\phi_r(-t)$$

$$\mathrm{K}\,\phi_a(t) = -\left(\phi_a + i\partial_t \phi_r\right)(-t)$$

$\mu(\psi) \equiv \partial_t \phi_r + h_r$

- The KMS symmetries in the two schemes are physically distinct.*
- Field redefinitions affect the renormalisation structure of physical observables.

*In fact, $\mathrm{K_C}\,\psi_r = \mathrm{K_{GC}}\,\psi_r$, but $\mathrm{K_C}\,\psi_a \overset{\text{onshell}}{=} -\mathrm{K_{GC}}\,\psi_a$.

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$\mu(\psi) \equiv \partial_t \phi_r + h_r$

- The KMS symmetries in the two schemes are physically distinct.*
- Field redefinitions affect the renormalisation structure of physical observables.

$\Longrightarrow$ One or both SK-EFT schemes may be incompatible with hydrostatics due to interactions.

*In fact, $\mathrm{K_C}\,\psi_r = \mathrm{K_{GC}}\,\psi_r$, but $\mathrm{K_C}\,\psi_a \overset{\text{onshell}}{=} -\mathrm{K_{GC}}\,\psi_a$.

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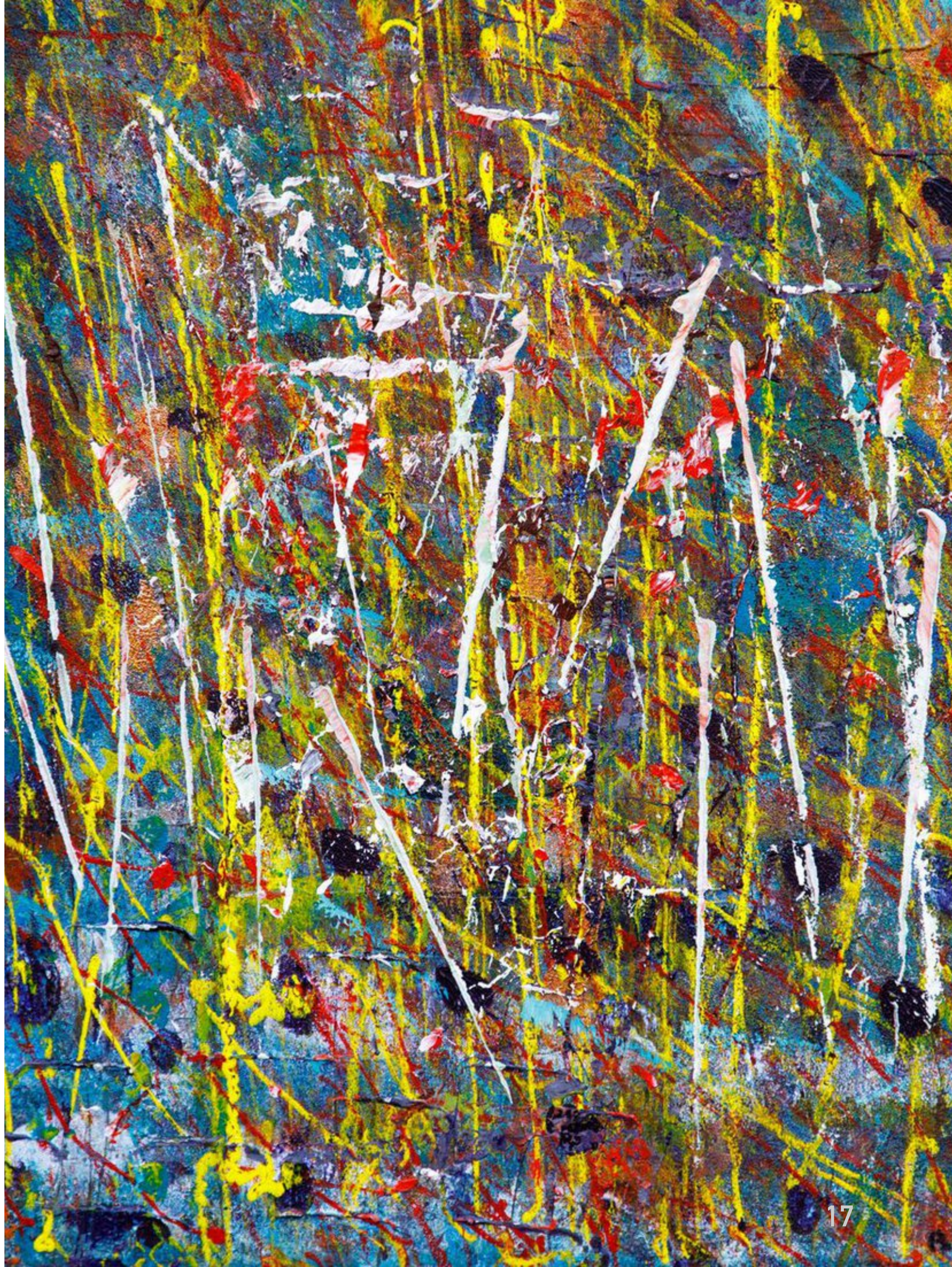


Stochastic Feynman rules

- Despite thier differences, the two SK-EFT schemes are qualitatively similar.
- Free propagators:

Ψ_r

Ψ_a

 $= 0$

 $\sim \dfrac{1}{\Omega - i\omega}$

 $\sim \dfrac{1}{(\Omega - i\omega)(\Omega + i\omega)}$

Pole(s) only at $\mathrm{Im}\,\omega < 0$

Pole(s) at both $\mathrm{Im}\,\omega < 0$ and $\mathrm{Im}\,\omega > 0$

$\Omega_{(A)} \equiv (a_2 + \gamma\,\mathbf{k}^2)\,\Gamma_0$

$\Omega_{(B)} \equiv (a_2 + \gamma\,\mathbf{k}^2)\,\sigma_0\,\mathbf{k}^2$

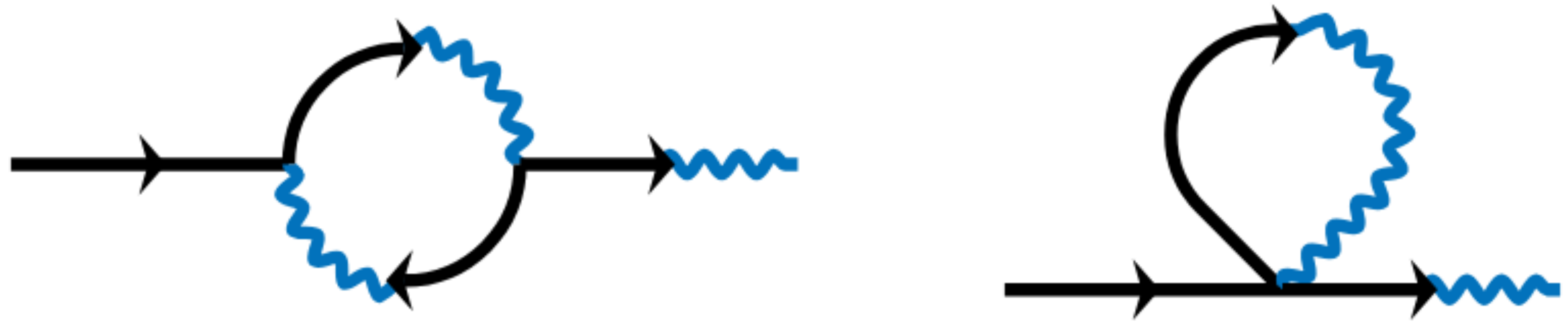
- Interactions:*

- These reproduce the classical retarded and symmetric correlation functions at tree-level.

*There are several other interactions involving background field insertions, which we do not explicitly mention here.

Non-unitary diagrams and GGL prescription

- The non-unitary diagrams:



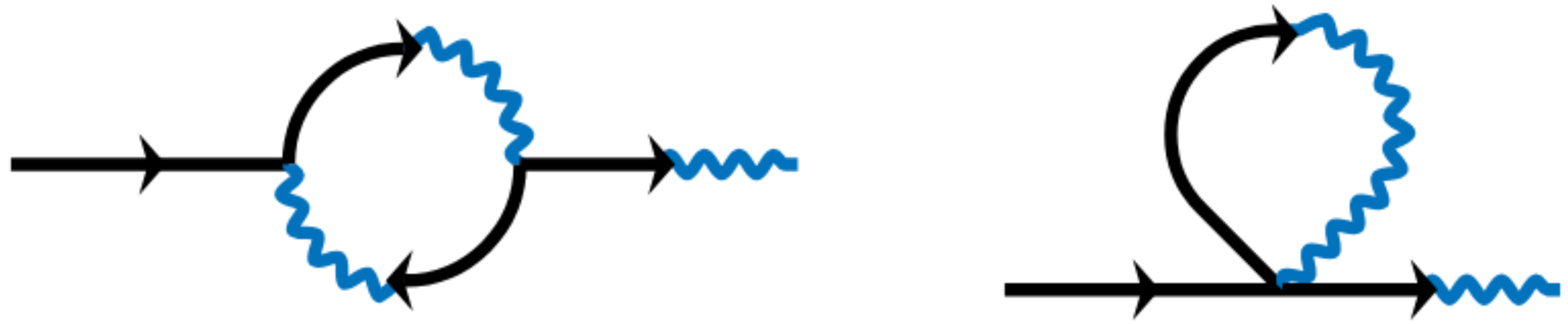
Poles of $G^{\text{ret}}(\omega, \mathbf{k})$ ~~lie at $\text{Im}\,\omega < 0$~~

$$G^{\text{ret}}(t - t', \mathbf{x} - \mathbf{x}') \not\propto \Theta(t - t')$$

- **What went wrong?** Requiring $S|_{h_a, \Psi_a \to 0} = 0 \not\Longrightarrow \mathcal{Z}|_{h_a \to 0} = 0$ beyond tree-level.

Non-unitary diagrams and GGL prescription

- The non-unitary diagrams:



Poles of $G^{\mathrm{ret}}(\omega, \mathbf{k})$ ~~lie at $\mathrm{Im}\,\omega < 0$~~

$G^{\mathrm{ret}}(t-t', \mathbf{x}-\mathbf{x}') \not\propto \Theta(t-t')$

- **What went wrong?** Requiring $S|_{h_a, \Psi_a \to 0} = 0 \;\not\Longrightarrow\; \mathscr{Z}|_{h_a \to 0} = 0$ beyond tree-level.
- **Quick fix:** Regularise the retarded propagator (GGL prescription) [Gao, Glorioso, Liu 2018]

$$\sim \frac{1}{\Omega - i\omega\,(1 - i\omega/M)^p} \qquad \Longleftrightarrow \qquad (1 + 1/M\,\partial_t)^p\,\partial_t \psi + \Gamma\left(\mu(\psi) - h\right) = 0$$

$\Longrightarrow$ *retarded loops vanish*

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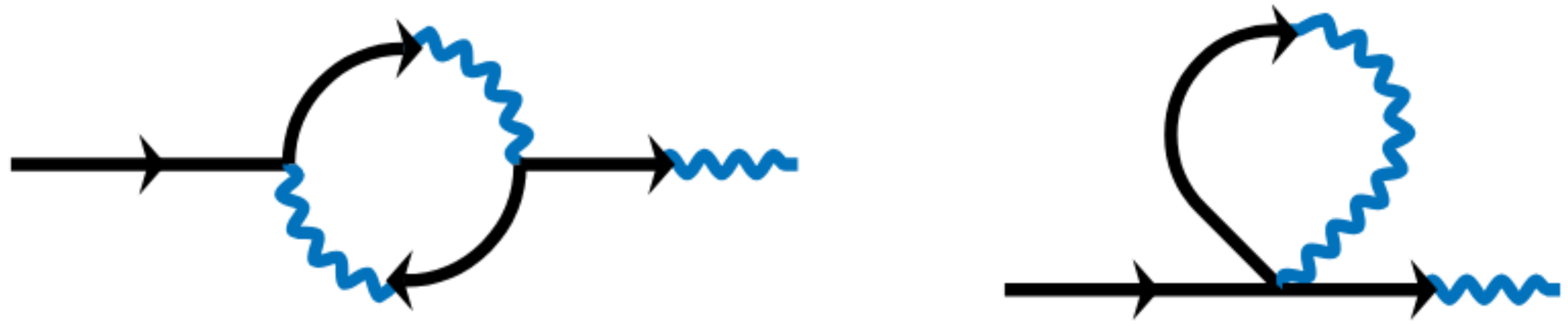

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➤ **Caveat:** $\mathrm{Im}\,S \not\geq 0$:

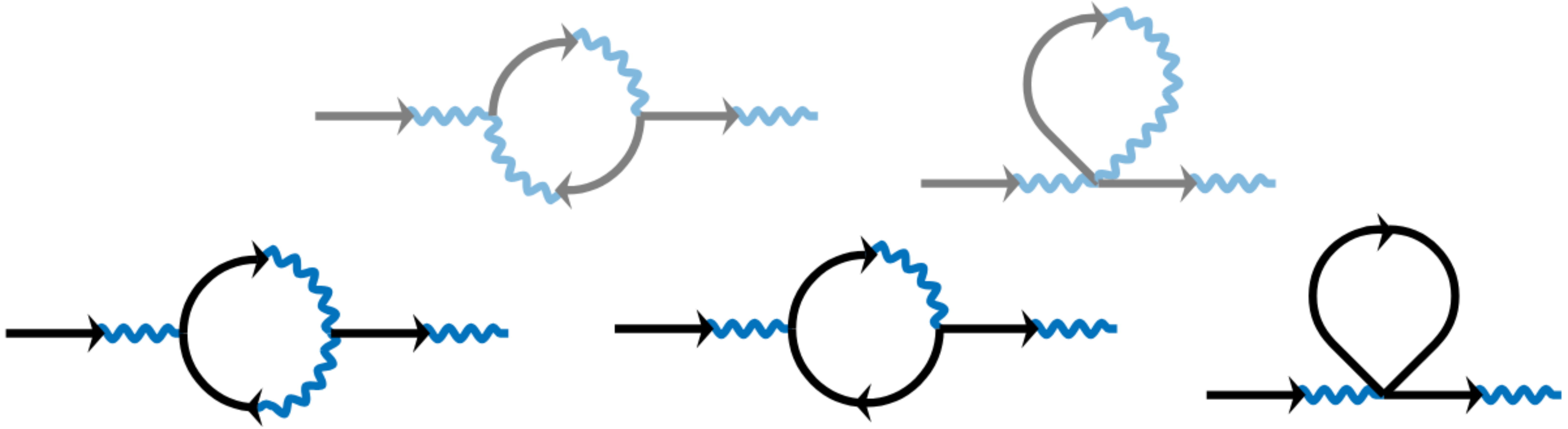
$$i\psi_a \frac{1}{\Gamma}\left(1 - \frac{p(p-1)}{2M^2}\omega^2 + \dots\right)\psi_a \qquad i\phi_a \Gamma\left(1 - \frac{p}{\Omega M}\omega^2 + \dots\right)\phi_a$$

(dictated by KMS)

$G^{\mathrm{sym}}(\omega, \mathbf{k}) \not\geq 0$

*Analogous problem arises in relativistic hydrodynamics in BDNK frame. [Jain, Kovtun 2023] [Mullins et al. 2023]

One-loop results and long-time tails



$$\mu(\psi) = \left(a_2 - \gamma \nabla^2\right)\psi + \tfrac{1}{2}a_3\psi^2 + \tfrac{1}{6}a_4\psi^3$$

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$$\Omega \equiv a_2\Gamma_0$$

$$D \equiv a_2\sigma_0$$

- Cutoff-independent non-analytic contributions ($\gamma \to 0,\ d = 3$):

$$G^{\text{ret}}(\omega, \mathbf{k})_{(A)} = \frac{\Gamma_0}{\Omega - i\omega} + \mathcal{O}(\Lambda^3)$$

$$G^{\text{ret}}(\omega, \mathbf{k})_{(B)} = \frac{\sigma_0\,\mathbf{k}^2}{D\,\mathbf{k}^2 - i\omega} - \frac{\sigma_0^2\,\mathbf{k}^4}{\left(D\,\mathbf{k}^2 - i\omega\right)^2}\left(\frac{a_3}{a_2} + \frac{\sigma_1}{\sigma_0}\right)^2 \frac{i\omega}{32\pi D}\sqrt{\mathbf{k}^2 - \frac{2i\omega}{D}} + \mathcal{O}(\Lambda^3)$$

Long-Time Tails

Hydrodynamic modes decay polynomially rather than exponentially.

[Alder, Wainwright 1970]
[De Schepper, Van Beyeren, Ernst 1974]
[Forster, Nelson, Stephen 1977]
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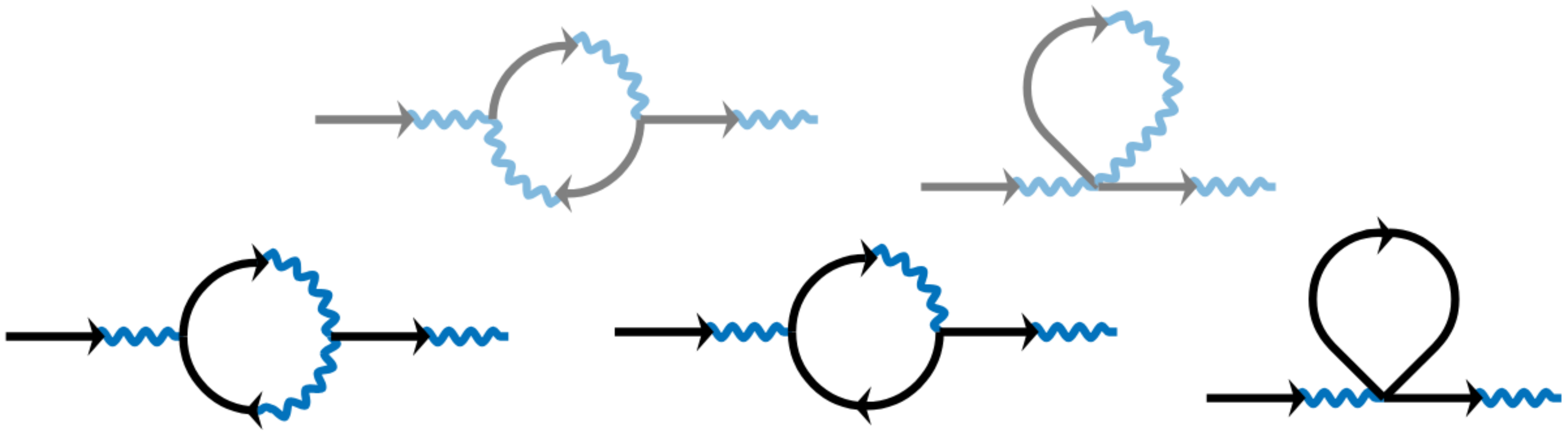

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- Weak hydrostatic compatibility:

$G^{\rm ret}(\omega = 0, \mathbf{k})$ is analytic* in $\mathbf{k}$

[AJ, Kovtun, Ritz, Shukla 2020]

*In the absence of spontaneous symmetry breaking.

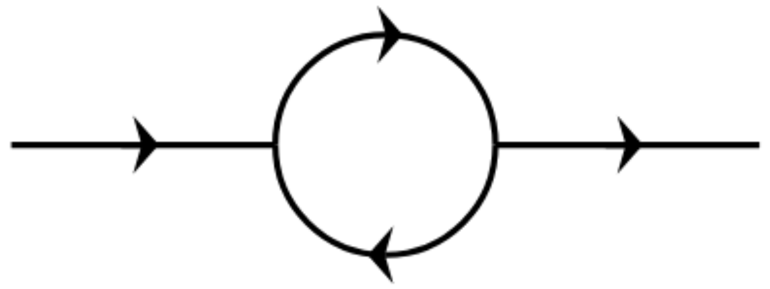
Hydrostatic compatibility

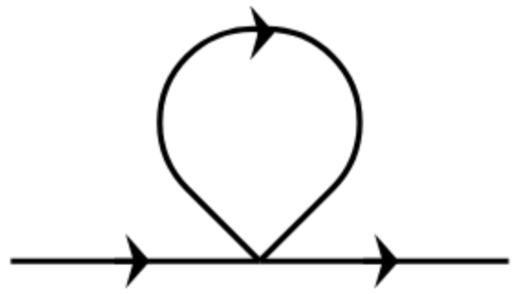
- The *correct* hydrostatic compatible results:

$$G^{\text{ret}}(\omega,\mathbf{k})_{(\text{A})} = \frac{\Gamma_0}{\Omega - i\omega} - \frac{\Gamma_0^2}{(\Omega - i\omega)^2}\left[\frac{a_4}{a_2} - \frac{2\Omega}{2\Omega - i\omega}\left(\frac{a_3}{a_2} + \frac{i\omega}{\Omega}\frac{\Gamma_1}{\Gamma_0}\right)^2 + \frac{i\omega}{\Omega}\left(\frac{\Gamma_2}{\Gamma_0} - \frac{2\Gamma_1^2}{\Gamma_0^2}\right)\right]\frac{\Lambda^3}{12\pi^2} + \dots$$

$$G^{\text{ret}}(\omega,\mathbf{k})_{(\text{B})} = \frac{\sigma_0\,\mathbf{k}^2}{D\,\mathbf{k}^2 - i\omega} - \frac{\sigma_0^2\,\mathbf{k}^4}{\left(D\,\mathbf{k}^2 - i\omega\right)^2}\left[\left(\frac{a_4}{a_2} - \frac{a_3^2}{a_2^2} + \frac{\sigma_2}{\sigma_0}\frac{i\omega}{D\,\mathbf{k}^2}\right)\frac{\Lambda^3}{12\pi^2} + \frac{i\omega}{D}\left(\frac{a_3}{a_2} + \frac{\sigma_1}{\sigma_0}\right)^2\left(\frac{1}{32\pi}\sqrt{\mathbf{k}^2 - \frac{2i\omega}{D}} - \frac{\Lambda}{8\pi^2}\right)\right] + \dots$$

$$\xrightarrow{\ \omega \to 0\ } \quad G^{\text{eqb}}(\mathbf{k}) = \frac{1}{a_2} - \frac{1}{a_2^2}\left(\frac{a_4}{a_2} - \frac{a_3^2}{a_2^2}\right)\frac{\Lambda^3}{12\pi^2} + \dots$$

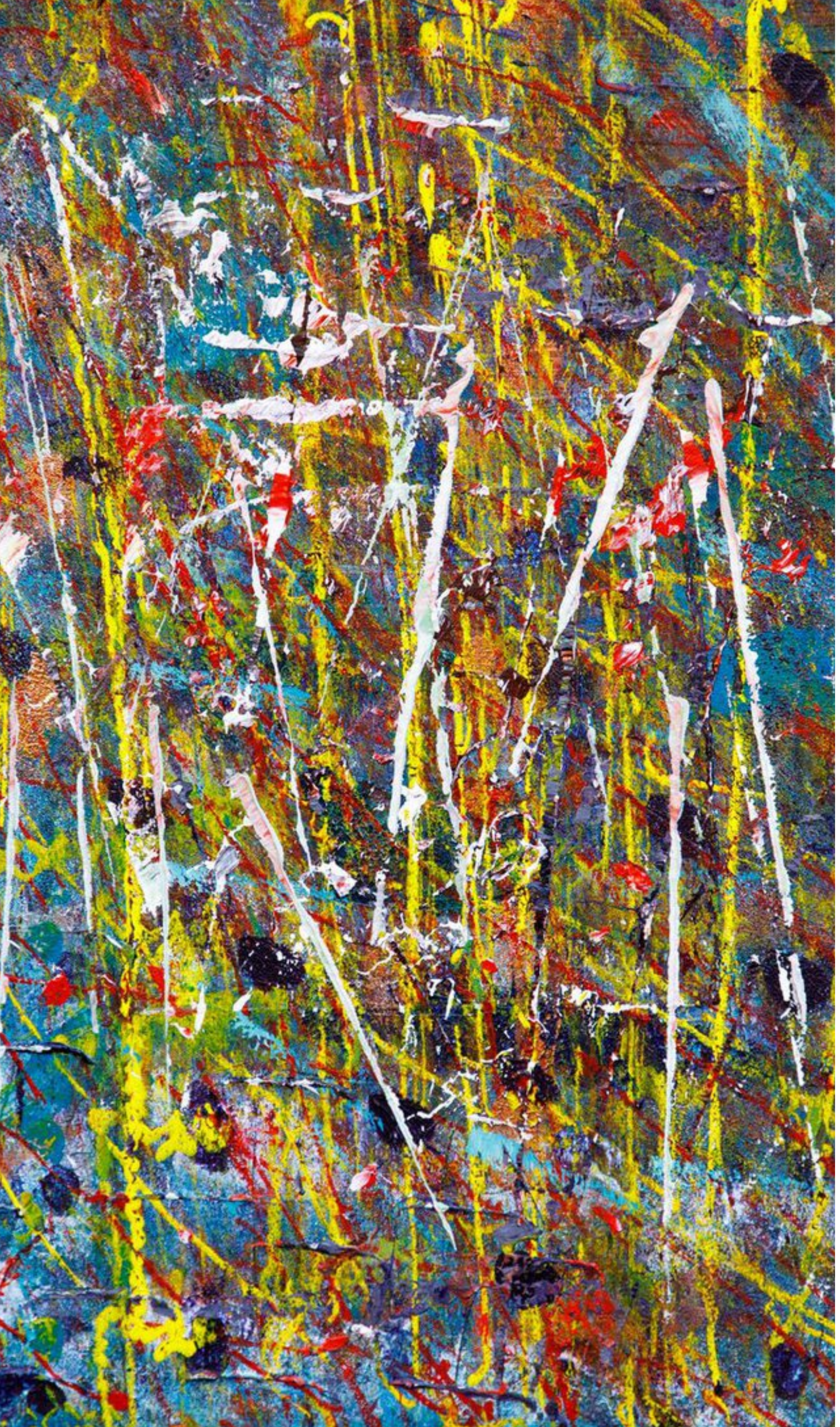




- The SK-EFT results (GGL prescription) differ by:

$$G^{\text{ret}}(\omega,\mathbf{k})_{(\text{A})} \sim \begin{cases} (\text{Can}): & 0 \\ (\text{GCan}): & \dfrac{-2\Gamma_0^2}{(\Omega - i\omega)^2}\left(\dfrac{\Gamma_2}{\Gamma_0} - \dfrac{\Gamma_1^2}{\Gamma_0^2}\right)\dfrac{\Lambda^3}{12\pi^2} \end{cases}$$

$$G^{\text{ret}}(\omega,\mathbf{k})_{(\text{B})} \sim \begin{cases} (\text{Can}): & \text{n/a} \\ (\text{GCan}): & \dfrac{-\sigma_0^2\,\mathbf{k}^4}{\left(D\,\mathbf{k}^2 - i\omega\right)^2}\dfrac{a_3}{a_2}\dfrac{\sigma_1}{\sigma_0}\dfrac{\Lambda^3}{12\pi^2} + \dots \end{cases}$$



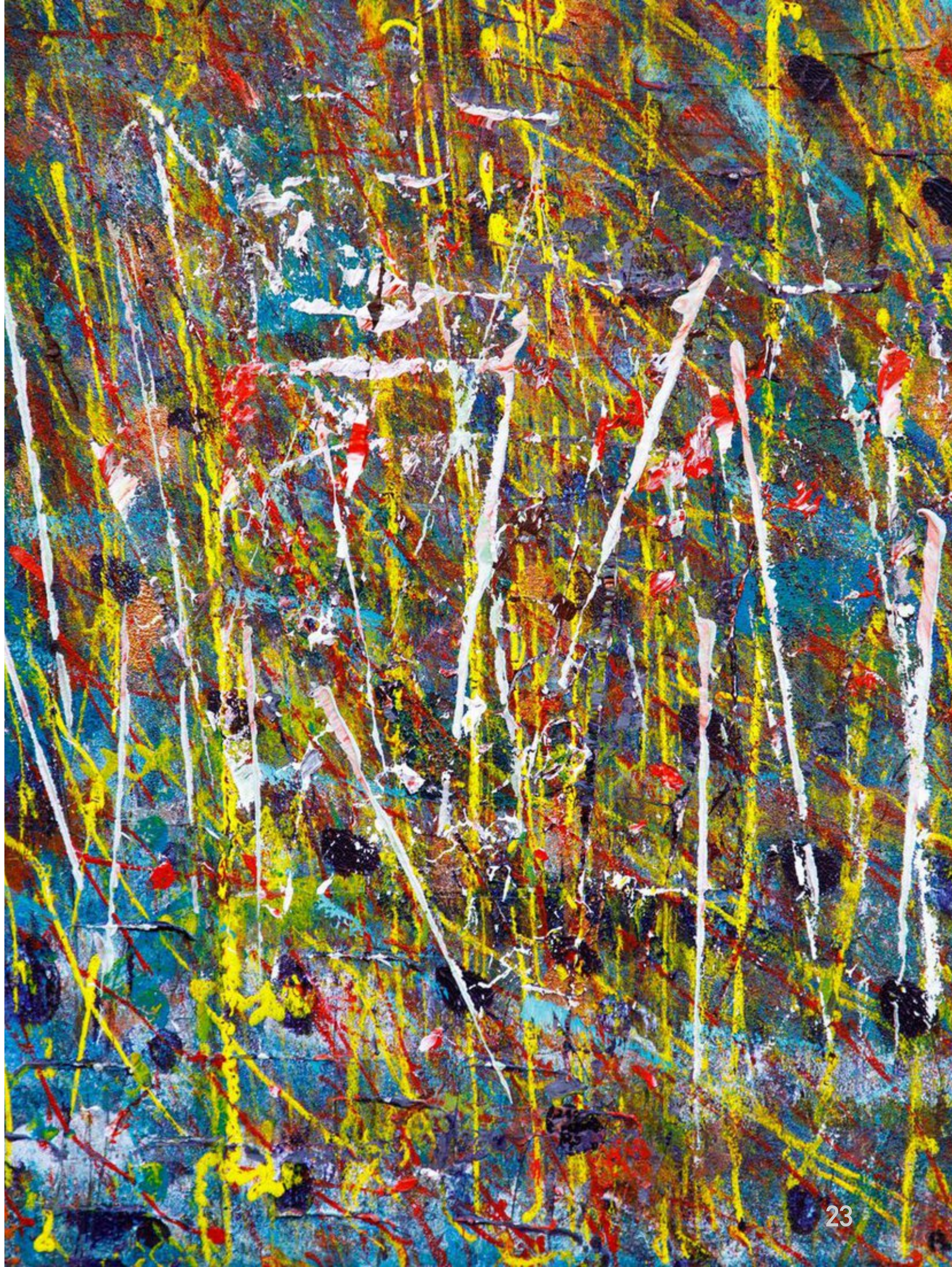
In case I made you sleep…

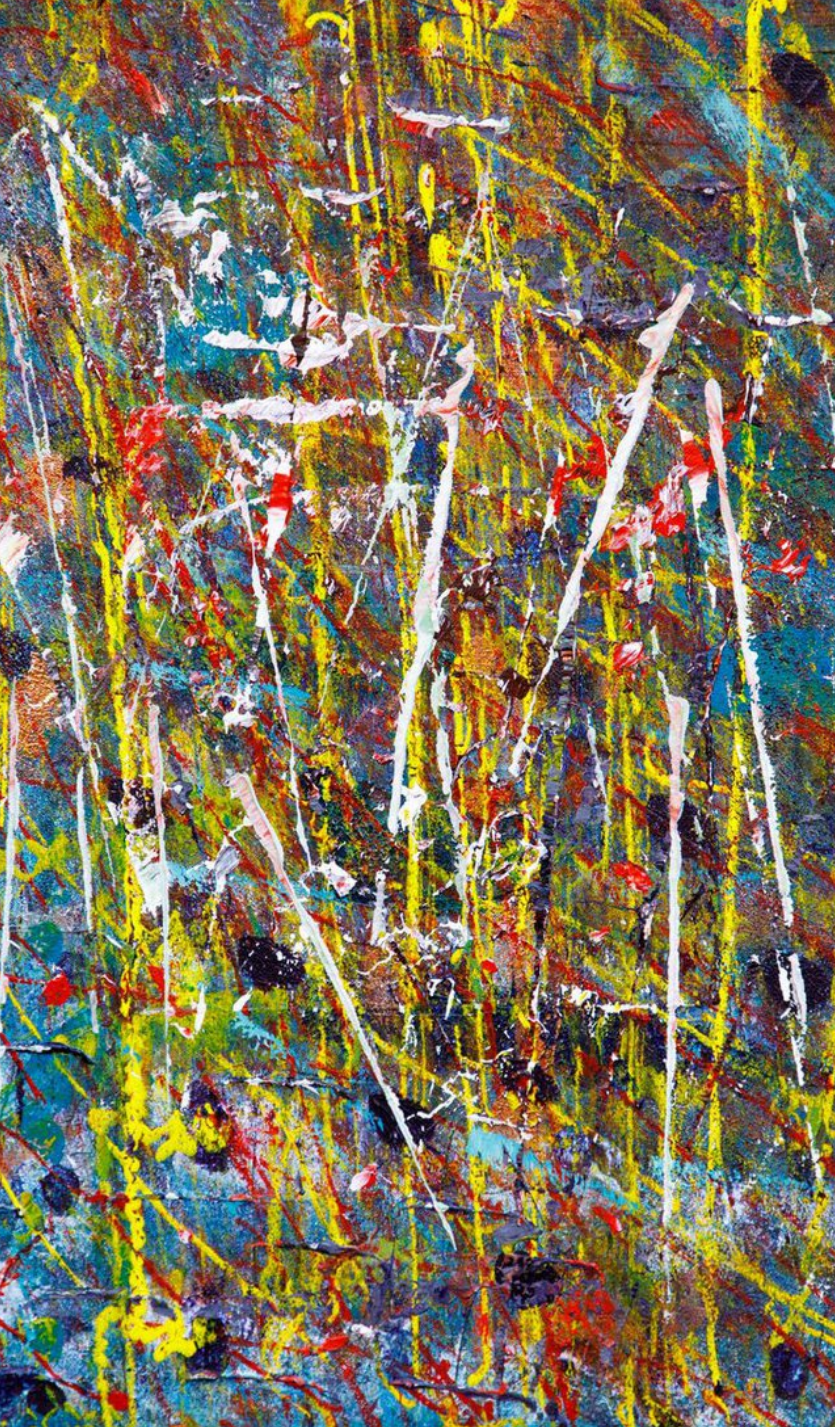
- Stochastic fluctuations induce loop corrections in retarded functions, which can be systmatically computed using Schwinger-Keldysh hydrodynamics.
- The naive SK-EFT results are incorrect because of non-unitary diagrams. This can be fixed using the GGL regularisation prescription, at the expense of positivity of symmetric functions.
- There may be multiple SK-EFT models corresponding to the same classical hydrodynamic model.
- Hydrostatic-compatibility with GGL regularisation:

	Can Scheme (GGL)	**GCan Scheme** (GGL)
Model A	✓	✗
Model B	—	✗

Outline

- Hydrostatic compatibility
- Schwinger-Keldysh hydrodynamics
- Hydrostatic compatibility at one-loop
- **BRST ghosts**
- Lessons and outlook





BRST ghosts

- The GGL prescription gives correct hydrostatic-compatible results in the "Can" scheme, but not in the "GCan" scheme.
- However, the "Can" scheme does not apply to Model B. Therefore, the GGL prescription does not yield *any* hydrostatic-compatible SK-EFT for Model B (i.e. for conserved degrees of freedom).
- The proper way to address these issues is introducing BRST ghosts, which systmatically cancel the unphysical contributions arising from non-unitary diagrams.

[Arenas, Barci 2010]
[Haehl, Loganayagam, Rangamani 2016]
[Gao, Liu 2017]
[Jensen, Pinzani-Fokeeva, Yarom 2017]

Ghosts in MSR path integral

- Technically, the MSR generating functional involves a functional determinant, which gives rise to ghosts:

$$\mathcal{Z}[h_r, h_a] = \int \mathcal{D}\theta\, \mathcal{D}\psi_r \; \det\left[\frac{\delta(E+f\theta)}{\delta\psi_r}\right] \delta\left[E+f\theta\right] \exp\left(i \int_x \frac{i}{4}\theta^2 + h_a \psi_r \right)$$

$$\int \mathcal{D}c_r \mathcal{D}c_a \exp\left(i \int_x c_a \frac{\delta(E+f\theta)}{\delta\psi_r} c_r \right)$$

- The theory is invariant under BRST symmetry:

$$\mathrm{Q}\,\psi_r = \epsilon\, c_r \qquad \mathrm{Q}\, c_a = \epsilon\, \psi_a \qquad \Longrightarrow \qquad \mathcal{Z}\,\big|_{h_a \to 0} = 0$$

- The MSR ghost sector *does not* necessarily furnish a KMS symmetry.
- The ghost sector is ambiguous*: it changes if we use ϕ_r (with $\mu(\psi_r) = \partial_t \phi_r + h_r$) as a degree of freedom.

*This ambiguity is related to the well-known ambiguity in discretising stochastic Langevin equations.

Supersymmetric BRST ghosts

- Introduce BRST ghost fields to cancel the unphysical non-unitary contributions

$$\text{Physical/Noise Fields}: \Psi_{r,a} = \left(\psi_{r,a}, \phi_{r,a}\right) \qquad \text{Anticommuting Ghost Fields}: C_{r,a} = \left(c_{r,a}, d_{r,a}\right)$$

- BRST symmetry:* $\mathrm{Q}\,\Psi_r = \epsilon\, C_r \qquad \mathrm{Q}\, C_a = \epsilon\, \Psi_a \qquad \Longrightarrow \qquad \mathscr{Z}\,|_{h_a \to 0} = 0$

- KMS symmetry: $\mathrm{K}\,\Psi_r(t) = \pm\,\Psi_r(-t) \qquad \mathrm{K}\,\Psi_a(t) = \pm\left(\Psi_a + i\partial_t \Psi_r\right)(-t)$

$$\mathrm{K}\, C_r(t) = \mp\, C_a(-t) \qquad \mathrm{K}\, C_a(t) = \pm\, C_r(-t)$$

- Emergent supersymmetry: $\left\{\mathrm{Q}, \bar{\mathrm{Q}}\right\} = i\partial_t \qquad \bar{\mathrm{Q}} \equiv \mathrm{K}\,\mathrm{Q}\,\mathrm{K}^{-1}$

Supersymmetry enables us to construct the BRST ghost sector from scratch.

[Arenas, Barci 2010]
[Haehl, Loganayagam, Rangamani 2016]
[Gao, Liu 2017]
[Jensen, Pinzani-Fokeeva, Yarom 2017]

We also need to introduce “background ghost fields” that mix with physical/noise background fields under BRST transformations.

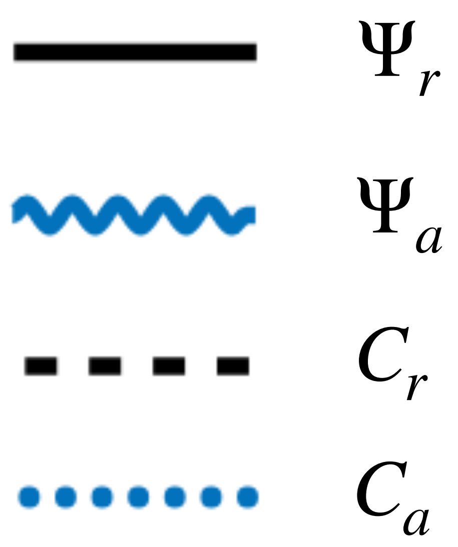
Cancellation of non-unitary diagrams

➤ Propagators:

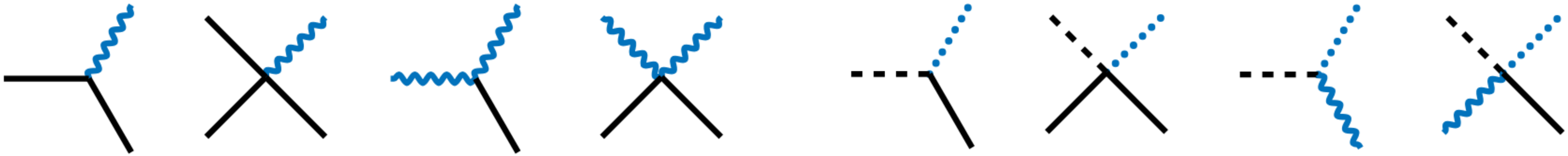
$$\sim \frac{1}{\Omega - i\omega}$$

$$\sim \frac{1}{\Omega - i\omega}$$

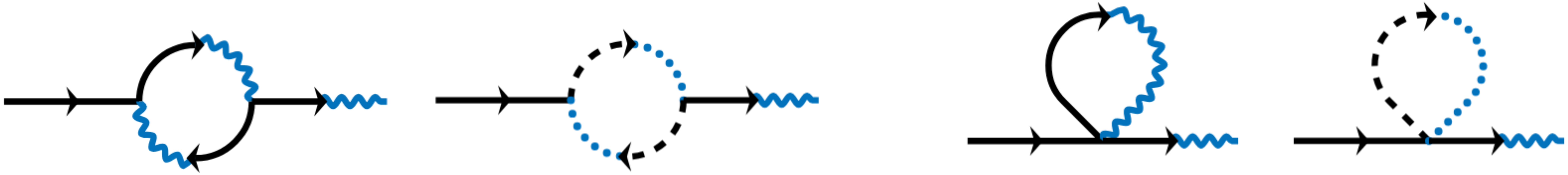
$$\sim \frac{1}{(\Omega - i\omega)(\Omega + i\omega)}$$



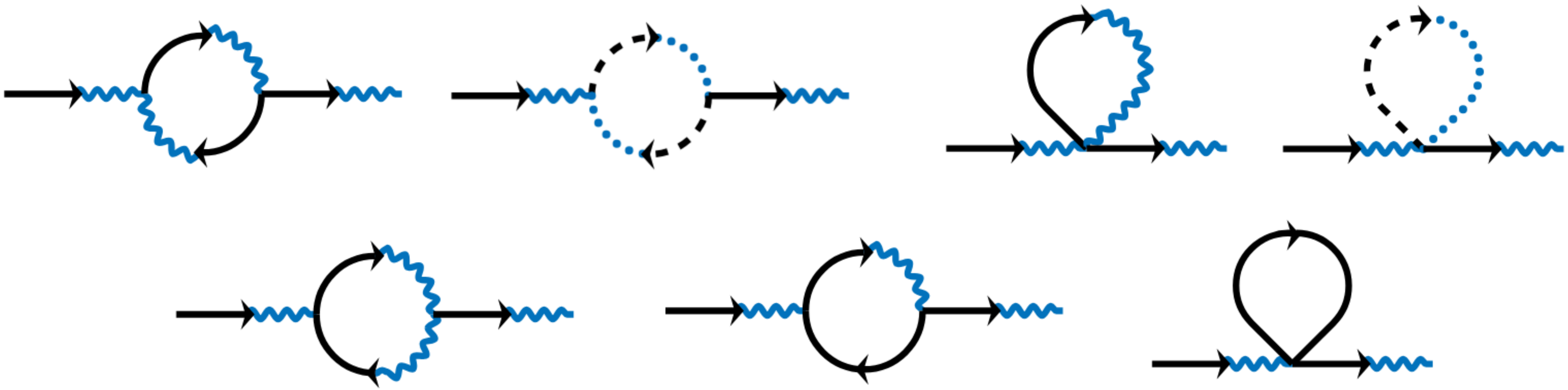
➤ Interactions:



➤ Non-unitary counter-diagrams:

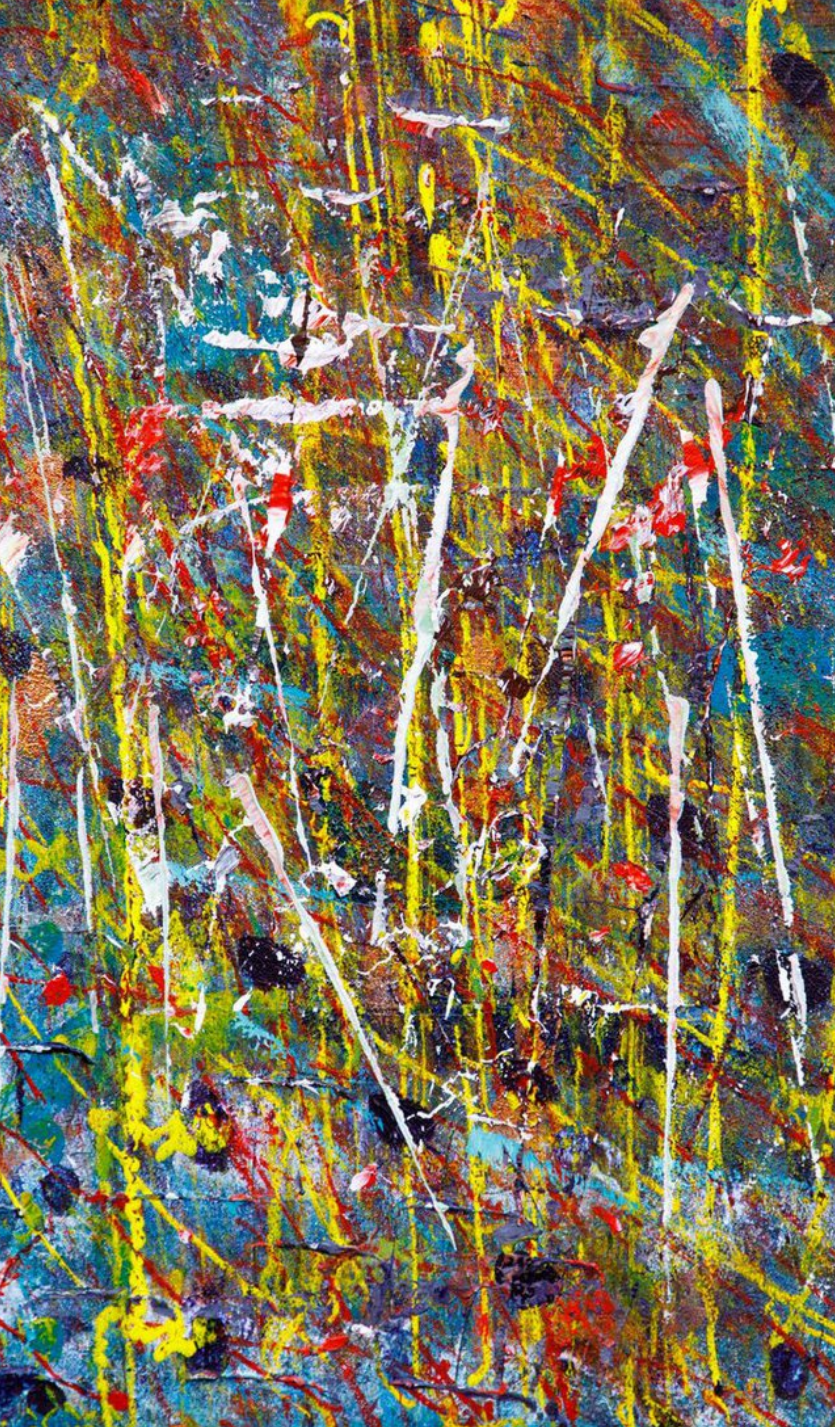


Hydrostatic compatibility



- The SK-EFT results (ghost prescription) deviate from the hydrostatic-compatible results by:

$$G^{\text{ret}}(\omega,\mathbf{k})_{(\text{A})} \sim \begin{cases} (\text{Can}): & \dfrac{\Gamma_0^2}{(\Omega - i\omega)^2}\left(\dfrac{\Gamma_2}{\Gamma_0} - \dfrac{\Gamma_1^2}{\Gamma_0^2}\right)\dfrac{\Lambda^3}{12\pi^2} \\ (\text{GCan}): & 0 \end{cases} \qquad G^{\text{ret}}(\omega,\mathbf{k})_{(\text{B})} \sim \begin{cases} (\text{Can}): & \text{n/a} \\ (\text{GCan}): & 0 \end{cases}$$



In case I made you sleep, again...

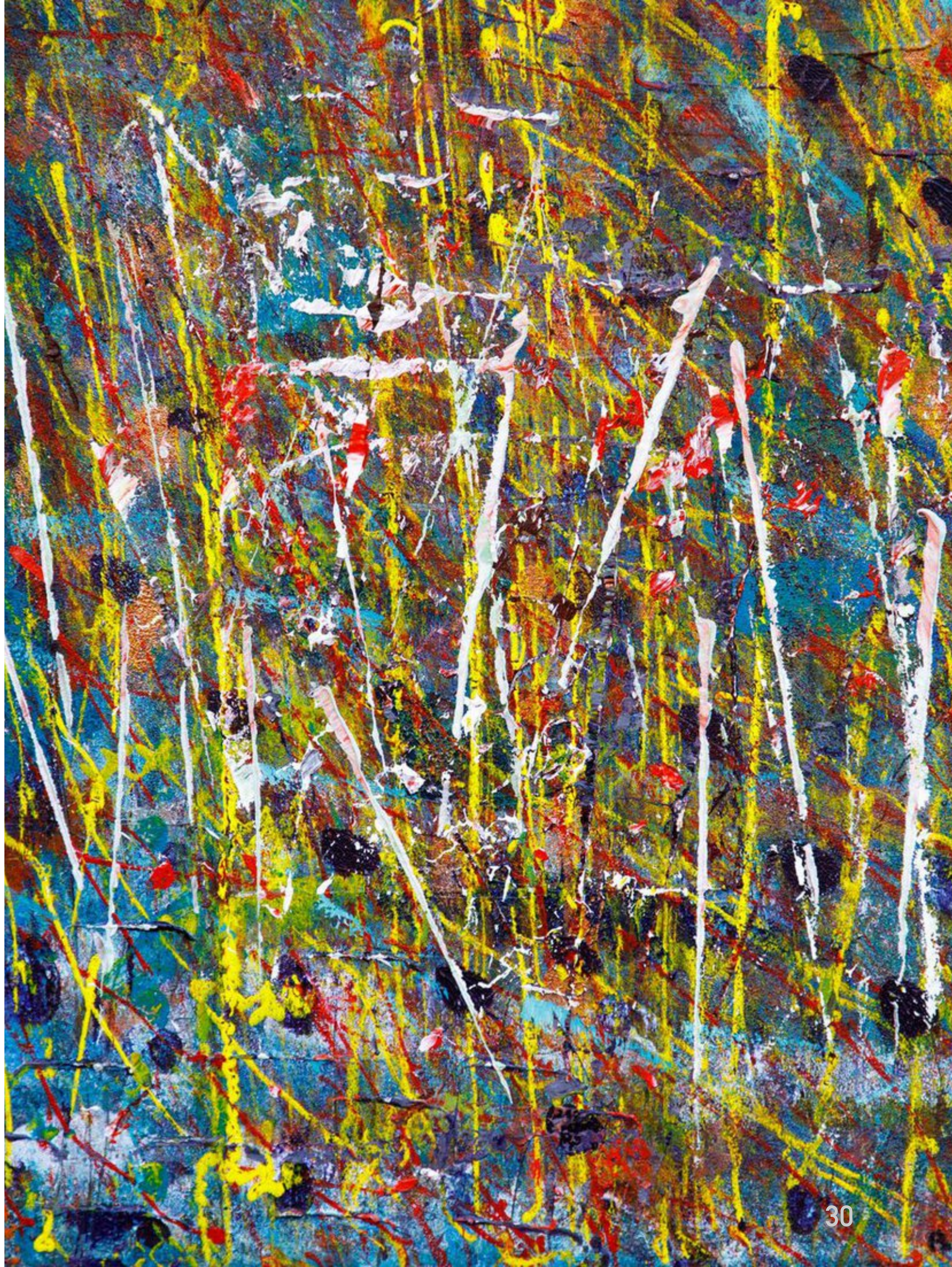
- The proper mechanism to cancel the non-unitary diagrams is by introducing BRST ghost fields.
- BRST symmetry combined with KMS symmetry leads to an emergent supersymmetry is SK-EFTs.
- Hydrostatic-compatibility with GGL and ghost regularisations:

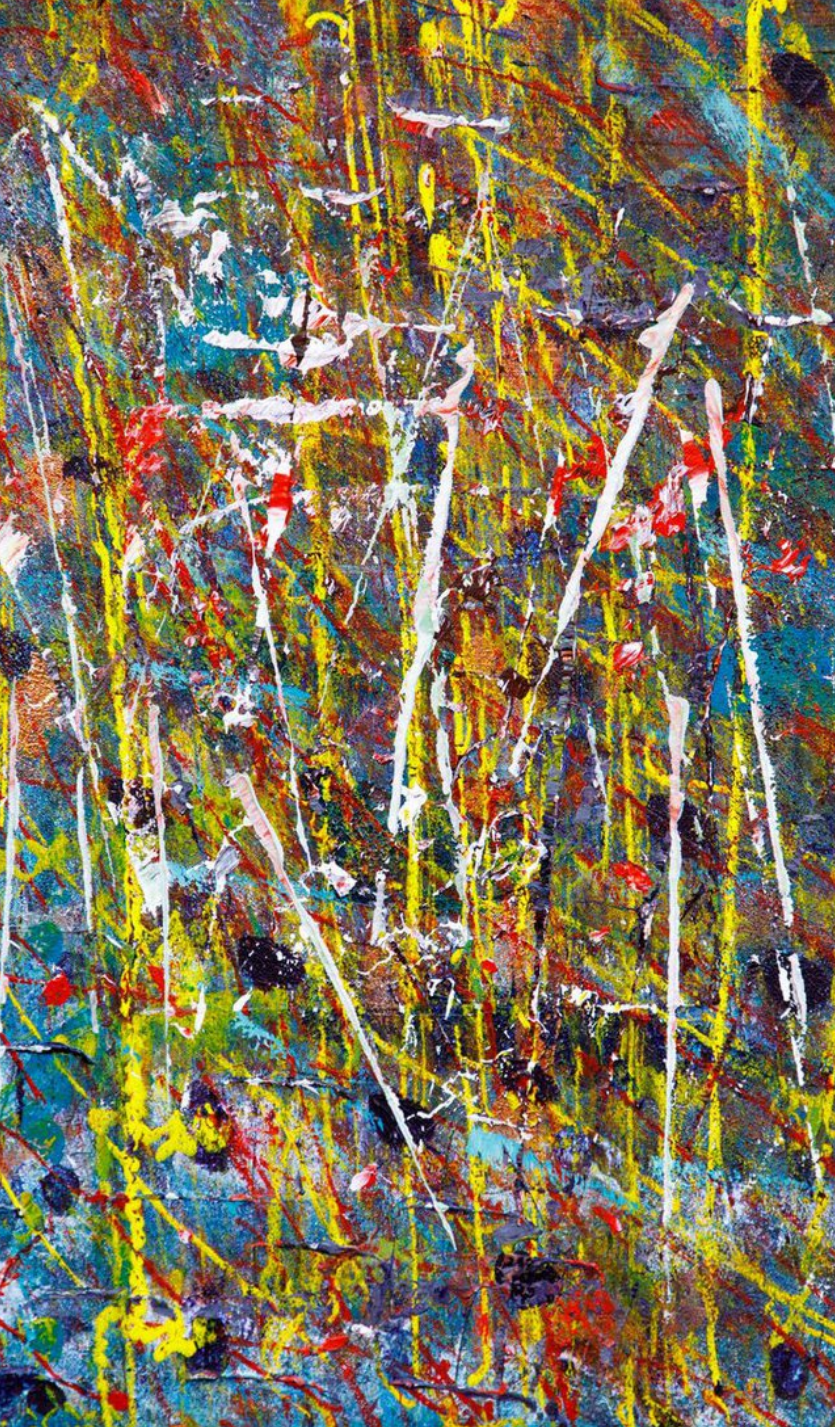
	Can Scheme (GGL)	**GCan Scheme** (GGL)	**Can Scheme** (ghosts)	**GCan Scheme** (ghosts)
Model A	✓	✗	✗	✓
Model B	—	✗	—	✓

- Unfortunately, neither of the regularisation prescriptions guarantee hydrostatic-compatibility.

Outline

- ➤ Hydrostatic compatibility
- ➤ Schwinger-Keldysh hydrodynamics
- ➤ Hydrostatic compatibility at one-loop
- ➤ BRST ghosts
- ➤ **Lessons and outlook**



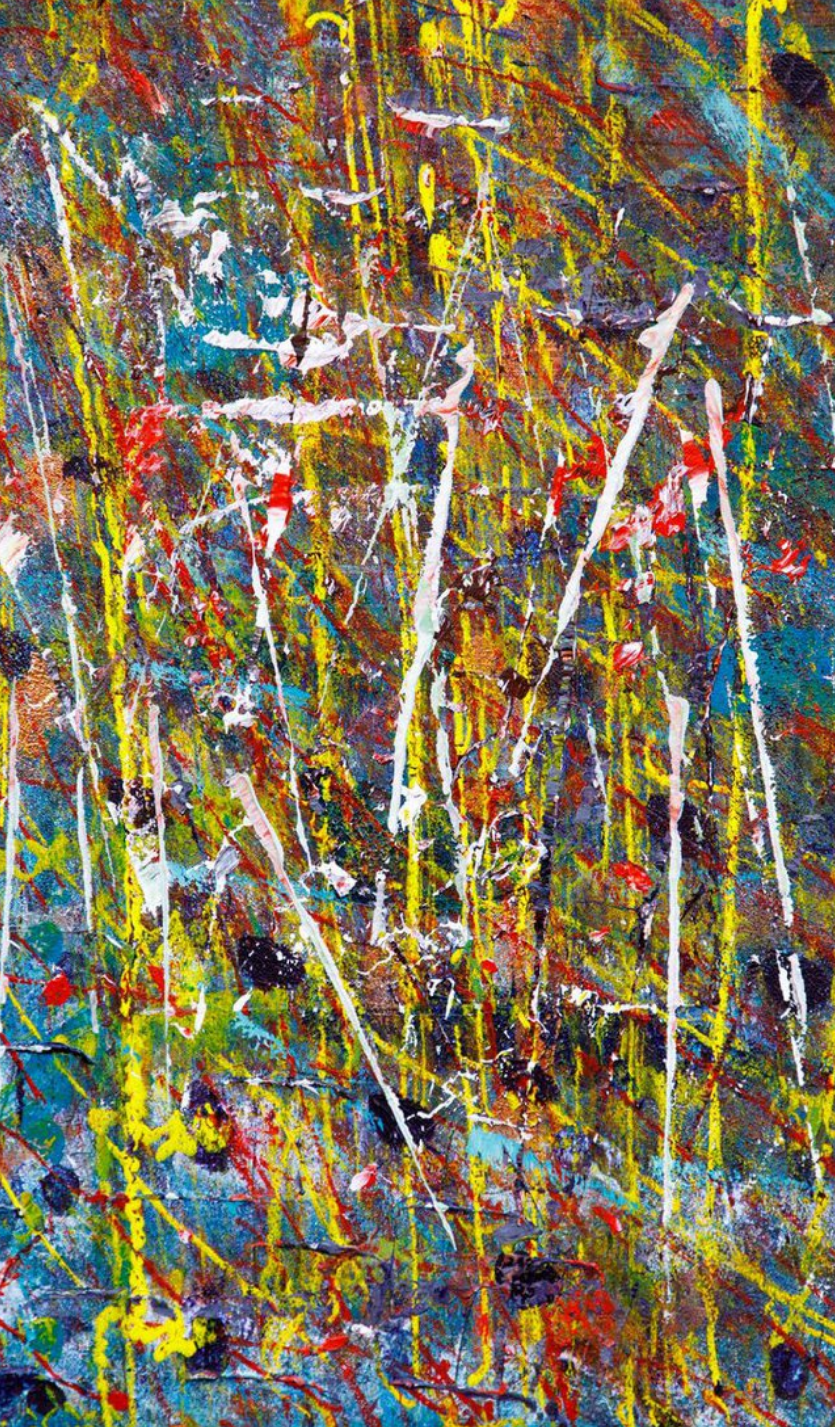


Lessons

- Hydrostatic compatibility of Schwinger-Keldysh hydrodynamics is not automatic. It depends on the regularisation prescription implemented to eliminate the unphysical non-unitary diagrams.
- Circumventing the need for ghosts using the GGL prescription might cause violations of unitarity in symmetric correlation functions.
- There may be multiple Schwinger-Keldysh models corresponding to the same classical hydrodynamic model.
- **What fares best:** Grand Canonical (GCan) scheme with BRST ghost regularisation is hydrostatic compatible and works for both Model A and Model B.

Hopefully, this continues to hold for full hydrodynamics with conserved energy and momentum degrees of freedom.

Wishful thinking!



Food for thought

- The "GCan" scheme breaks down near critical points

$$\mu(\psi) = \frac{\delta}{\delta\psi}\int_{\mathbf{x}}\left(\frac{1}{2}\partial^i\psi\,\partial_i\psi + \cancel{\frac{a_2}{2}\psi^2} + \frac{a_4}{24}\psi^4\right)$$

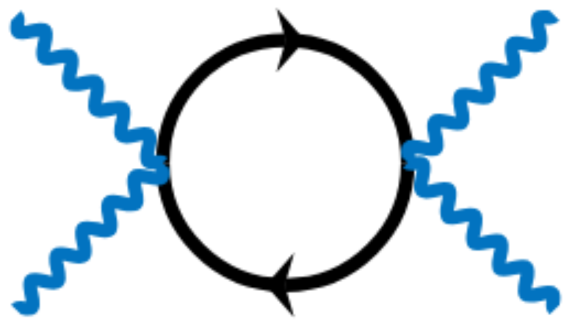
$$\mu(\psi) = -\,\partial_i\partial^i\psi + \ldots$$

$$\psi(\mu) = -\,\frac{1}{\partial_i\partial^i}\mu + \ldots$$

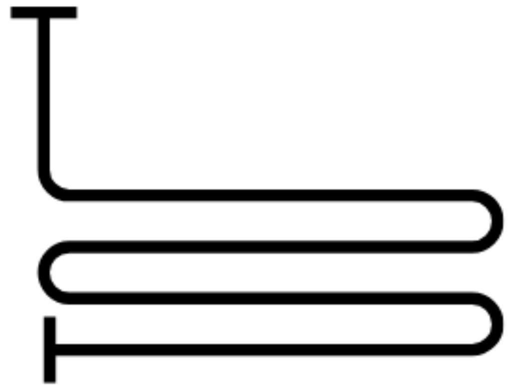
while "Can" scheme only works for Model A.
Is there a consistent hydrostatic-compatible SK-EFT for Model B?
How does this fit with conventional stochastic hydrodynamics?

- Can we run real-time Wilsonian RG on SK-EFTs?
Will need non-Gaussian stochastic transport.

[Jain, Kovtun 2020]

$\phi_a^4 \sim$ 

- Can we extend the BRST ghost structure to out-of-time-ordered (OTO) SK-EFTs?
Higher-$\mathcal{N}$ emergent supersymmetry?
Interesting repucursions for Quantum Chaos?



THANK YOU

References

[250x.xxxxx] AJ, Kovtun
[2011.03691] AJ, Kovtun, Ritz, Shukla

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DIEP

UvA

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