Symmetry and causality constraints on Fermi liquids

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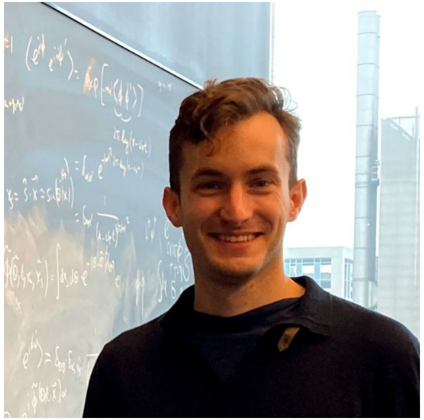
Umang Mehta



Luca V. Delacretaz

Outline

- Motivation
- Fermi liquid theory: Review and a modern perspective
- Symmetry and causality constraints
- Non-linear response and constraints
- Conclusion

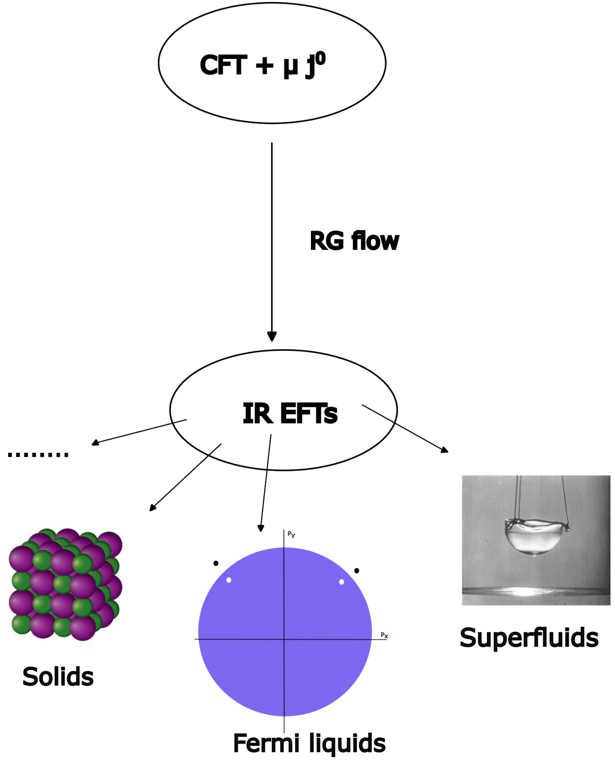
Motivation

Conformal field theories are ubiquitous in both high energy physics and condensed matter.

CFT + doping as a way to generate new phases of matter

Motivation

- Conformal Superfluids from CFTs
  [Ratazzi et al. 2015]
- Conformal solids from CFTs
  [Sachdev 2012]
- Conformal Fermi liquids from CFTs
  [Geracie et al. 2015]



Swampland of compressible phases

Q: Can any compressible IR phase be reached in this manner?

Swampland of compressible phases

Q: Can any compressible IR phase be reached in this manner?

A: No! Symmetry and causality constrain the EFT parameters.

Focus on Fermi liquid IR phase in 2+1 dimensions

(see also [Creminelli et al. 2022, 2024])

Fermi liquids

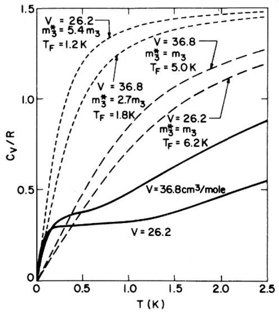
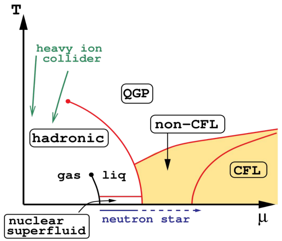
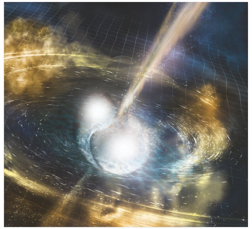
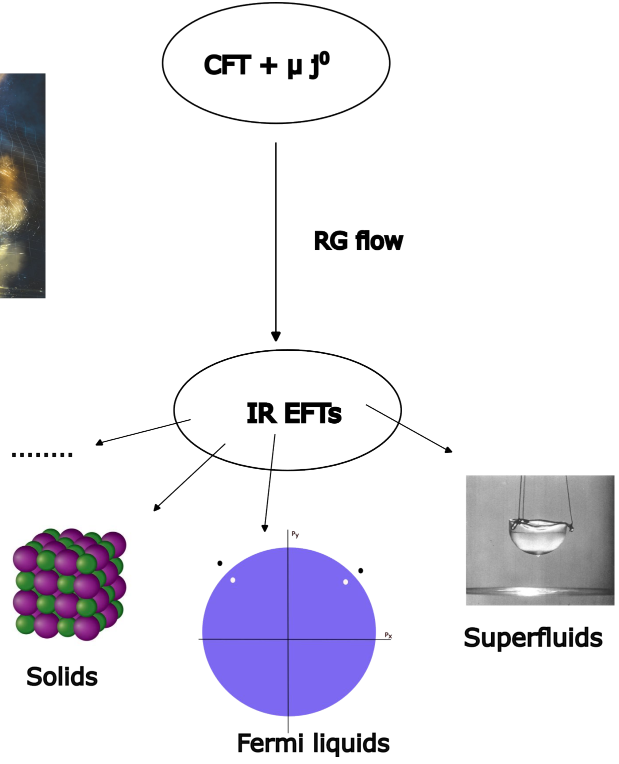


FIG. 6. Smoothed results for the ^{3}He specific heat (in units of the gas constant R) measured at molar volumes corresponding to nominal sample pressures of 0 and 29 bar. For comparison, long-dashed curves show the ideal-Fermi-gas specific heat at the same two densities. Short-dashed curves were also computed using the ideal-gas relations but with the particle mass adjusted to give the correct limiting slopes at $T = 0$.





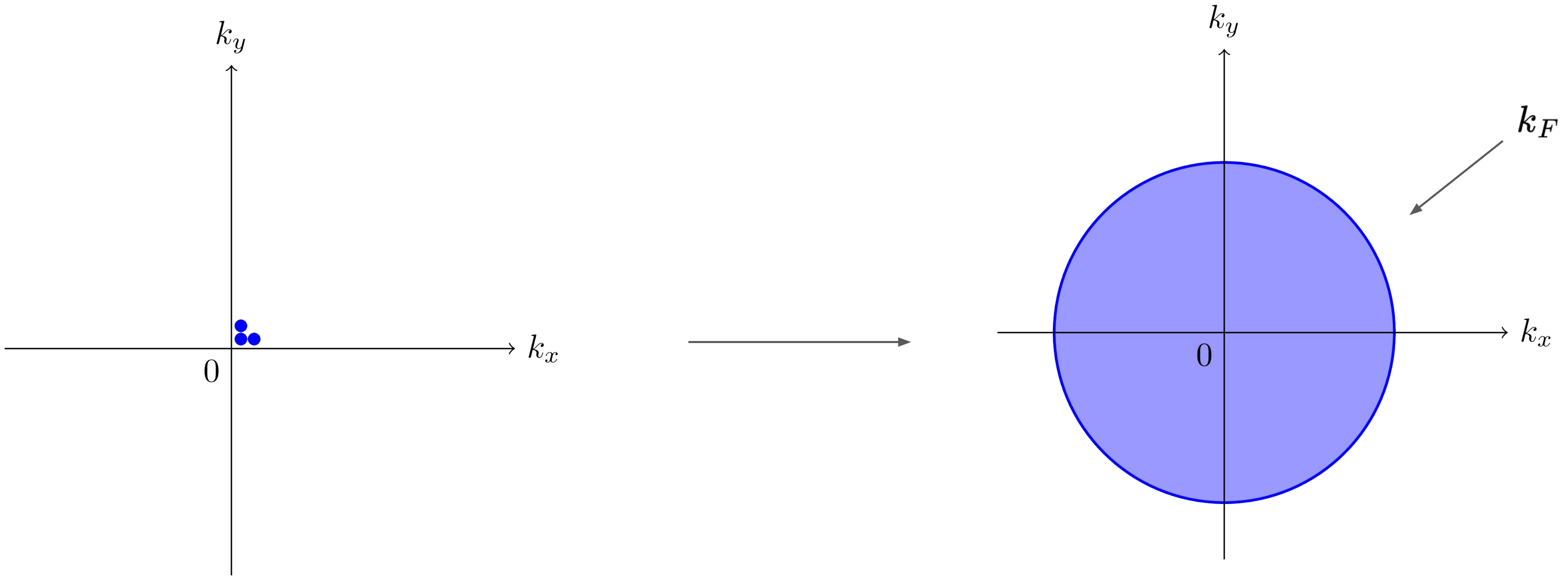


- $C_V \propto T$, Large Entanglement entropy, ...

- Unbroken U(1), translations and rotations

Landau Fermi liquid theory: Review

Let us consider free non-relativistic fermions: Fermi gas.

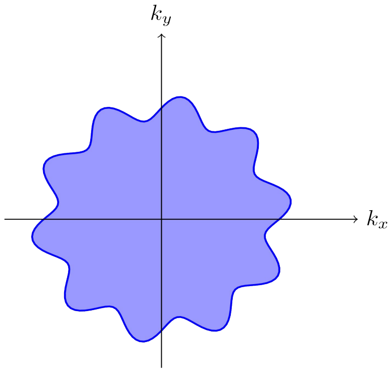


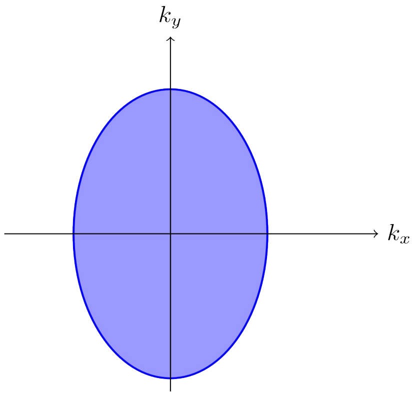
Pauli exclusion principle.

$$f_0(p) = \begin{cases} 0 \ \text{if} \ p > k_F \\ 1 \ \text{if} \ p < k_F \end{cases}$$

LFLT: Review

Fluctuations / particle-hole excitations about the FS: Boltzmann Kinetic equation





Local FS per volume element $L \gg \frac{1}{k_F}$

$$\partial_t f(t,x,p) + \nabla_p H \cdot \nabla_x f - \nabla_x H \cdot \nabla_p f = 0$$

$$H = \epsilon(p) + V(x)$$

$$f(t=0,x,p) \equiv f_0(x,p) = \Theta(k_F - p)$$

LFLT: Review

Turn on interactions adiabatically: D.O.F s remain unchanged

Particle $\|\vec{p}\| > k_F$	Quasi-particle $\|\vec{p}\| > k_F$
Hole $\|\vec{p}\| < k_F$	Quasi-hole $\|\vec{p}\| < k_F$
Always stable	Only stable at low energies (EFT)

Landau parameters

$$\partial_t f(t,x,p) + \nabla_p H \cdot \nabla_x f - \nabla_x H \cdot \nabla_p f = 0$$

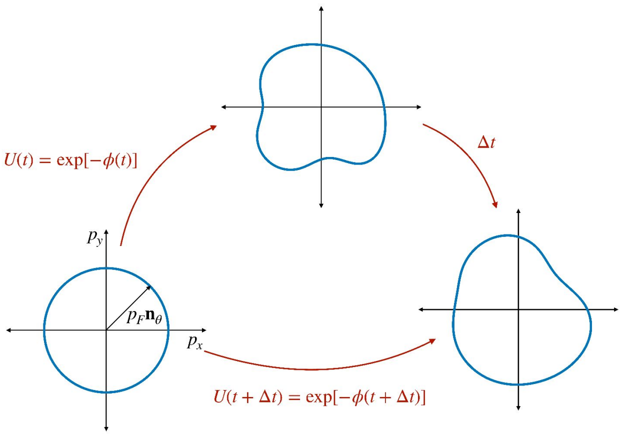
$$H = \tilde{\epsilon}(p,x) + V(x)$$

$$\tilde{\epsilon}(p,x) = \epsilon(p) + 2\int_{\vec{p}'} F^{(2,0)}(p,p') f(t,p',x) + \cdots$$

EFT

"An EFT of shapes" - write down an action that reproduces the Landau Kinetic equation (LKE).

[Polchinski 1992,…Delacretaz et al. 2022]



Fluctuating shapes obtained as canonical transformations of the spherical ground state

EFT: Gaussian action

The most general distribution function takes the form,

$$f \equiv U f_0 U^{-1} = f_0 - \{\phi, f_0\} + \frac{1}{2}\{\phi, \{\phi, f_0\}\} + \cdots .$$

Not all canonical transformations lead to a new state, $\phi(t, \vec{x}, \vec{p}) = \phi(t, \vec{x}, \theta)$,

i.e., restricted to dependance only on direction on FS.

Finally the gaussian action that reproduces linearised LKE,

$$S = -\frac{k_F}{2}\int \frac{dt d^2x d\theta}{(2\pi)^2} \nabla_n \phi \left(\dot{\phi} + v_F \nabla_n \phi + v_F \int \frac{d\theta'}{(2\pi)^2} F^{(2,0)}(\theta - \theta')\nabla_{n'}\phi'\right) + \cdots ,$$

$$\nabla_n = \hat{n}(\theta) \cdot \nabla, \qquad \hat{n} = \binom{\cos(\theta)}{\sin(\theta)}$$

[Luther 1979,…Delacretaz et al. 2022]

Landau parameters: Symmetry constraints

Must non-linearly realise spontaneously broken Lorentz and conformal symmetries.

This is in general difficult: instead we constrain observables using ward identities corresponding to these symmetries (for now).

Observables: Correlation functions of conserved operators.

$$\rho = \frac{p_F}{2\pi} \int \frac{d\theta}{2\pi} \nabla_n \phi \,, \qquad T^{0i} = \frac{p_F^2}{2\pi} \int \frac{d\theta}{2\pi} \hat{n}^i \nabla_n \phi \,.$$

Warmup: Galilean invariance

$$\chi_{\pi^i\pi^i} = \lim_{|\vec{q}|\to 0, \omega\to 0} G^R_{T^{0i}T^{0i}}(\omega, \vec{q})$$

Gaussian action

Turn on sources in the linearised action,

$$\chi_{\pi^i\pi^i} = \frac{k_F^3}{4\pi v_F(1+F_1)}$$

$$F^{(2,0)}(\theta-\theta') = 2\pi\sum_\ell F_\ell e^{i\ell(\theta-\theta')}\,,$$

$$F_{-\ell} = F_\ell^* = F_\ell\,.$$

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Symmetry

$$\chi_{\pi^i\pi^i} = m\langle\rho\rangle = \frac{mk_F^2}{4\pi}$$

m is the ``bare mass''/ the central charge of the Galilean algebra.

$$1+F_1 = \frac{k_F}{v_F m}\,.$$

[Landau 1957]

Lorentz invariance

$$\chi_{\pi^i\pi^i} = \lim_{|\vec{q}|\to 0, \omega\to 0} G^R_{T^{0i}T^{0i}}(\omega, \vec{q})$$

Gaussian action

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$$F_{-\ell} = F_\ell^* = F_\ell\,.$$

Symmetry

Using relativistic ward identities,

$$\nabla_\mu \langle T^{\mu\nu}\rangle = F^{\nu\lambda}\langle j_\lambda\rangle, \qquad \nabla_\mu\langle j^\mu\rangle = 0\,,$$

$$\chi_{\pi^i\pi^i} = \frac{\mu k_F^2}{4\pi}$$

[Policastro 2005,...]

$$1 + F_1 = \frac{k_F}{v_F \mu}\,.$$

[Baym et al. 1976, Delacretaz, SDC, Mehta 2025]

Scale invariance

$$\chi_{\rho\rho} = \lim_{|\vec{q}|\to 0, \omega\to 0} G^R_{\rho\rho}(\omega, \vec{q})$$

Gaussian action

Turn on sources in the linearised action,

$$\chi_{\rho\rho} = \frac{k_F}{2\pi v_F(1+F_0)}$$

$$F^{(2,0)}(\theta - \theta') = 2\pi \sum_\ell F_\ell e^{i\ell(\theta-\theta')} \,,$$

$$F_{-\ell} = F_\ell^* = F_\ell \,.$$

Scale invariance

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Symmetry

Scale invariance implies,

$$\langle\rho\rangle \propto \mu^{\frac{2}{z}} \,, \qquad \chi_{\rho\rho} = \frac{2\langle\rho\rangle}{z\mu} = \frac{k_F^2}{2z\pi\mu}$$

$$1 + F_0 = \frac{\mu z}{v_F k_F} \,.$$

Delacretaz, SDC, Mehta 2025

Landau parameters: Causality constraints

Causality implies for space-like separation: $[\mathcal{O}_1, \mathcal{O}_2] = 0$

↓

Retarded Green's function is analytic in momentum space for $\mathrm{Im}\,\omega \geq |\mathrm{Im}\,p|$

If polynomially bounded

$$G^R(t, \vec{x}) = \int_{\omega, \vec{p}} e^{-i\omega t + i\vec{p}\cdot\vec{x}} G^R(\omega, \vec{p}),$$

All non-analyticities therefore must obey: $\mathrm{Im}\,\omega \leq |\mathrm{Im}\,p|$

Landau parameters: Causality constraints

Even in absence of interactions, the system exhibits non-analyticities

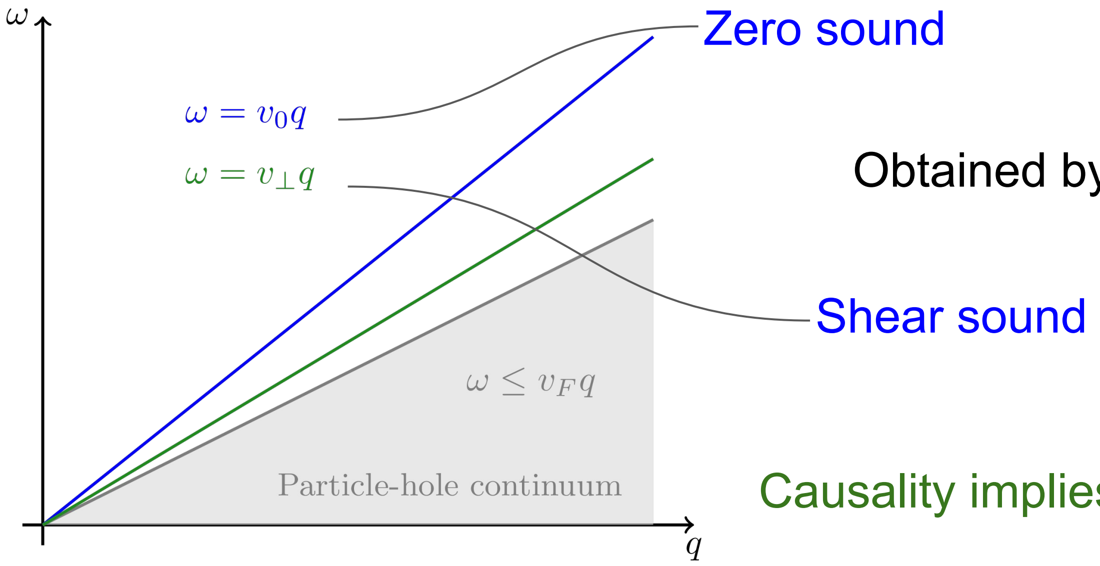
$$G^R_{\rho\rho}(\omega,\vec{q}) = -i\frac{p_F}{2\pi v_F}\left[-1+\frac{s}{\sqrt{(s+i0^+)^2-1}}\right], \qquad s\equiv\frac{\omega}{v_F|\vec{q}|}, \qquad S^{\phi}_{\theta,\theta'} = \frac{1}{k_F}\frac{i(2\pi)^2\delta(\theta-\theta')}{q_n(\omega-v_F q_n)}\,.$$

$\omega = v_F q$ denotes the start of particle-hole continuum.

Causality implies: $v_F \leq 1$ —Conformal FL→ $F_0F_1 + F_0 + F_1 \geq 0\,.$

Landau parameters: Causality constraints

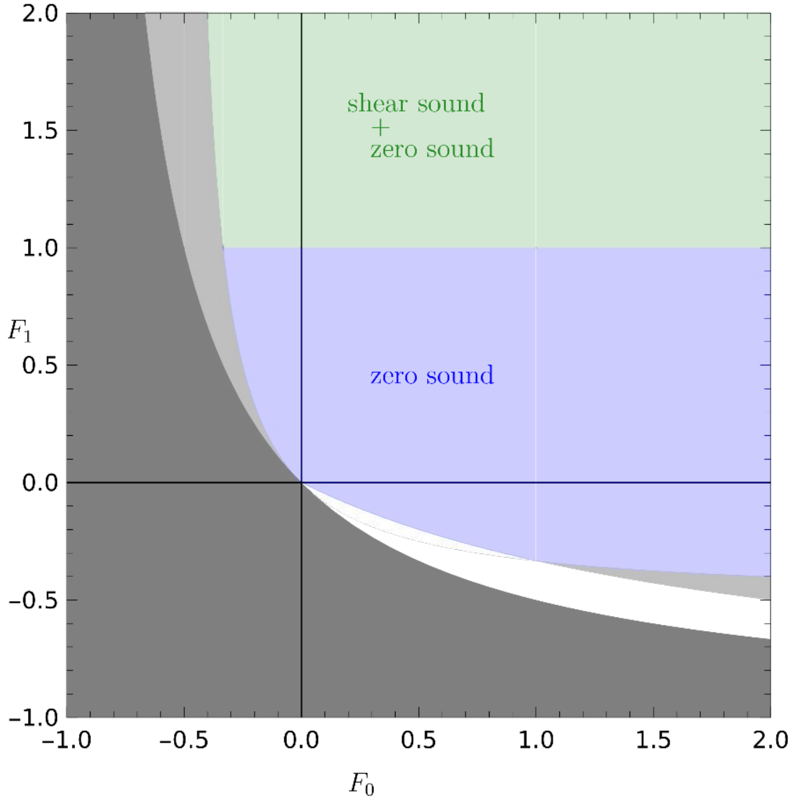
Interactions also lead to collective excitations (for simplicity $F_{\ell\geq 2}=0$)



Obtained by solving the interacting LKE

Causality implies: $v_\perp \leq 1, \qquad v_0 \leq 1$

Landau parameters: Causality constraints



Stronger than thermodynamic constraints!

$$F_\ell \geq -1$$

Bound on marginal EFT parameters

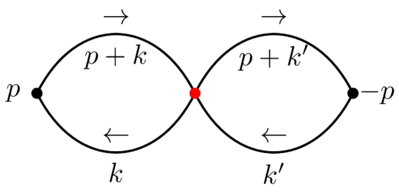
Delacretaz, SDC, Mehta 2025

Examples

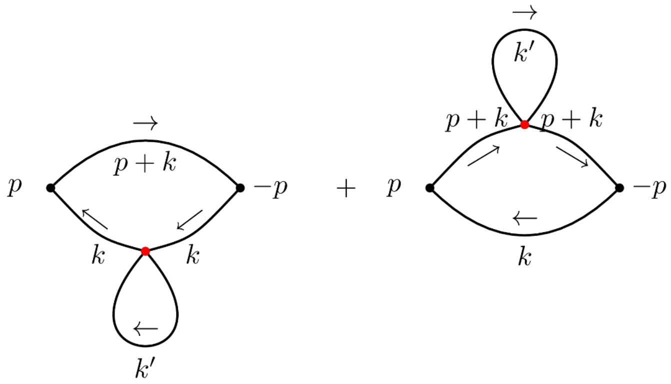
- Perturbative check of Lorentz constraint:

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m + \mu\gamma^0\right)\psi + \lambda(\bar{\psi}\psi)^2\,.$$

Compute relevant diagrams contributing to density two point function.



Contributions to Landau parameter



Corrections to Fermi momentum

Examples

- Non-perturbative check: Chern simons theories coupled to massless matter at finite chemical potential:

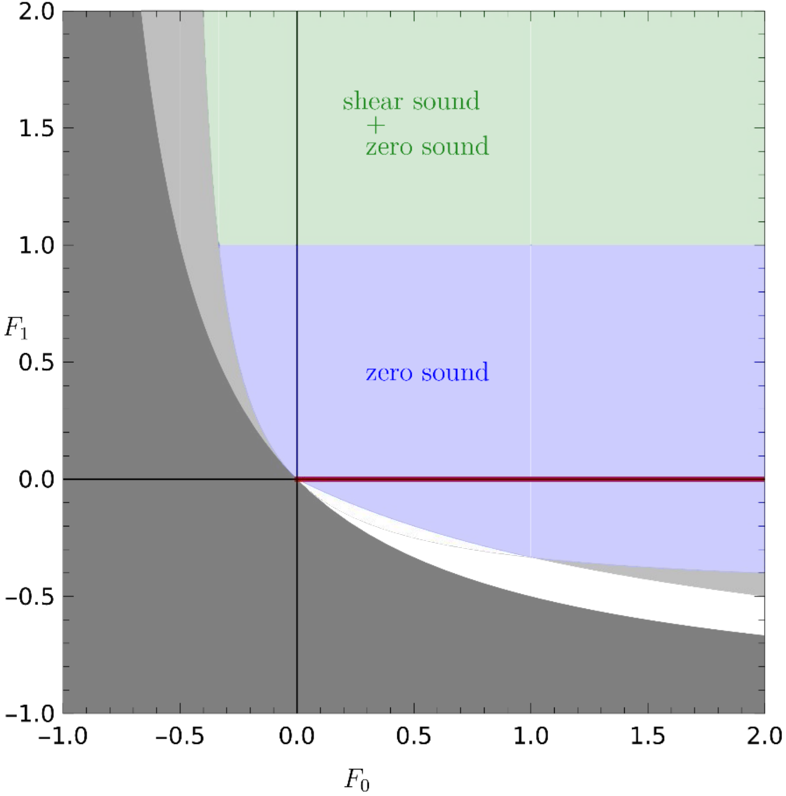
$$\mathcal{L} = N\left(\frac{i}{4\pi\lambda}\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\,\left(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\right) + \bar{\psi}\gamma^\mu D_\mu\psi - \mu\bar{\psi}\gamma^3\psi\right).$$

In the large κ, N limit, these theories can be solved to all orders in the finite 't Hooft coupling $\lambda = N/\kappa$

$$F_0 = \frac{\lambda^2}{1-\lambda^2} + O\left(\frac{1}{N}\right), \qquad F_{\ell\geq 1} = O\left(\frac{1}{N}\right), \qquad v_F = \frac{p_F}{\mu} = \sqrt{1-\lambda^2}, \qquad |\lambda| \leq 1\,.$$

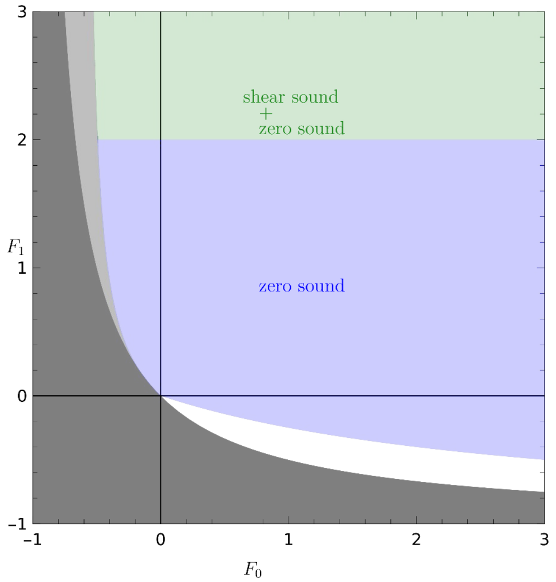
[Geracie et al. 2015]

Causal conformal Fermi liquid!


shear sound
+
zero sound
zero sound
F_1
F_0
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

d=3+1 and moduli space of FL

Can do the same exercise for d=3+1



Interesting Limits:

$$\lim_{F_0,F_1\to\infty} v_0 = \frac{1}{\sqrt{2}} \quad \lim_{F_0,F_1\to\infty} v_F = 0$$

Speed of sound in a conformal superfluid!

$$\lim_{F_1\to\infty} v_0^2 = \frac{1}{2} - v_\perp^2 \quad \lim_{F_1\to\infty} v_F = 0$$

Obeyed in a conformal solid!

[Esposito et al. 2017]

LFLT: Time to get serious!

In true spirit of EFT….

WZW

H

$$S = \int dt\, \mathrm{Tr}\left[f_0 U^{-1}(\partial_t - \epsilon)U\right],$$
$$- \int_{txpp'} F^{(2,0)}(p,p')\delta f(x,p)\delta f(x,p') + \dots,$$
$$- \int_{txpp'p''} F^{(3,0)}(p,p',p'')\delta f(x,p)\delta f(x,p')\delta f(x,p'') + \dots.$$

Linearised analysis involves just these terms

$$f \equiv U f_0 U^{-1} = f_0 - \{\phi, f_0\} + \frac{1}{2}\{\phi,\{\phi,f_0\}\} + \cdots.$$

$$\delta f(x,p) = f(x,p) - f_0(p)$$

[Delacretaz et al. 2022]

Generalised Landau parameters

LFLT: Non-linear response

$$S = \int dt\, \mathrm{Tr}\left[f_0 U^{-1}(\partial_t - \epsilon)U\right],$$
$$- \int_{txpp'} F^{(2,0)}(p,p')\delta f(x,p)\delta f(x,p') + \ldots ,$$
$$- \int_{txpp'p''} F^{(3,0)}(p,p',p'')\delta f(x,p)\delta f(x,p')\delta f(x,p'') + \ldots .$$

Contribute at leading order to higher point correlation functions

- Multipartite entanglement entropy
  [Kane et al. 2022]
- Precision tests of Fermi liquids
  [Beane et al. 2022]
- Non-linear response in topological materials
  [Bradlyn et al. 2024]
- ....

Generalised Landau parameters: Non-linear response

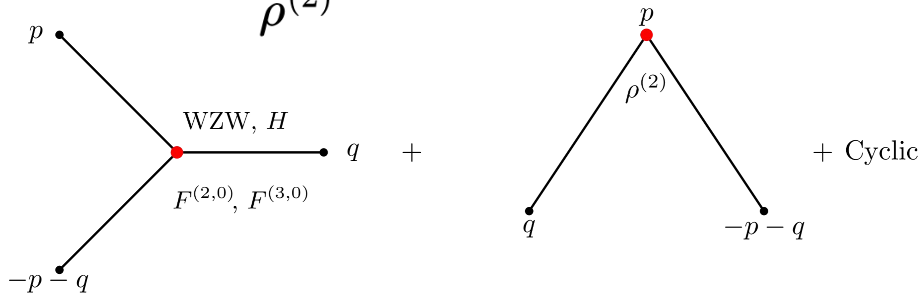
EFT allows us to construct operators to non-linear orders using Noether procedure,

$$\rho(t,x) = k_F \int \frac{d\theta}{(2\pi)^2}\left[\nabla_n\phi + \frac{1}{2k_F}\nabla_s\left(\partial_\theta\phi\nabla_n\phi\right)\right] + O(\phi^3)\,.$$

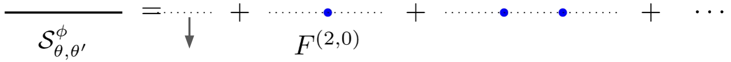
Similarly for $j^i, T^{\mu\nu}$

$\rho^{(2)}$

$$\langle\rho\rho\rho\rangle \simeq$$



Evaluate closed form expressions perturbatively



$$\frac{1}{k_F}\frac{i(2\pi)^2\delta(\theta-\theta')}{q_n(\omega-v_F q_n)}\,.$$

Generalised Landau parameters: Symmetry constraints

What are the symmetry constraints on the generalised Landau Parameters?

Symmetry	**Operator identity**	**Phase space implementation**
Poincare + Galilean	$\partial_\mu T^{\mu\nu} = 0, T^{0i} = mj^i$	Yes
Poincare + Schrodinger	$\partial_\mu T^{\mu\nu} = 0, T^{0i} = mj^i, 2T^t_t + T^i_i = 0$	Yes
Poincare + Lorentz	$\partial_\mu T^{\mu\nu} = 0, T^{\mu\nu} = T^{\nu\mu}$	No
Poincare + Lorentz + Scale	$\partial_\mu T^{\mu\nu} = 0, T^{\mu\nu} = T^{\nu\mu}, T^\mu_\mu = 0$	No

Recover linear constraints + new non-linear constraints

Delacretaz, SDC, Mehta 2025

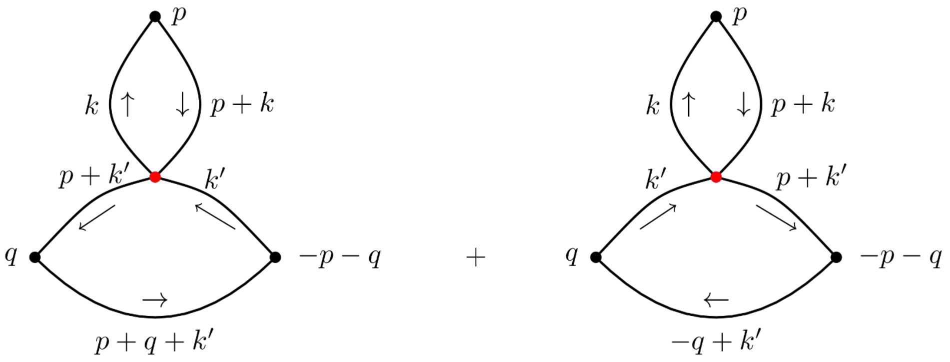
Examples

Lorentz constraint takes the following form at non-linear level,

$$\left\{2\nabla_p^i(\epsilon_p F^{(2,0)}(p,p')) - 2\int_{p''} F^{(2,0)}(p'',p')F^{(2,0)}(p,p'')\nabla^i_{p''} f^0_{p''} - 3\int_{p''}\epsilon_{p''}F^{(3,0)}(p,p',p'')\nabla^i_{p''}f^0_{p''}\right\}_{k_F} = 0\,.$$

Perturbative check in our model

$$\langle \rho\rho\rho \rangle \simeq$$



$+$ Cyclic.

Conclusion

- Manifest Lorentz invariance?
- Spin in d=3+1 Fermi liquid?
- Using dispersion relations to bootstrap Landau parameters?