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Branch-cut in the shear-stress response function of massless scalar particles in Kinetic Theory

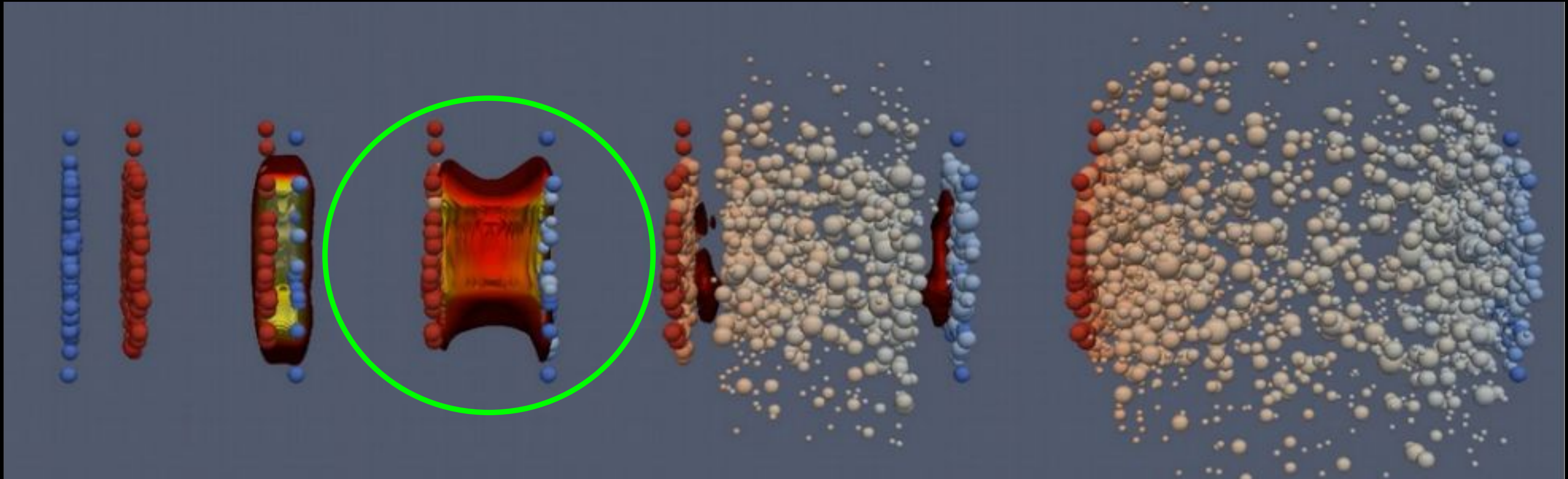
Gabriel Soares Rocha (*gabrielsr@id.uff.br*)
based on *PRD 110 (2024) 7, 076003 [arXiv:2404.04679];*
with I. Danhoni, K. Ingles, G. S. Denicol and J. Noronha
Foundations and Applications of Relativistic Hydrodynamics
Florence, Italy
May 13th, 2025

Outline

- Hydrodynamics – generic definition and Israel-Stewart-like theories
- Kinetic theory, hydro and the debate on transport coefficients
- Linear response theory for φ^4 interaction within Kinetic Theory and the branch cut
- The Gavassino theorem and the interpretation for long-lived modes

Introduction

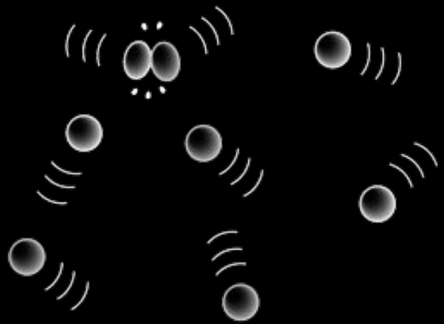
- Hydrodynamics is a crucial element in studying the evolution of matter in ultrarelativistic heavy-ion collisions;



Hybrid models for heavy-ion collisions (credit: MADAI collaboration)

Hydrodynamics

- Effective theory for long wavelength, low frequency dynamics
  "hydro modes survive at late times"
- Separation of scales ($\ell_{micro} << L_{macro}$) + near equilibrium:
  general dynamics $\mapsto$ hydrodynamics
- Dynamics described in terms of coarse grained variables $\varepsilon_0(x)$ $u^\mu(x)$
  (temperature/energy density, velocity)



Hydrodynamics

- Dynamics of coarse grained macroscopic variables (temperature/energy density, velocity);
- Fundamental equations of motion: local conservation laws $\partial_\mu T^{\mu\nu} = 0$

Equilibrium part

Dissipative part

$$T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + T^{\mu\nu}_{\text{diss}}$$

$$\varepsilon_0 u^\mu u^\nu - P_0(\varepsilon_0)\Delta^{\mu\nu} \qquad - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu}$$

energy density

pressure

bulk viscous pressure

anisotropic pressure

Thermodynamic properties:
e.g. Equation of state

Dynamics involves transport properties

Hydrodynamics

- Dynamics of coarse grained macroscopic variables (temperature/energy density, velocity);
- Fundamental equations of motion: local conservation laws $\partial_\mu T^{\mu\nu} = 0$

Not enough for eq + diss
(4 equations, 10 variables)

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + T^{\mu\nu}_{\mathrm{diss}} = \varepsilon_0 u^\mu u^\nu - [P_0(\varepsilon_0) + \Pi]\Delta^{\mu\nu} + \pi^{\mu\nu}$$

Further dynamical information is needed

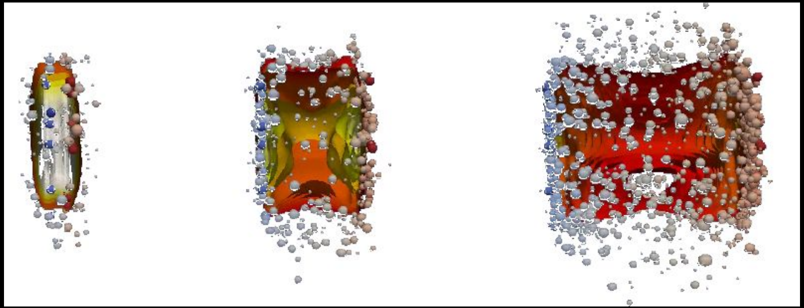
- Constitutive relations/independent EoMs for closure → ***Hydrodynamic theories*** (Navier-Stokes, *Israel-Stewart*, BDNK etc).

GSR, Wagner, Denicol, Noronha, and Rischke Entropy 26, 189 (2024)

Israel-Stewart-like theories

$$\tau_\Pi D\Pi + \Pi = -\zeta\theta + \cdots ,$$

$$\tau_\pi D\pi^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots ,$$



Israel & Stewart, (1976); DNMR (2012)

Solved in codes such as MUSIC, CLvisc, VISHNU

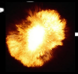
$$\theta \equiv \nabla_\mu u^\mu$$

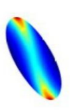
expansion rate

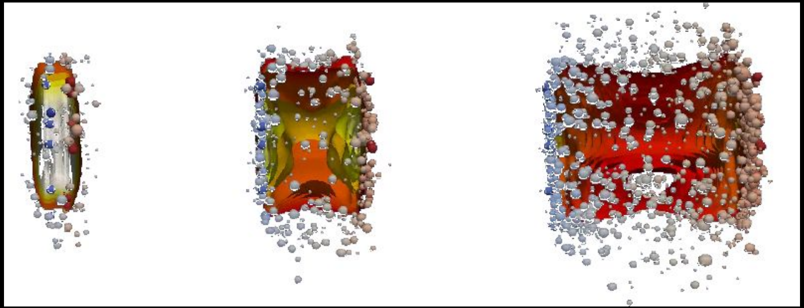
$$\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\nabla_\alpha u_\beta$$

traceless-symmetric projection

Israel-Stewart-like theories

 $D\Pi + \Pi = -\zeta$  $+ \cdots ,$

 $D\pi^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta$  $^{\nu} + \cdots ,$



Israel & Stewart, (1976); DNMR (2012)

Solved in codes such as MUSIC, CLvisc, VISHNU

- ➢ In case τ= 0: relativistic Navier-Stokes – acausal and unstable
- ➢ Procedures exist to derive ζ(T), η(T) from Kinetic Theory

Kinetic Theory and Hydrodynamics

Kinetic Theory

GSR, Wagner, Denicol, Noronha, and Rischke Entropy 26, 189 (2024)

- Non-equilibrium dynamics: Boltzmann equation

$$p^\mu \partial_\mu f_{\mathbf{p}} = C[f] = \frac{1}{2}\int dK\ dK'\ dP' W_{pp' \leftrightarrow kk'} (f_{\mathbf{k}} f_{\mathbf{k}'} - f_{\mathbf{p}} f_{\mathbf{p}'}),$$

spacetime dependence

collision rate: conservation of momentum + cross-section

- Compatibility with conservation laws

$$T^{\mu\nu} = \int dP\ p^\mu p^\nu f_p, \quad \overset{\text{Boltzmann}}{\Longrightarrow} \quad \partial_\mu T^{\mu\nu} = 0$$

Kinetic Theory

GSR, Wagner, Denicol, Noronha, and Rischke Entropy 26, 189 (2024)

- Non-equilibrium dynamics: Boltzmann equation

$$p^\mu \partial_\mu f_{\mathbf{p}} = C[f] = \frac{1}{2}\int dK\ dK'\ dP' W_{pp'\leftrightarrow kk'}(f_{\mathbf{k}} f_{\mathbf{k}'} - f_{\mathbf{p}} f_{\mathbf{p}'}),$$

spacetime dependence

collision rate: conservation of momentum + cross-section

- Near equilibrium dynamics & hydro – linearized collision term

$$C[f] \simeq f_{0\mathbf{p}} \hat{L}\phi_{\mathbf{p}} = \frac{1}{2}\int dK\ dK'\ dP' W_{pp'\leftrightarrow kk'} f_{0\mathbf{p}} f_{0\mathbf{p}'}(\phi_{\mathbf{k}} + \phi_{\mathbf{k}'} - \phi_{\mathbf{p}} - \phi_{\mathbf{p}'})$$

$$\phi_{\mathbf{p}} = \delta f_{\mathbf{p}}/f_{0\mathbf{p}} = (f_{\mathbf{p}} - f_{0\mathbf{p}})/f_{0\mathbf{p}}$$

relative deviation from equilibrium

- Non-trivial, RTA usually employed

Anderson & Witting, Physica 74, 466 (1974)
GSR, Denicol, Noronha PRL 127, 042301 (2021)

φ^4 theory – an important example

- Massless scalar particles

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{\lambda\varphi^4}{4!}, \quad \sigma_T(s) = \frac{g}{s}, \quad g \equiv \frac{\lambda^2}{32\pi},$$

s – (total energy in center of momentum frame)^2

- The *exact* spectrum of $\hat{L}$ is known

Denicol and Noronha, Phys.Lett.B 850 (2024) 138487

$$\boxed{\hat{L}\,|n,\ell\rangle = \chi_{n,\ell}\,|n,\ell\rangle}$$

$$|n,\ell\rangle = L_n^{(2\ell+1)} p^{\langle\mu_1} \cdots p^{\mu_\ell\rangle}$$

assoc. Laguerre polynomials

fully orthogonal, traceless projection

$$\chi_{n,\ell} = -\frac{g n_0 \beta^2}{4}\left(\frac{n+\ell-1}{n+\ell+1} + \delta_{n0}\delta_{\ell 0}\right)$$

Rare

see talk by G. S. Denicol for more results

Shear equations of motion – φ^4 theory

- After a power-counting procedure, we obtain GSR, de Brito, Denicol PRD 108 (2023) 3, 036017

$$\tau_\pi D\pi^{\langle\lambda\mu\rangle} + \pi^{\lambda\mu} = 2\eta\sigma^{\lambda\mu} + \varphi_2 \nu^{\langle\lambda}\nu^{\mu\rangle} - \delta_{\pi\pi}\pi^{\lambda\mu}\theta - \tau_{\pi\nu}\nabla^{\langle\lambda}P_0\,\nu^{\mu\rangle}$$
$$+ \ell_{\pi\nu}\nabla^{\langle\lambda}\nu^{\mu\rangle} + \lambda_{\pi\nu}\nabla^{\langle\lambda}\alpha\,\nu^{\mu\rangle} - 2\tau_\pi\omega_\nu{}^{\langle\lambda}\pi^{\mu\rangle\nu} - \tau_{\pi\pi}\sigma_\nu{}^{\langle\lambda}\pi^{\mu\rangle\nu},$$

- All coefficients are analytically obtained, I highlight

$$\eta = \frac{48}{g\beta^3} \qquad \tau_\pi = \frac{6\eta}{\varepsilon_0 + P_0}$$

The debate for transient hydro

- Three concurrent interpretations for τ_π etc: D. Wagner and L. Gavassino, PRD 109, 016019 (2024)
  - mere ultraviolet regulators; sole purpose: provide causality and stability; no applicability extension of Navier-Stokes; R. P. Geroch, J. Math. Phys. 36, 4226 (1995) L. Lindblom, Annals Phys. 247, 1 (1996)

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  - embodiment the microscopic interaction, extending Navier-Stokes theory to a transient regime

Denicol, Niemi, Molnar, and Rischke, PRD 85, 114047 (2012)
Denicol and Rischke, Microscopic Foundations of Relativistic Fluid Dynamics (Springer, 2021)

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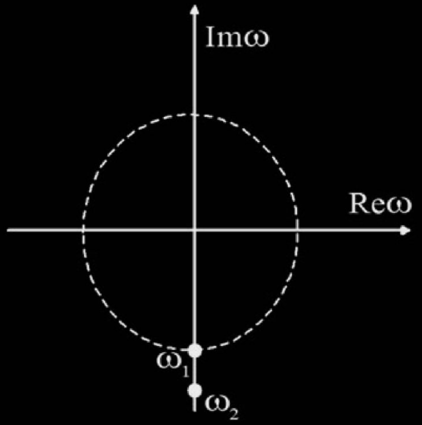
$$\frac{1}{\tau} \sim$$ *Im*(pole closest to the origin of the ret. Green function)

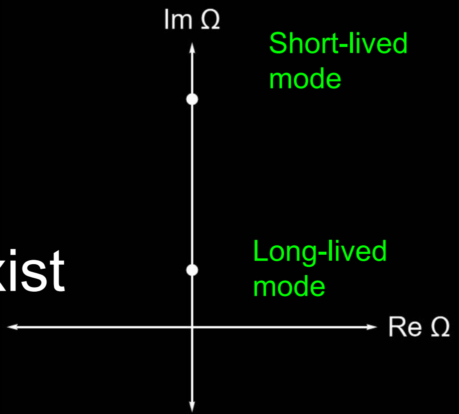
Denicol, Noronha, Niemi, and Rischke, PRD 83, 074019 (2011)



Challenge to the microscopic extension interpretation

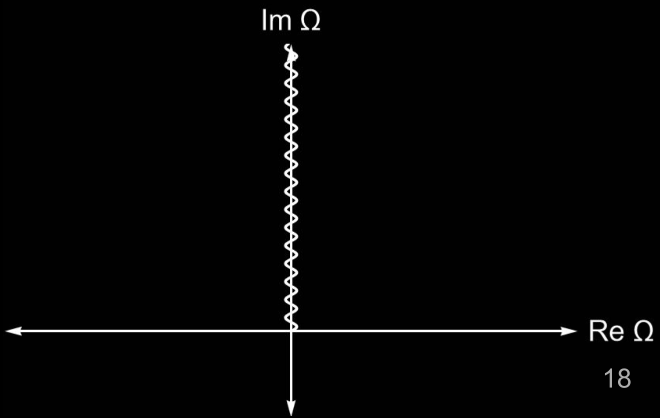
- $1/\tau \sim$ Im(pole closest to the origin)

  Denicol, Noronha, Niemi, and Rischke, PRD 83, 074019 (2011)



- However, evidence for origin-touching branch-cut exist
  - Radiation hydro Gavassino PRL 133, 032302 (2024)
  - φ^4 interaction, numerical Moore, JHEP 05, 084 (2018)
    Ochsenfeld and Schlichting, JHEP 09, 186 (2023)

- Next, we shall see what happens in linear response within kinetic theory



Linear response theory in Kinetic theory – φ^4 interaction

Linear response function for shear in Kinetic Theory

- Dynamics $p^\mu \partial_\mu f_{0\mathbf{p}} + p^\mu \partial_\mu \delta f_{\mathbf{p}} = f_{0\mathbf{p}} \hat{L} \phi_{\mathbf{p}}$

Linear response function for shear in Kinetic Theory

- Dynamics $$p^\mu \partial_\mu f_{0\mathbf{p}} + p^\mu \partial_\mu \delta f_{\mathbf{p}} = f_{0\mathbf{p}} \hat{L} \phi_{\mathbf{p}}$$

- Simplifying assumptions:

(i) Shear-only source $$p^\mu \partial_\mu f_{0\mathbf{p}} \simeq -f_{0\mathbf{p}} \beta p^{\langle\mu} p^{\nu\rangle} \sigma_{\mu\nu}$$

(ii) Space-Homogeneity $$p^\mu \partial_\mu \delta f_{\mathbf{p}} = E_{\mathbf{p}} u^\mu \partial_\mu \delta f_{\mathbf{p}} + p^\mu \nabla_\mu \delta f_{\mathbf{p}} \simeq E_{\mathbf{p}} \frac{d}{d\tau} \delta f_{\mathbf{p}},$$

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Linear response function for shear in Kinetic Theory

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Fourier space

$$f_{0\mathbf{p}} \left(E_{\mathbf{p}} i\Omega - f_{0\mathbf{p}} \hat{L} \right) \tilde{\phi}_{\mathbf{p}} = f_{0\mathbf{p}} \beta p^{\langle\mu} p^{\nu\rangle} \tilde{\sigma}_{\mu\nu}$$

Important interplay

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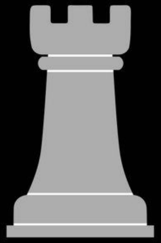
Linear response function for shear in Kinetic Theory

$$f_{0\mathbf{p}}\left(E_{\mathbf{p}} i\Omega - f_{0\mathbf{p}}\hat{L}\right)\widetilde{\phi}_{\mathbf{p}} = f_{0\mathbf{p}}\beta p^{\langle\mu}p^{\nu\rangle}\widetilde{\sigma}_{\mu\nu}$$

- Two approaches: numerical (moments tower), analytical (trotterization);



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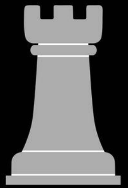
Numerical approach – Moments tower



- Integrating the linearized Boltzmann equation with $|n, 2\rangle$

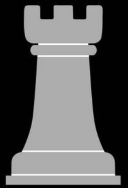
$$\boxed{\sum_m \mathcal{S}_{nm} \Phi_m^{\alpha\beta} = \frac{16}{g\beta^3} \tilde{\sigma}^{\alpha\beta} \delta_{n,0} \, ,}$$

$$\Phi_n^{\mu\nu} \equiv \int dP L_{n\mathbf{p}}^{(5)} p^{\langle\mu} p^{\nu\rangle} \delta f_{\mathbf{p}} .$$

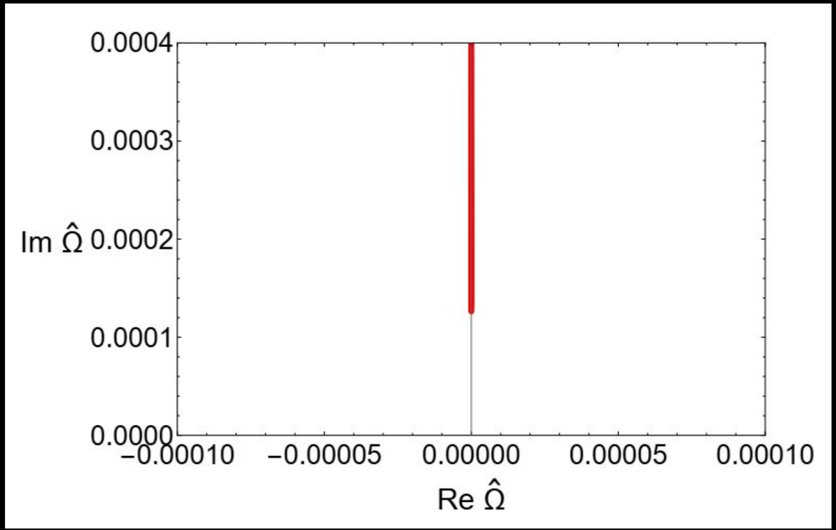
$$\pi^{\alpha\beta} = \Phi_0^{\alpha\beta}$$

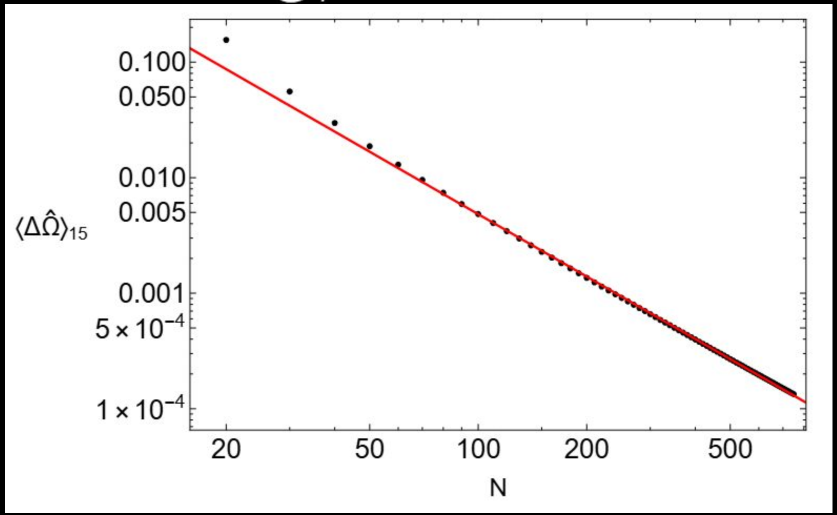
Truncations are analytically invertible (tridiagonal matrix)

Numerical approach – Moments tower

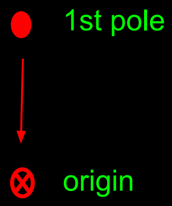


$$\pi^{\alpha\beta} = 2\eta(\widehat{\Omega})\widetilde{\sigma}^{\alpha\beta} \qquad \eta(\widehat{\Omega}) = \frac{8}{g\beta^3}(\mathcal{S}^{-1})_{00}$$

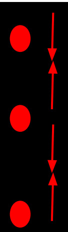




Poles for N = 1000
Closest pole distance ~ 1/N



Average distance between first 15 poles ~ $1/N^{1.8}$



Analytical approach – Trotterization

- From linear BE in Fourier, we can express $\widetilde{\pi}^{\mu\nu}$ as

$$\widetilde{\pi}^{\mu\nu} = \beta \int dP f_{0\mathbf{p}} p^{\langle\mu} p^{\nu\rangle} (E_{\mathbf{p}} i\Omega - \hat{L})^{-1} p^{\langle\alpha} p^{\beta\rangle} \widetilde{\sigma}_{\alpha\beta} .$$

Analytical approach – Trotterization

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$$\widetilde{\pi}^{\mu\nu} = \beta \int dP f_{0\mathbf{p}} p^{\langle\mu} p^{\nu\rangle} (E_{\mathbf{p}} i\Omega - \hat{L})^{-1} p^{\langle\alpha} p^{\beta\rangle} \widetilde{\sigma}_{\alpha\beta}.$$

- Computed employing the Trotterization technique

Wu,Byrd & Lidar, PRL 89, 057904 (2002).
Hall, Lie Groups, Lie Algebras, and Representations (Springer, Cham, 2015)

$$\hat{G}^{-1} = \int_0^\infty dx\, e^{-xG}. \qquad e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} (e^{\hat{A}/n} e^{\hat{B}/n})^n.$$

Then $$\widetilde{\pi}^{\mu\nu} = \lim_{n\to\infty} \widetilde{\pi}^{\mu\nu}_n,$$

Analytical approach – Trotterization

Then $$\widetilde{\pi}^{\mu\nu} = \lim_{n\to\infty} \widetilde{\pi}^{\mu\nu}_{n},$$

$$\widetilde{\pi}^{\mu\nu}_{n} = \beta\widetilde{\sigma}_{\alpha\beta} \int_0^\infty dx \int dP f_{0\mathbf{p}}\, p^{\langle\mu} p^{\nu\rangle} \left[\exp\left(-i\Omega\frac{x}{n}E_{\mathbf{p}}\right)\exp\left(\frac{x}{n}\hat{L}\right)\right]^n p^{\langle\alpha} p^{\beta\rangle}.$$

$|0,2\rangle$

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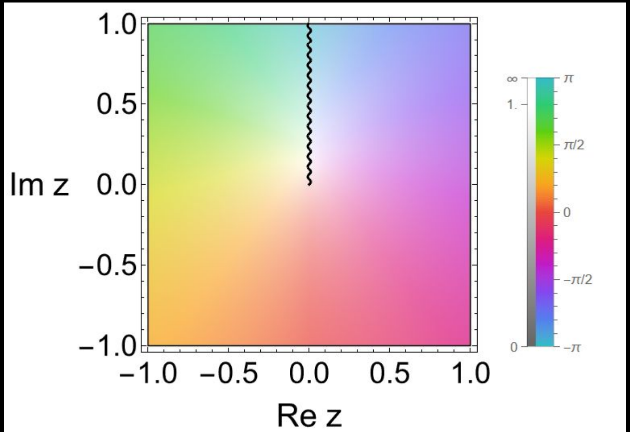
Analytical approach – 1st Trotterization truncation

$$\widetilde{\pi}_1^{\mu\nu} = 2\eta_1(\widehat{\Omega})\widetilde{\sigma}^{\mu\nu}$$

$$\eta_1(\widehat{\Omega}) = \frac{8}{g\beta^3}\frac{|\widehat{\chi}_{02}|^5}{(i\widehat{\Omega})^6} U\left(6, 6, \frac{|\widehat{\chi}_{02}|}{i\widehat{\Omega}}\right)$$

Tricomi hypergeometric function



$$(iz)^{-6}U(6, 6, i/z)$$

$$\widehat{\Omega} = 2\Omega/(gn_0\beta^2) \qquad \widehat{\chi}_{n2} = -\frac{1}{2}\frac{n+1}{n+3}.$$

Branch-cut in $i\,0^+ < z < +i\,\infty$

$$U(a, b, z) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{d}t\, e^{-zt} t^{a-1}(1+t)^{b-a-1}$$

Branch cut in $-\infty <$ Re $z < 0$

Analytical approach – nth Trotterization truncation

$$\widetilde{\pi}_n^{\mu\nu} = 2\eta_n(\widehat{\Omega})\widetilde{\sigma}^{\mu\nu}$$

More involved, sum of Tricomi U's

$$\eta_n(\widehat{\Omega}) = \frac{8n}{g\beta^3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{k_1+k_2} \sum_{k_4=0}^{\infty} \sum_{k_5=0}^{k_3+k_4} \cdots \sum_{k_{2n-4}=0}^{\infty} \sum_{k_{2n-3}=0}^{k_{2n-5}+k_{2n-4}} \binom{k_{2n-3}+5}{k_{2n-3}} a_{k_{2n-3}}^{(k_{2n-5},k_{2n-4})} \cdots a_{k_5}^{(k_3,k_4)} a_{k_3}^{(k_1,k_2)} \left(\frac{|\widehat{\chi}_{02}| + \sum_{j=1}^{n-1} |\widehat{\chi}_{k_{2j-1}2}|}{i\widehat{\Omega}} \right)^{6n-1}$$
$$\times \frac{\Gamma(K_n+1)}{i\widehat{\Omega}} U\left(K_n + 6n, 6n, \frac{|\widehat{\chi}_{02}| + \sum_{j=1}^{n-1} |\widehat{\chi}_{k_{2j-1}2}|}{i\widehat{\Omega}} \right),$$

Branch-cut in $i\,0^+ < z < +i\,\infty$

$$K_n = k_1 + \sum_{j=1}^{n-2} k_{2j} + k_{2n-3}$$

$$L_{k_1,\mathbf{p}}^{(5)} L_{k_2,\mathbf{p}}^{(5)} \equiv \sum_{j_1=0}^{k_1+k_2} a_{j_1}^{(k_1,k_2)} L_{j_1,\mathbf{p}}^{(5)}.$$

Analytical approach – nth Trotterization truncation

$$\widetilde{\pi}_n^{\mu\nu} = 2\eta_n(\widehat{\Omega})\widetilde{\sigma}^{\mu\nu}$$

More involved, sum of Tricomi U's

$$\eta_n(\widehat{\Omega}) = \frac{8n}{g\beta^3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{k_1+k_2}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{k_3+k_4}\cdots\sum_{k_{2n-4}=0}^{\infty}\sum_{k_{2n-3}=0}^{k_{2n-5}+k_{2n-4}}\binom{k_{2n-3}+5}{k_{2n-3}}a_{k_{2n-3}}^{(k_{2n-5},k_{2n-4})}\cdots a_{k_5}^{(k_3,k_4)}a_{k_3}^{(k_1,k_2)}\left(\frac{|\widehat{\chi}_{02}|+\sum_{j=1}^{n-1}|\widehat{\chi}_{k_{2j-1}2}|}{i\widehat{\Omega}}\right)^{6n-1}$$
$$\times\frac{\Gamma(K_n+1)}{i\widehat{\Omega}}U\left(K_n+6n,6n,\frac{|\widehat{\chi}_{02}|+\sum_{j=1}^{n-1}|\widehat{\chi}_{k_{2j-1}2}|}{i\widehat{\Omega}}\right),$$

Branch-cut in $i\,0^+ < z < +i\,\infty$

$$K_n = k_1 + \sum_{j=1}^{n-2}k_{2j} + k_{2n-3}$$

$$L_{k_1,\mathbf{p}}^{(5)}L_{k_2,\mathbf{p}}^{(5)} \equiv \sum_{j_1=0}^{k_1+k_2}a_{j_1}^{(k_1,k_2)}L_{j_1,\mathbf{p}}^{(5)}.$$

Is the presence of these long-lived modes particular to φ^4 theory?

The Gavassino theorem

Gavassino PR Research 6, L042043 (2024)

- Supposing that $\sigma(s,\Theta) \leq \frac{A}{s^a} + \frac{B}{s}, \quad 0 < a < 1, \quad A, B \geq 0 \quad \forall (s,\Theta)$ one can construct:

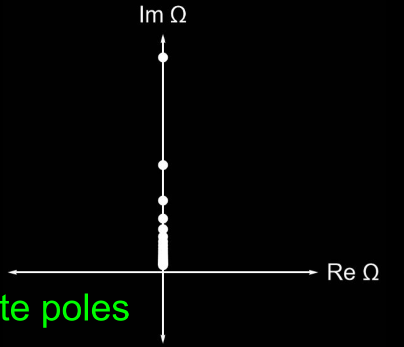
True for screened gauge theories

The Gavassino theorem

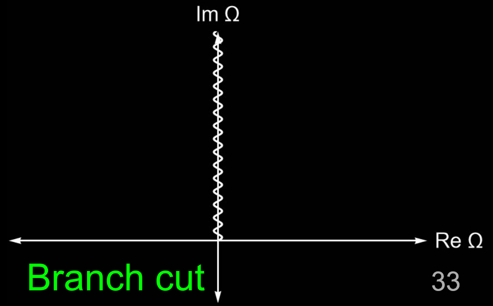
Gavassino PR Research 6, L042043 (2024)

- Supposing that $\sigma(s,\Theta) \leq \frac{A}{s^a} + \frac{B}{s}, \quad 0 < a < 1, \quad A, B \geq 0 \quad \forall (s,\Theta)$
  one can construct:
  - Arbitrarily long-lived non-hydro excitations;
  - Non-hydrodynamic states that produce negligibly small entropy;
  - Non-hydrodynamic states such that $\left|(1/E_{\mathbf{p}})\hat{L}\phi_{\mathbf{p}}\right| << |\phi_{\mathbf{p}}|$;
- Within the space of non-hydro modes, **Ω**=0 is an accumulation point

All of the four statements above are equivalent



Discrete poles



Branch cut

Reconciliation

$$\widetilde{\pi}_n^{\mu\nu} = 2\eta_n(\widehat{\Omega})\widetilde{\sigma}^{\mu\nu}$$

In the asymptotic $\Omega \to 0$ limit

$$\widetilde{\pi}_n^{\mu\nu} \simeq 2\eta_n(0)[1 - 36i\widehat{\Omega}]\widetilde{\sigma}^{\mu\nu}$$

$$\downarrow$$

$$[\tau_\pi D + 1]\pi_n^{\mu\nu} \simeq 2\eta_n(0)\sigma^{\mu\nu}$$

$$\tau_\pi = 72/(g n_0 \beta^2) \qquad \eta_n(0) = 48/(g\beta^3)$$

Recovers GSR, C.V.P. de Brito, G. S. Denicol PRD 108 (2023) 3, 036017

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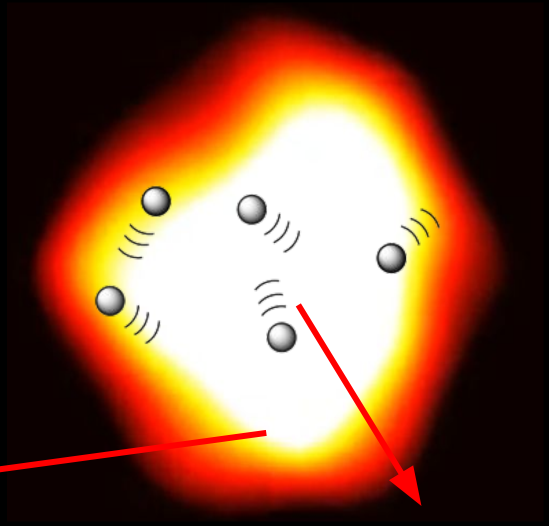
Recovers GSR, C.V.P. de Brito, G. S. Denicol PRD 108 (2023) 3, 036017

This is true for all of the truncations even though Ω=0 is a branch point

Interpretation

- There are long-lived states which are not hydrodynamic
- Origin: $\sigma_T(s) = \frac{g}{s}$ – high-energy colliding particles don't see each other
- These long-lived states have also been found in non-relativistic scenarios
  Gavassino PR Research 6, L042043 (2024)
  Caflish Commun. Math. Phys. 74, 71 (1980); *id*. 74, 97 (1980)
  Strain Commun. Math. Phys.529 (2010)
- Partitioning during the evolution:
  free streaming modes vs. hydro modes



Conclusions

- We have computed analytically the shear response function from kinetic theory and demonstrated the existence of a branch-cut for φ^4 theory;
- Non-trivial evolution: partition between free-streaming and hydro modes; this cannot be captured by e.g. RTA
- This is true for many relevant interactions given the Gavassino theorem;
- It does not mean that hydro breaks down.

THAT'S ALL FOR TODAY!

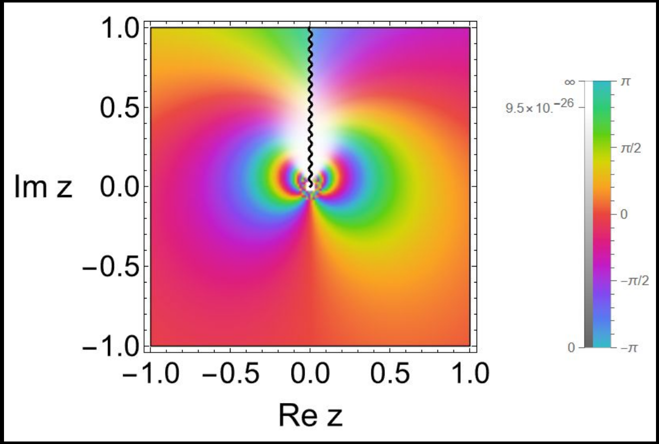
BACKUP

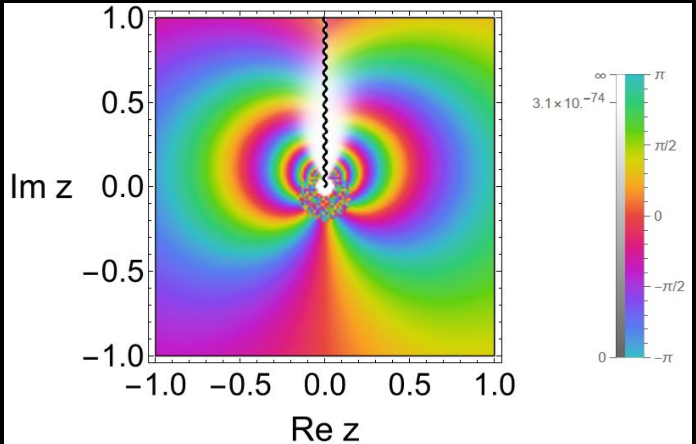
Analytical approach – higher truncations

Second truncation

$$\tilde{\pi}_2^{\mu\nu} = 2\eta_2(\widehat{\Omega})\tilde{\sigma}^{\mu\nu}$$

$$\eta_2(\widehat{\Omega}) = \frac{16}{g\beta^3}\sum_{k=0}^{\infty}\binom{k+5}{k}\frac{\Gamma(2k+1)}{i\widehat{\Omega}}\left(\frac{|\widehat{\chi}_{02}|+|\widehat{\chi}_{k,2}|}{i\widehat{\Omega}}\right)^{11} U\left(2k+12, 12, \frac{|\widehat{\chi}_{02}|+|\widehat{\chi}_{k,2}|}{i\widehat{\Omega}}\right)$$





Analytical approach – higher truncations

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Higher truncations

$$\widetilde{\pi}_n^{\mu\nu} = 2\eta_n(\widehat{\Omega})\widetilde{\sigma}^{\mu\nu}$$

Branch-cut in $i0^+ < z < +i\infty$

$$\eta_n(\widehat{\Omega}) = \frac{8n}{g\beta^3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{k_1+k_2}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{k_3+k_4}\cdots\sum_{k_{2n-4}=0}^{\infty}\sum_{k_{2n-3}=0}^{k_{2n-5}+k_{2n-4}}$$
$$\times\binom{k_{2n-3}+5}{k_{2n-3}} a_{k_{2n-3}}^{(k_{2n-5},k_{2n-4})}\cdots a_{k_5}^{(k_3,k_4)} a_{k_3}^{(k_1,k_2)}\left(\frac{|\widehat{\chi}_{02}|+\sum_{j=1}^{n-1}|\widehat{\chi}_{k_{2j-1}2}|}{i\widehat{\Omega}}\right)^{6n-1}$$
$$\times\frac{\Gamma(K_n+1)}{i\widehat{\Omega}} U\left(K_n+6n, 6n, \frac{|\widehat{\chi}_{02}|+\sum_{j=1}^{n-1}|\widehat{\chi}_{k_{2j-1}2}|}{i\widehat{\Omega}}\right),$$

$$K_n = k_1 + \sum_{j=1}^{n-2} k_{2j} + k_{2n-3}$$

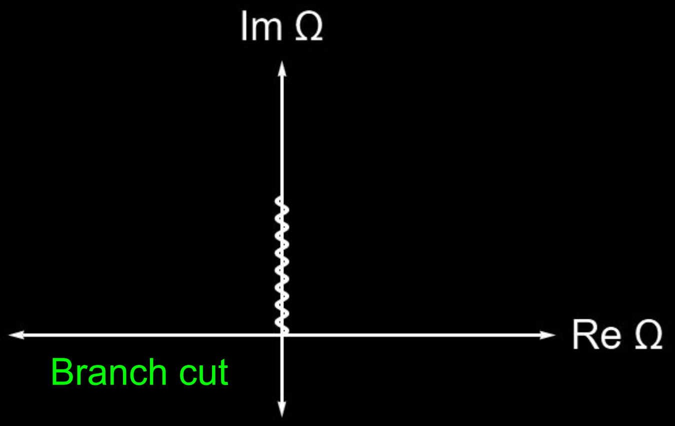
$$L_{k_1,\mathbf{p}}^{(5)} L_{k_2,\mathbf{p}}^{(5)} \equiv \sum_{j_1=0}^{k_1+k_2} a_{j_1}^{(k_1,k_2)} L_{j_1,\mathbf{p}}^{(5)}.$$

Another theorem

M. Dudynski & Ekiel-Jezewska Commun. Math. Phys. 115 607-629 (1988)

- For $\sigma(s,\Theta) \leq \dfrac{B \sin^c \Theta}{(s-4m^2)^a}$

$0 < a < 2, B > 0, c > -2 \quad m \neq 0$



- Ω=0 is an accumulation point, all other properties follow

Gavassino PR Research 6, L042043 (2024)

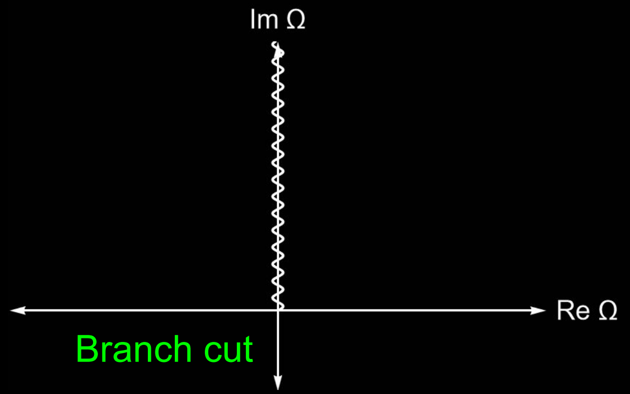
RTA-like approximation

$$\tilde{\pi}^{\mu\nu} = \beta \int dP f_{0\mathbf{p}} p^{\langle\mu} p^{\nu\rangle} \left(E_{\mathbf{p}} i\Omega - \hat{L}\right)^{-1} p^{\langle\alpha} p^{\beta\rangle} \tilde{\sigma}_{\alpha\beta}.$$

- Approximation: $\hat{L} \simeq \chi \mathbb{1}$, interesting, because $-1/2 < \widehat{\chi}_{n2} \leq -1/6$

$$\tilde{\pi}^{\mu\nu} = \beta \int dP f_{0\mathbf{p}} p^{\langle\mu} p^{\nu\rangle} \frac{1}{i\Omega E_{\mathbf{p}} - \chi} p^{\langle\alpha} p^{\beta\rangle} \tilde{\sigma}_{\alpha\beta} \equiv 2\eta\left(\Omega\right) \tilde{\sigma}^{\mu\nu},$$

$$\eta\left(\Omega\right) \equiv \frac{\beta}{15} \int_0^\infty \frac{dE_{\mathbf{p}}}{2\pi^2} f_{0\mathbf{p}} \frac{E_{\mathbf{p}}^5}{i\Omega E_{\mathbf{p}} - \chi}.$$



Israel Stewart-like hydro from Boltzmann

Boltzmann eqn. dynamics → Moments dynamics

DNMR, PRD 85, 114047 (2012)
de Brito Denicol, PRD 110 (2024) 3, 036017

$$\rho_r^{\mu_1 \cdots \mu_\ell} = \int dP E_{\mathbf{p}}^r p^{\langle \mu_1} \cdots p^{\mu_\ell \rangle} \delta f_{\mathbf{p}},$$

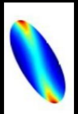
fully traceless projection

→ **deviation from local equilibrium**

$$\delta f_{\mathbf{p}} \equiv f_{\mathbf{p}} - f_{0\mathbf{p}}$$

$$\pi^{\mu\nu} = \rho_0^{\mu\nu}$$

$$D\rho_r^{\langle\mu\nu\rangle} + \cdots + \alpha_r^{(2)} \sigma^{\mu\nu} = -\sum_n \mathcal{L}_{rn}^{(2)} \rho_n^{\mu\nu}$$



-1

Hydrodynamics: power counting

$$\rho_r^{\mu_1 \cdots \mu_\ell} \longrightarrow \{\Pi, \nu^\mu, \pi^{\mu\nu}\}$$

DNMR, PRD 85, 114047 (2012)
Wagner, Palermo, Ambrus PRD 106 (2022) 1, 016013

Israel Stewart-like hydro from Boltzmann

Boltzmann eqn. dynamics→ Moments dynamics

DNMR, PRD 85, 114047 (2012)
de Brito Denicol, PRD 110 (2024) 3, 036017

$$\rho_r^{\mu_1\cdots\mu_\ell} = \int dPE_{\mathbf{p}}^r p^{\langle\mu_1}\cdots p^{\mu_\ell\rangle}\delta f_{\mathbf{p}},$$

→ **deviation from local equilibrium**

$$\delta f_{\mathbf{p}} \equiv f_{\mathbf{p}} - f_{0\mathbf{p}}$$

$$f_{0\mathbf{p}} \equiv \exp(\alpha - \beta u_\mu p^\mu).$$

In particular, $\Pi = \frac{1}{3}(\rho_2 - m^2\rho_0) \quad \nu^\mu = \rho_0^\mu \quad \pi^{\mu\nu} = \rho_0^{\mu\nu}$

Hydrodynamics:
reduction of d.o.f's $\rho_r^{\mu_1\cdots\mu_\ell} \longrightarrow \{\Pi, \nu^\mu, \pi^{\mu\nu}\}$ **(Landau matching)**