CSIC
Consejo Superior de Investigaciones Científicas

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Excelencia María de Maeztu

Modelling relativistic turbulence: covariant approach to LES

Collaborators: *N. Andersson, I. Hawke, M.J. Hatton, G. Comer*

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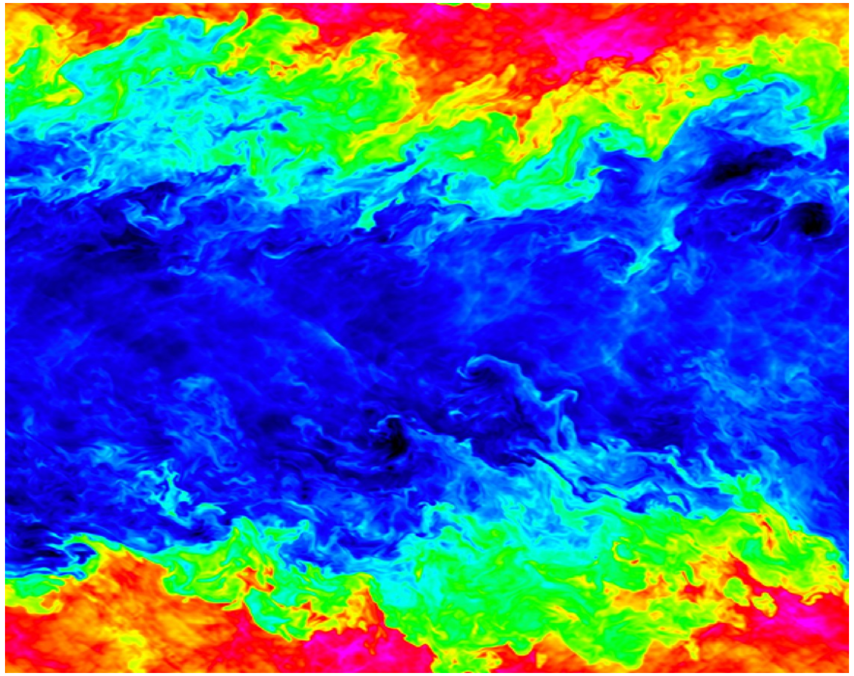
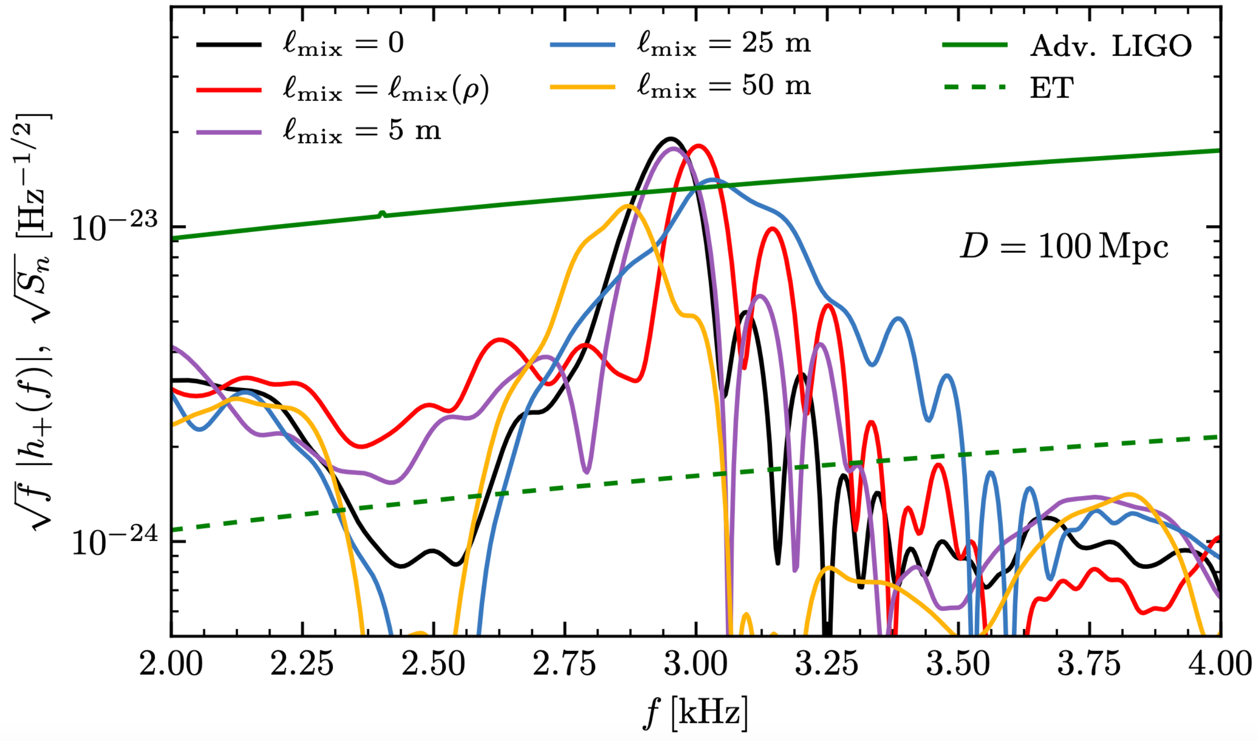
[PRD: 104 084090 , PRD: 110 123040 , PRD: 110 123039]

Turbulence impact on BNS mergers

Turbulence develops in mergers due to, e.g., the Kelvin-Helmholtz instability (KHI) at the slip-line between merging NS. Turbulence modelling impacts on the merger dynamics, the magnetic field amplification and the post-merger gravitational wave spectrum.



[Carrasco+ (2020)]



[Radice (2020)]

Turbulence and Large Eddy Simulations

Turbulence is a manifestation of the intrinsic non-linearity of hydrodynamics. The small scales can impact on the large-scale dynamics via non-linear mode coupling.

DNS of BNS mergers are not feasible:

- Conservative estimate of dissipation scale ≈ 1 cm
- Best resolution in large-scale simulations ≈ 10 m

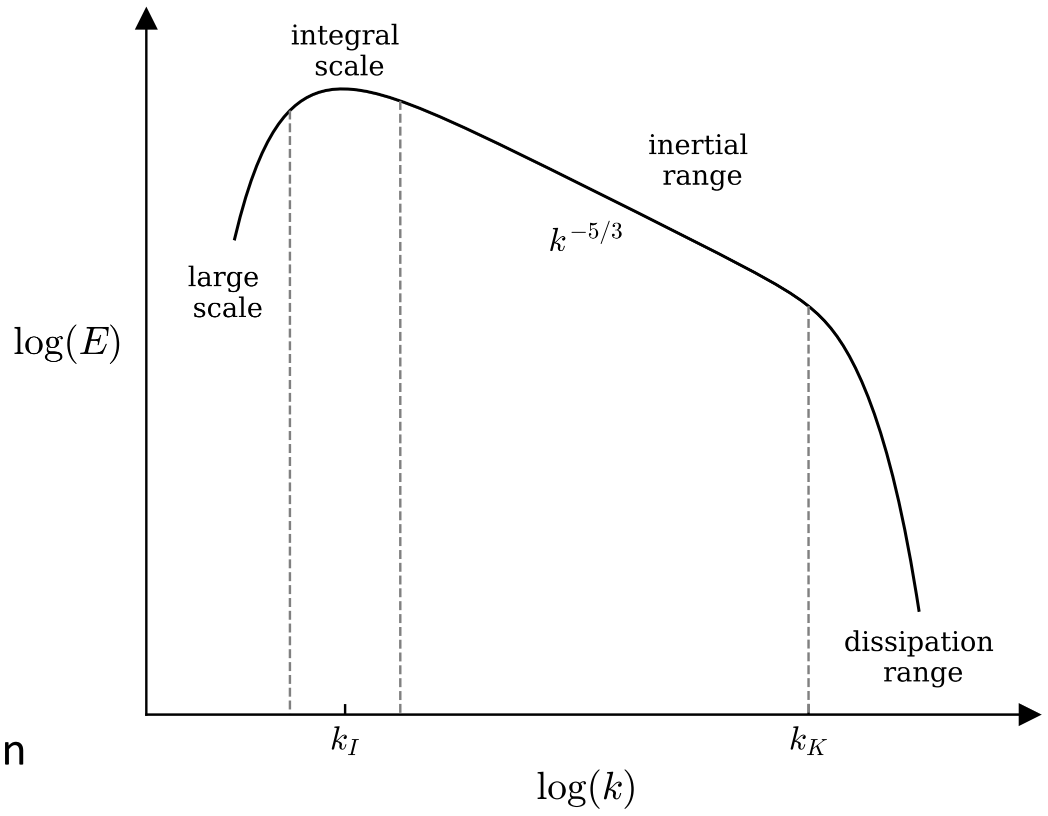
$$\mathrm{Re} \approx 10^6 \div 10^{15}$$

Accounting for the unresolved physics:

- Filtering to separate into resolved and unresolved
- Evolve large-scale dynamics, model the rest

A little bit of context:

- Applications to numerical relativity have shown impressive results, e.g. Aguilera-Miret+(2022)
- All practical implementations so far break covariance, both in filtering and in the closures Radice-Hawke(2024)



General relativistic LES: the issue of covariance

The 3+1 "operational" approach
Filtering as a simple set of rules to be applied directly on the 3+1 equations

Same as for Newtonian theory

Quick derivation of the coarse-grained equations

$$
\begin{aligned}
A &= \langle A \rangle + \delta A \\
\langle c \rangle &= c \\
\langle A + B \rangle &= \langle A \rangle + \langle B \rangle \\
\langle \partial_a A \rangle &= \partial_a \langle A \rangle
\end{aligned}
$$

This raises a number of "theory" questions...

- Spatial filtering in relativity: spatial w.r.t. whom?
- Filter op. does not commute with nonlinear terms. What about EFE?
- Integrals we need: input and output are tensors

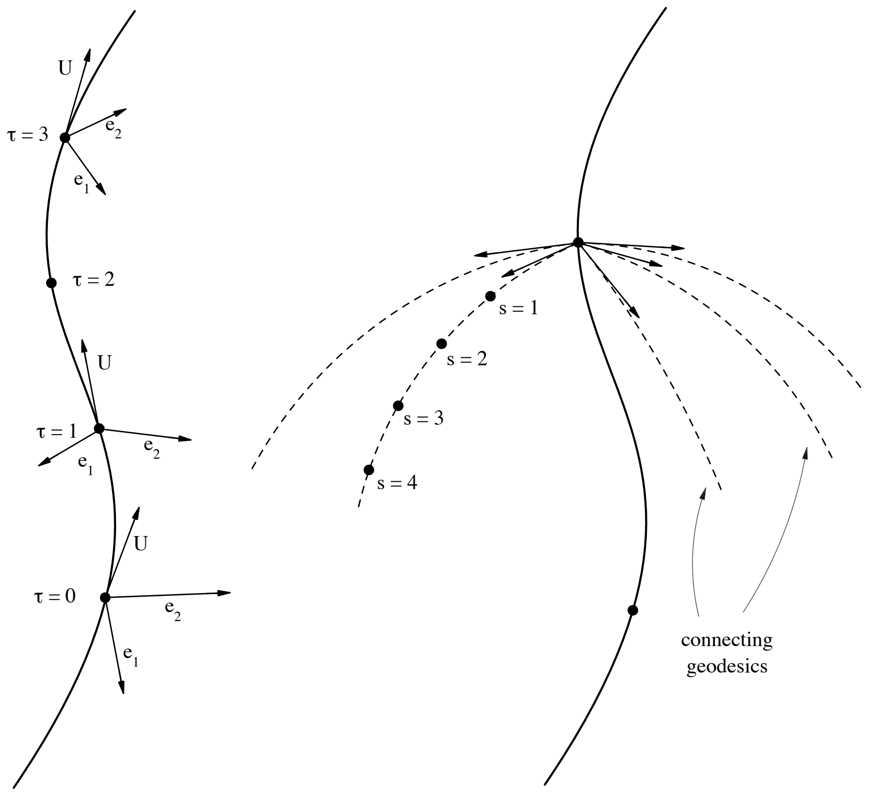
The issue of covariance: not only theory

"Non covariant choice of closure schemes can induce artificial (coordinate independent) artefacts. For example, one expects turbulent momentum transport to operate only when there is non-zero shear in a Local Lorentz frame, which is guaranteed only for covariant closures."

[Duez+ (2020)]

Fibration framework and Fermi coordinates

[PRD: 104 084090]



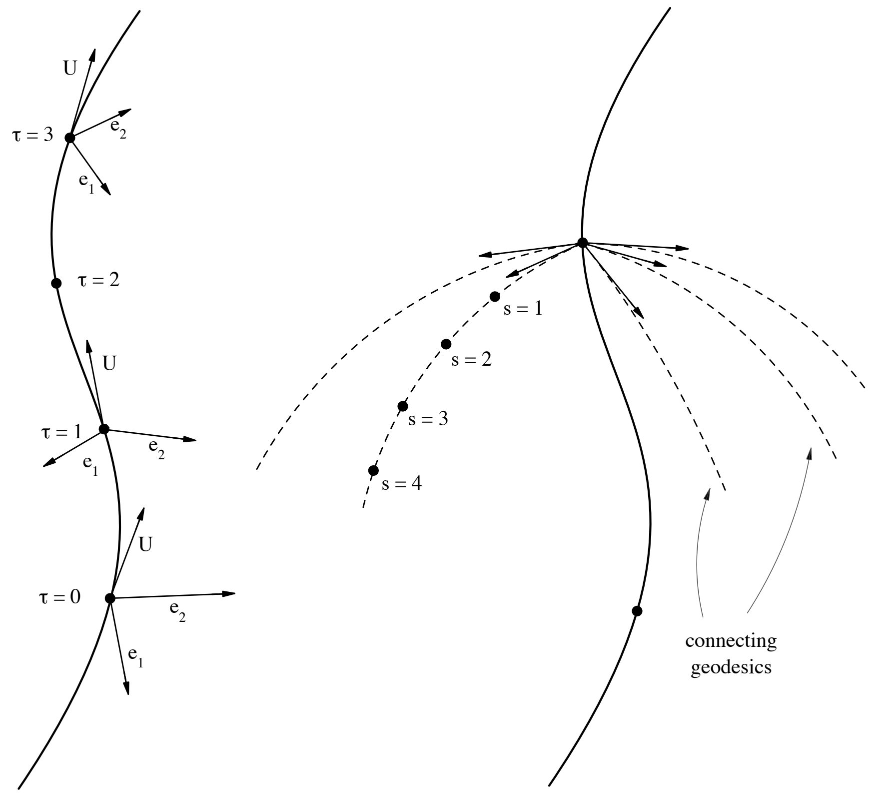
A step back: desirable properties:

1. Do not affect the "geometry sector": EFE are highly non-linear, but we want to stick to GR.
2. Keep the connection to the microphysics: important *per se* and for C2P (internal consistency of the scheme)
3. Generically, modelling errors associated with any LES scheme at finite resolution depend on i) sub-filter physics ii) ST metric iii) numerical scheme. We want to have control over these errors: aim is to use these to compensate for the numerical error.

<u>Any LES model is calibrated with some "choice" of the metric background</u>

Fibration framework and Fermi coordinates

[PRD: 104 084090]



Key ideas:

- Fluid worldlines provide a natural fibration of ST
- Fermi coordinates:
    1. meaningful ST split w.r.t. a local observer
    2. curvature terms are 2ns order in exp. away from central WL

Advantages of the framework:

- *Metric unaffected is shown, rather than assumed.* The geometry "sector" of the theory is untouched.
- Fibration observer filtering → lift from SR to GR "for free"
- Preserve link to the thermodynamics, which "lives" in the fibration.

Linking dissipative hydro to turbulence models

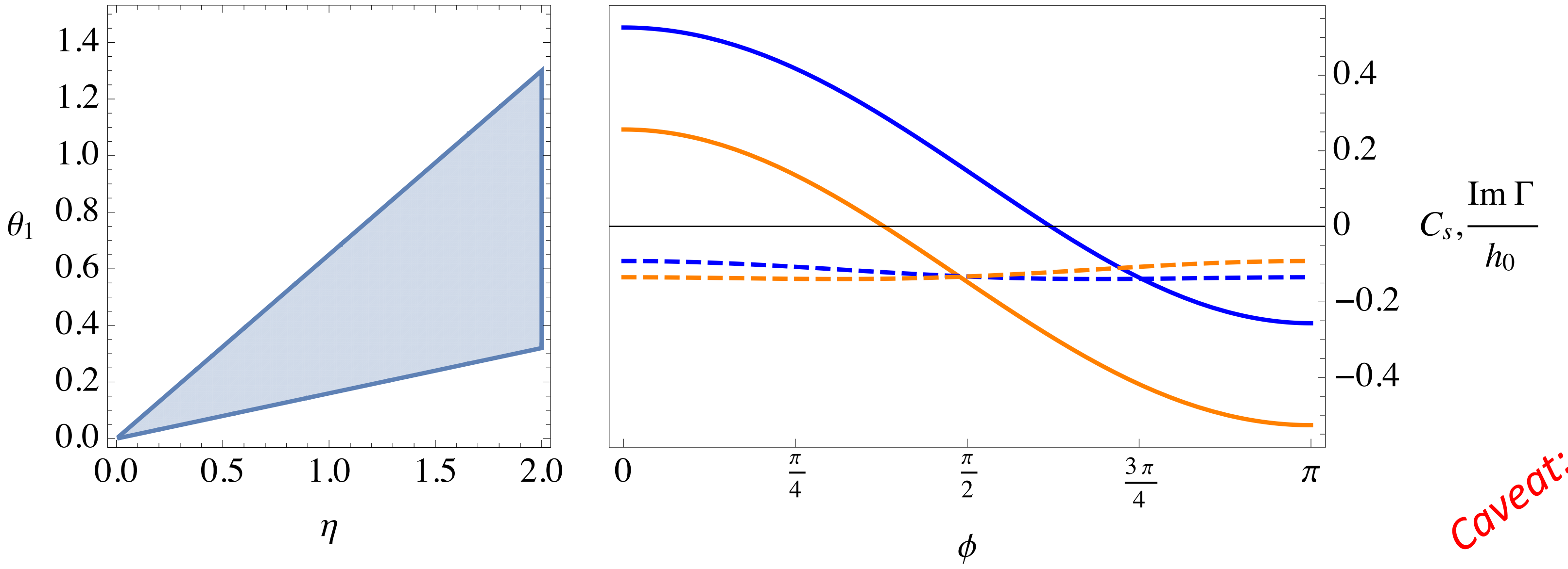
[PRD: 104 084090]

The "classic" Smagorinsky model is based on the assumption that turbulent stresses are *dissipative in the mean*. Effectively this is an Eckart-type model: need to ensure covariant stability in relativity.

$$\nabla_a \tilde{n} = \perp_a^b \nabla_b \tilde{n} - \tilde{u}_a \dot{\tilde{n}}$$
$$\nabla_a \tilde{\varepsilon} = \perp_a^b \nabla_b \tilde{\varepsilon} - \tilde{u}_a \dot{\tilde{\varepsilon}}$$
$$\nabla_a \tilde{u}_b = -\tilde{u}_a \tilde{a}_b + \tilde{\omega}_{ab} + \tilde{\sigma}_{ab} + \frac{1}{3}\tilde{\theta} \perp_{ab}$$

$\xrightarrow{\textit{BDNK-type modelling}}$

$$\tilde{\pi}^{ab} = \eta \tilde{\sigma}^{ab}$$
$$\tilde{\Pi} = \pi_1 \tilde{\theta} + \pi_2 \dot{\tilde{n}} + \pi_3 \dot{\tilde{\varepsilon}}$$
$$\tilde{q}^a = \theta_1 \tilde{a}^a + \theta_2 \perp^{ab} \nabla_b \tilde{n} + \theta_3 \perp^{ab} \nabla_b \tilde{\varepsilon}$$
$$\tilde{M} = \chi_1 \tilde{\theta} + \chi_2 \dot{\tilde{n}} + \chi_3 \dot{\tilde{\varepsilon}}$$



Caveat: not quite the same!

Validating the strategy: Lagrangian filtering

[PRD: 110 123040]

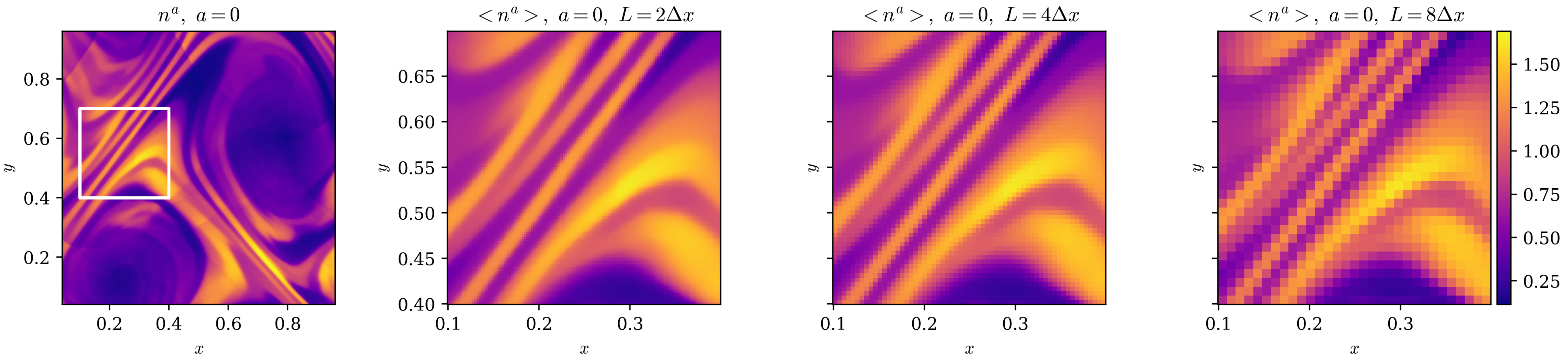
Key steps:

- run SR box simulations of KHI (e.g. METHOD [1])
- build filtering observers: minim. average particle drift
- perform **Lagrangian** filtering (tilted box)

$$E^a_{(1)} = e^a_{(x)} + U^a U_b e^b_{(x)} \ , \quad E^a_{(1)} E^{(1)}_a = 1$$

$$r_{(I)} = \int_{\mathcal{V}_L} E^a_{(I)} n_a \, d\mathcal{V}_L \ , \quad I = 1, 2, 3$$

$$\langle X \rangle = \int_{V_L} X \, dV_L$$



[1] https://github.com/AlexJamesWright/METHOD

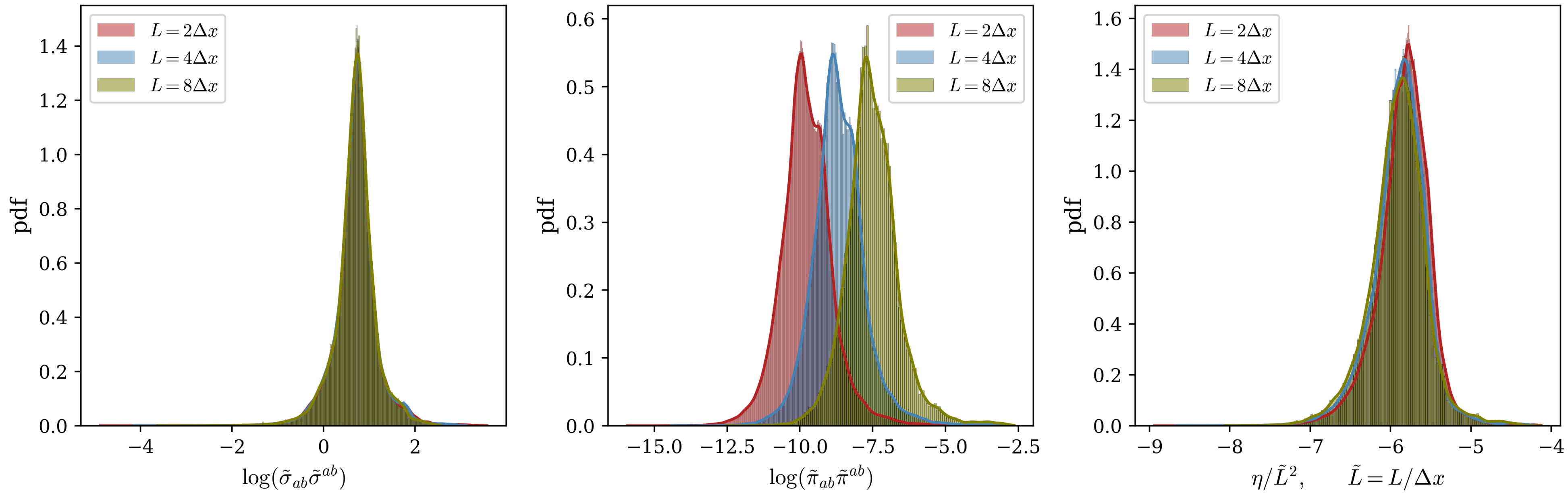
Impact on matter sector: effective dissipative terms

[PRD: 110 123040]

$$\langle T^{ab}\rangle = \underbrace{(\tilde{\varepsilon} + \langle p\rangle)\,\tilde{u}^a\tilde{u}^b + \langle p\rangle g^{ab}}_{\text{ideal terms}} + \overbrace{2\tilde{u}^{(a}\tilde{q}^{b)} + \tilde{s}^{ab}}^{\text{“dissipative” terms}}$$

- Residuals need modelling: closures!
- EoM: “effective” dissipative fluid

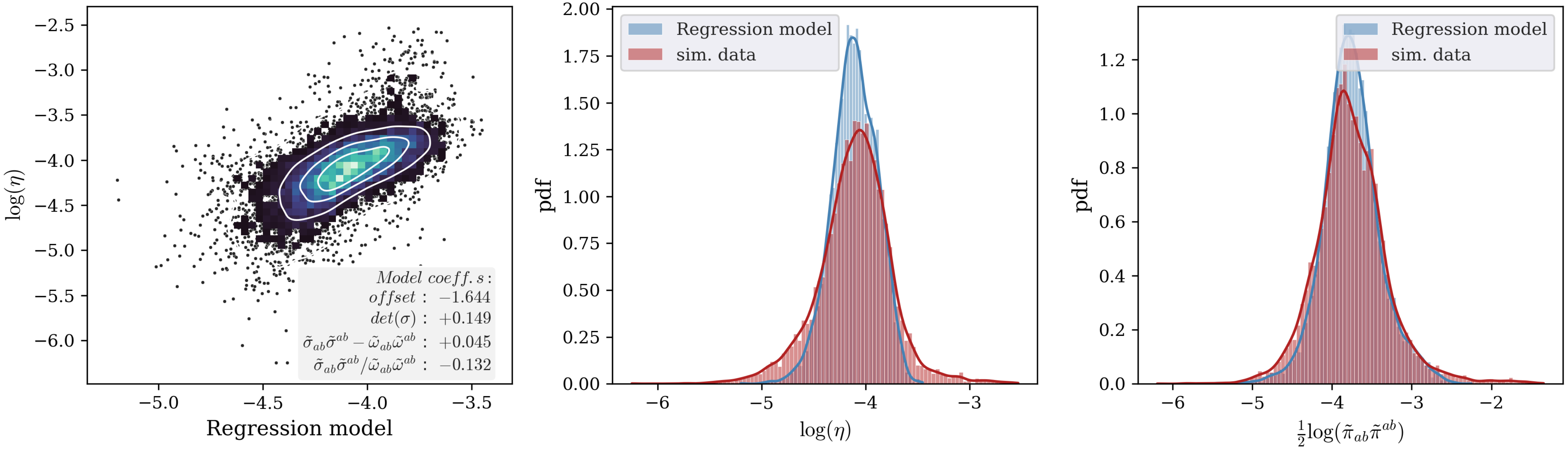
“Residuals” due to non-linearities, capturing the impact of sub-filter fluctuations



A simple linear regression model

[PRD: 110 123040]

- explanatory vars: $\left\{\tilde{T},\, \tilde{n},\, \tilde{\sigma}_{ab}\tilde{\sigma}^{ab},\, \det(\tilde{\sigma}),\, \tilde{\omega}_{ab}\tilde{\omega}^{ab},\, \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} - \tilde{\omega}_{ab}\tilde{\omega}^{ab},\, \tilde{\sigma}_{ab}\tilde{\sigma}^{ab}/\tilde{\omega}_{ab}\tilde{\omega}^{ab}\right\}$
- "Quality factor": $W_1(X,Y) = \sum_i ||X_{(i)} - Y_{(i)}||$



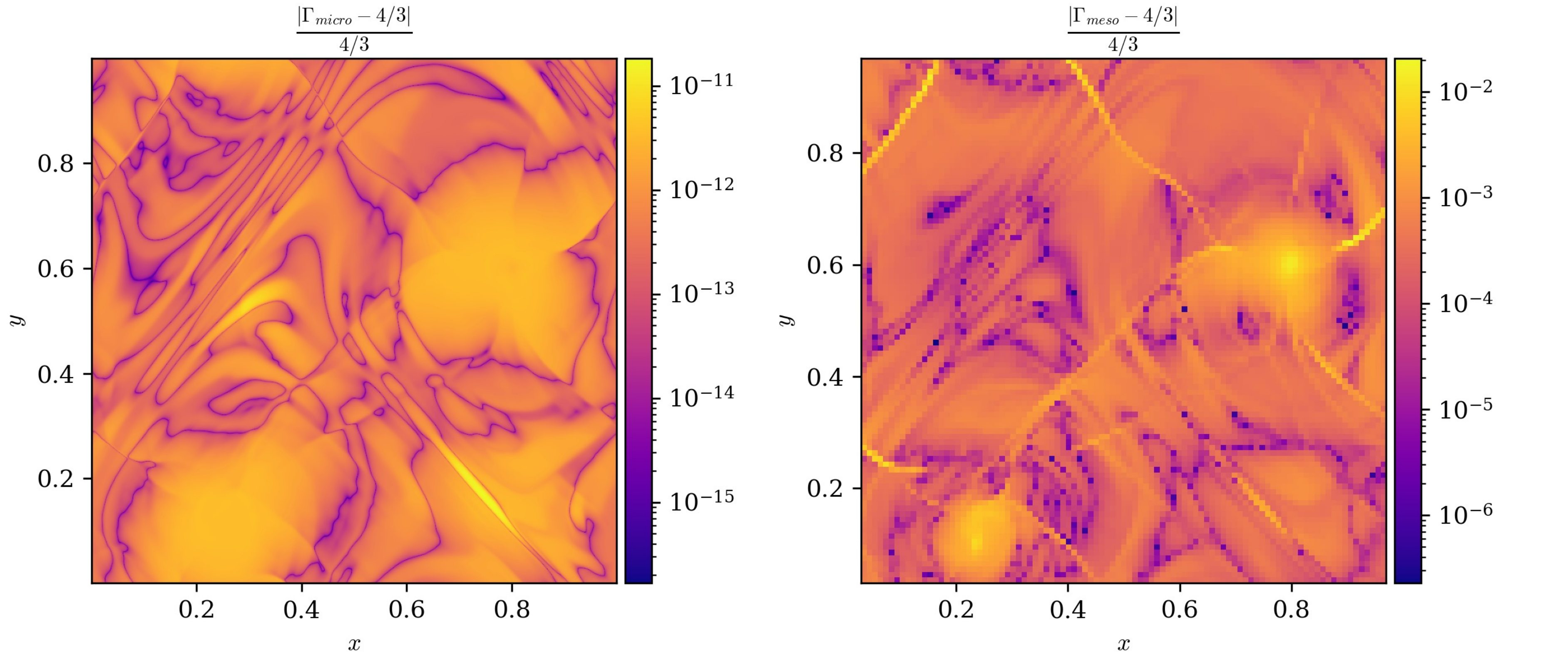
Impact on matter sector: thermodynamics

[PRD: 110 123040]

$$\langle p \rangle = -\tilde{\varepsilon} + \tilde{\mu}\tilde{n} + \tilde{T}\tilde{s} + M$$

- Pressure as a non-linear closure in NR: residuals?
- Neglected so far

Testing the null-hypothesis: what if I ignore the non-linearities in the pressure?



(Caveat: need 3D, realistic EoS, but...)

Recap/conclusions:

- Turbulence develops in mergers, with a quantitative impact on (many aspects of) dynamics
- We need to model it properly given the expected sensitivities of 3G detectors
- Modelling turbulence "requires" LES-type strategies
- Proposed a covariant framework to do so in general relativistic settings
- Practical implementation of the strategy: promising first results on **"a priori"** calibration
- For a discussion about extending the strategy to magnetized fluids, see PRD: 110 123039

Thank you for listening!

Back-up slides

Resolving (or not) the UV limit: bulk viscous case

Writing the equations in non-dimensional form we see that the reaction timescale is decoupled from the rest:

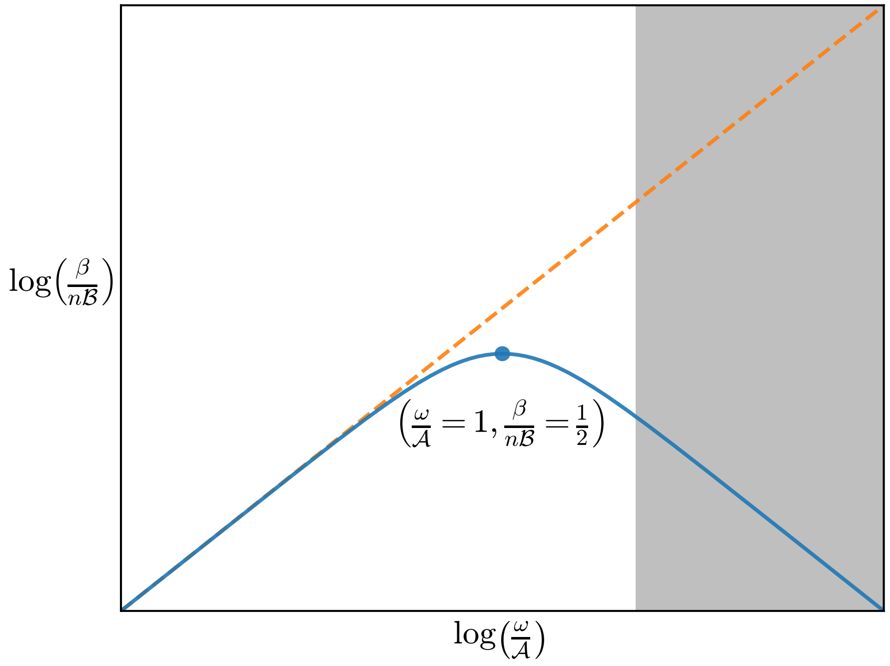
$$\frac{d\varepsilon}{dt} = -\frac{1}{\epsilon_{St}}(\varepsilon + c_r^2 p)\theta$$

$$a_b = -\frac{1}{\epsilon_{St}}\frac{1}{\epsilon_{Ma}^2}\frac{1}{\varepsilon + c_r^2 p}\perp_b^c \nabla_c p$$

$$\frac{dn}{dt} = -\frac{1}{\epsilon_{St}} n\theta$$

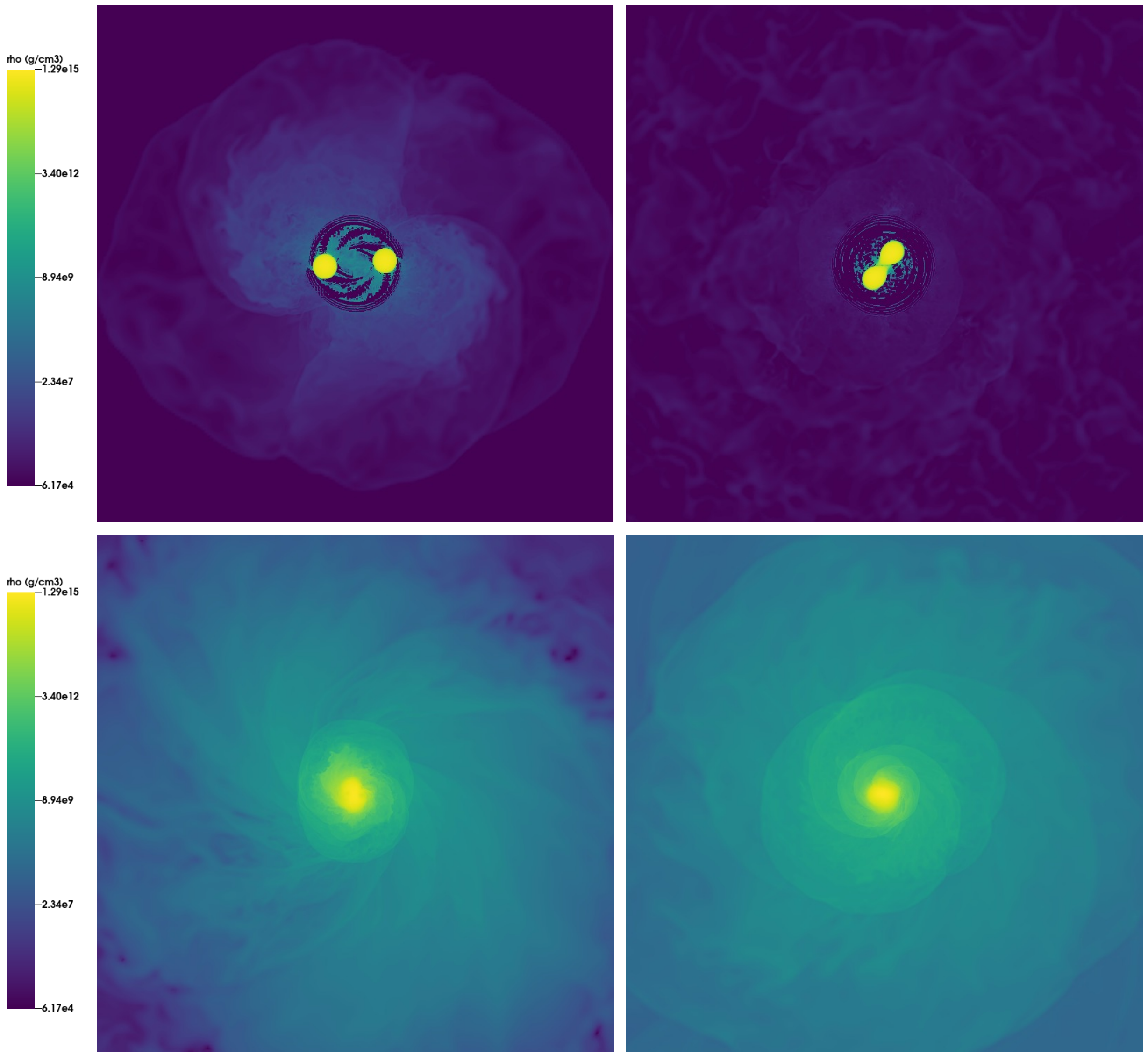
$$\frac{dY_\mathrm{e}}{dt} = -\frac{1}{\epsilon_A}(Y_\mathrm{e} - Y_\mathrm{e}^{\mathrm{eq}})$$

Integrating out the electron fraction via multi-scale methods, we obtain a NS-type bulk-viscous pressure:



Fast with respect to what? Resolving vs not-resolving the UV limit.

Tomography of a BNS merger



<u>The rich physics involved:</u>

1. "Extreme" temperatures and densities

$$T_{\max} \approx 10 \div 100\,\mathrm{MeV}$$
$$\rho_{\max} \approx 6\rho_0 = 1.5 \times 10^{15}\,\mathrm{g\,cm^{-3}}$$

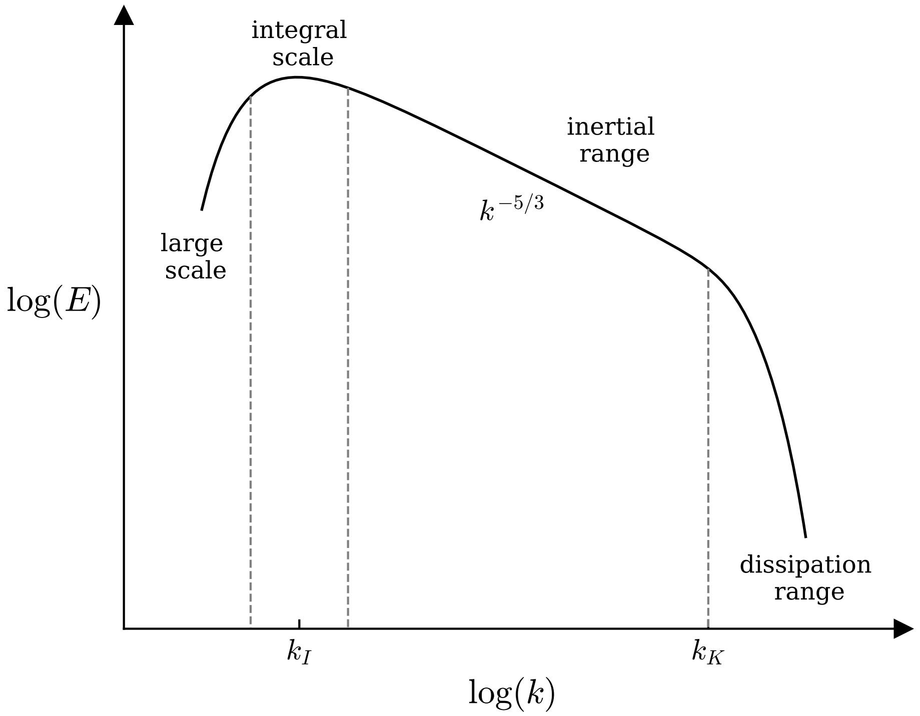
2. Reaction-sourced viscosity (and more...)

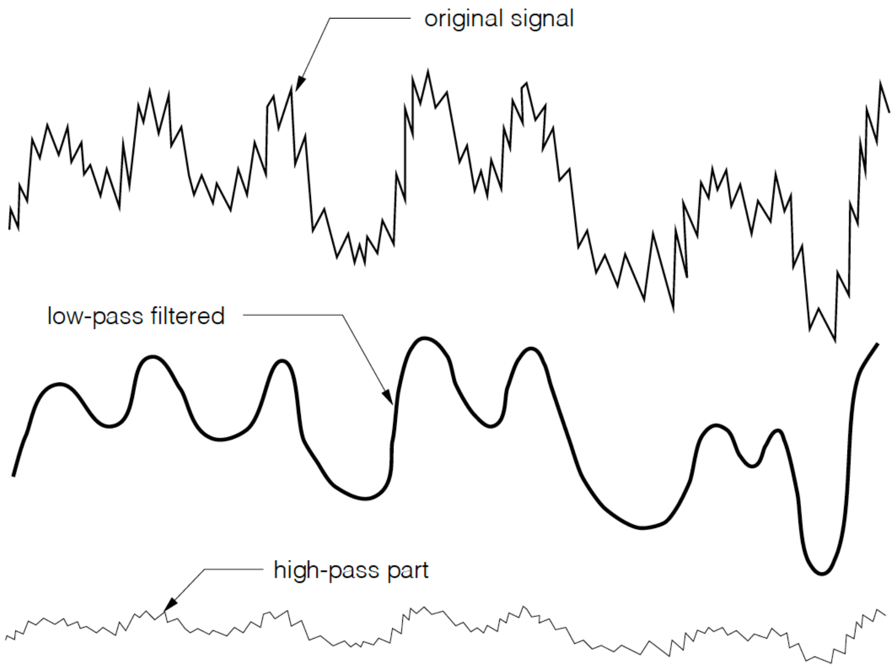
$$n \longleftrightarrow p + e^- + \overline{\nu}_e$$

3. Turbulence develops in the merger remnant
   - Kelvin-Helmholtz instability
   - Magneto-rotational instability (?)

4. Magnetic fields and neutrino radiation transport

LES as a "low-pass filter"





$$A(\mathbf{x}, t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\delta A(\mathbf{x}, t) = \sum_{|\mathbf{k}|\geq k_c} a_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$