Non-modal effects in holographic models

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THE ROYAL SOCIETY

STAG RESEARCH CENTRE

Foundations and Applications
of Relativistic Hydrodynamics
GGI, 16 May 2025

Plan

- Introduction
- Perturbations of Poiseuille flow
- Perturbations of black holes in AdS
- (QNM orthogonality relations)
- Summary

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This talk is about some basic questions regarding perturbations of black holes

Through AdS/CFT: AdS black holes provide real-time, non-perturbative description of a strongly coupled plasma

Focus on linear perturbations around thermal equilibrium

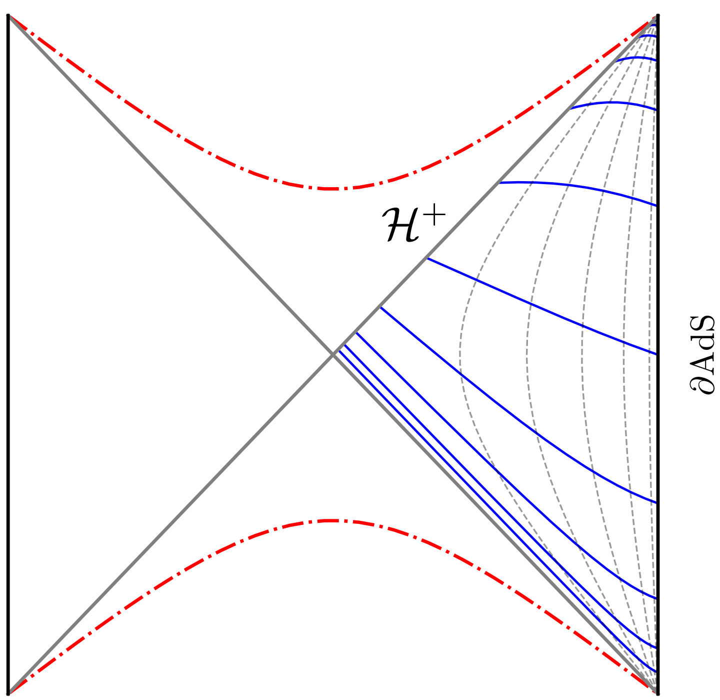
Often characterised by eigenfunctions of the Hamiltonian $\mathcal{H} = i\partial_t$

$$(\sim e^{-i\omega t})$$

$$\omega(k) = c_s k + \ldots$$ sound

$$\omega(k) = -iDk^2 + \ldots$$ shear

$$\omega(k) = -i\Gamma + \ldots$$ non-hydrodynamic



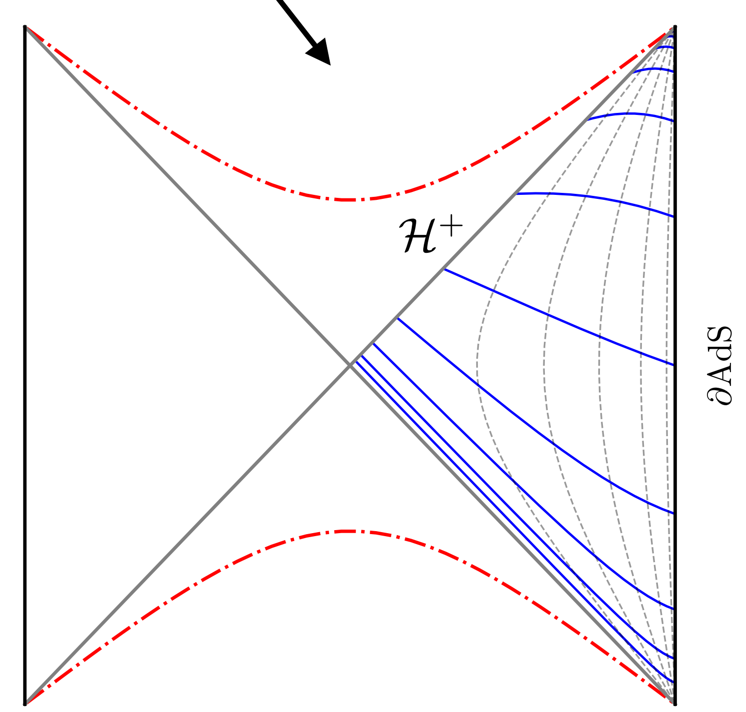
These are **quasinormal modes** (QNMs) of the black hole

- Ingoing at the horizon (regularity)
- Normalisable at the AdS boundary

For black holes $\mathcal{H} = i\partial_t$ is not Hermitian

This arises due to dissipation through the horizon

$\omega \in \mathbb{C}$



$\langle \cdot, \cdot \rangle$

$\langle a, \mathcal{H} b \rangle = \langle \mathcal{H} a, b \rangle + \text{fluxes through } \mathcal{H}^+$

So, for black holes $\mathcal{H} \neq \mathcal{H}^\dagger$

Moreover the Hamiltonian is **non-normal**: $\left[\mathcal{H}, \mathcal{H}^\dagger\right] \neq 0$

This is important because there exists a complete, orthonormal basis made of eigenfunctions iff $[\mathcal{H}, \mathcal{H}^\dagger] = 0$ (spectral theorem)

- QNMs are generically not orthogonal to each other
- Leads to transient effects that a spectral analysis misses

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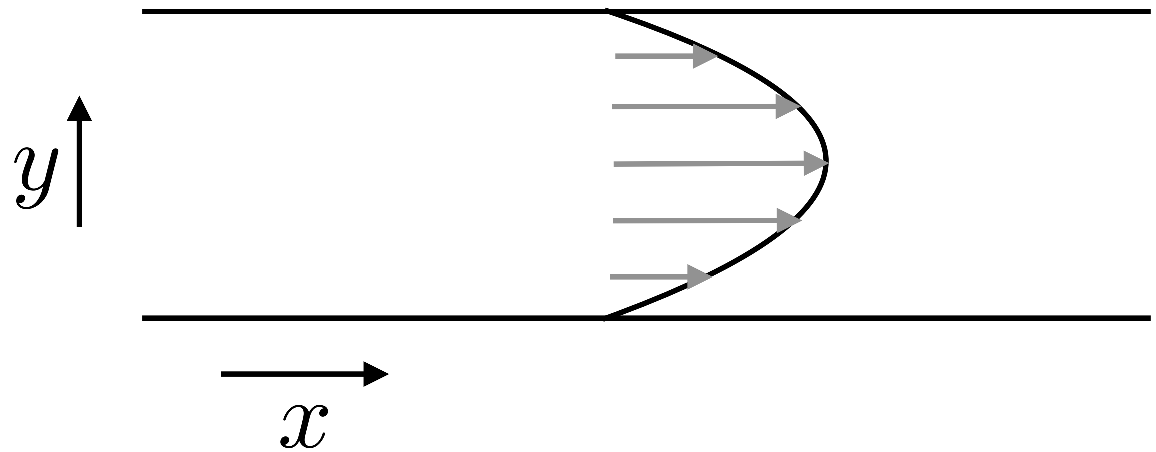
Plane Poiseuille flow



$$\vec{u} = (1 - y^2, 0, 0)$$

Linear perturbations

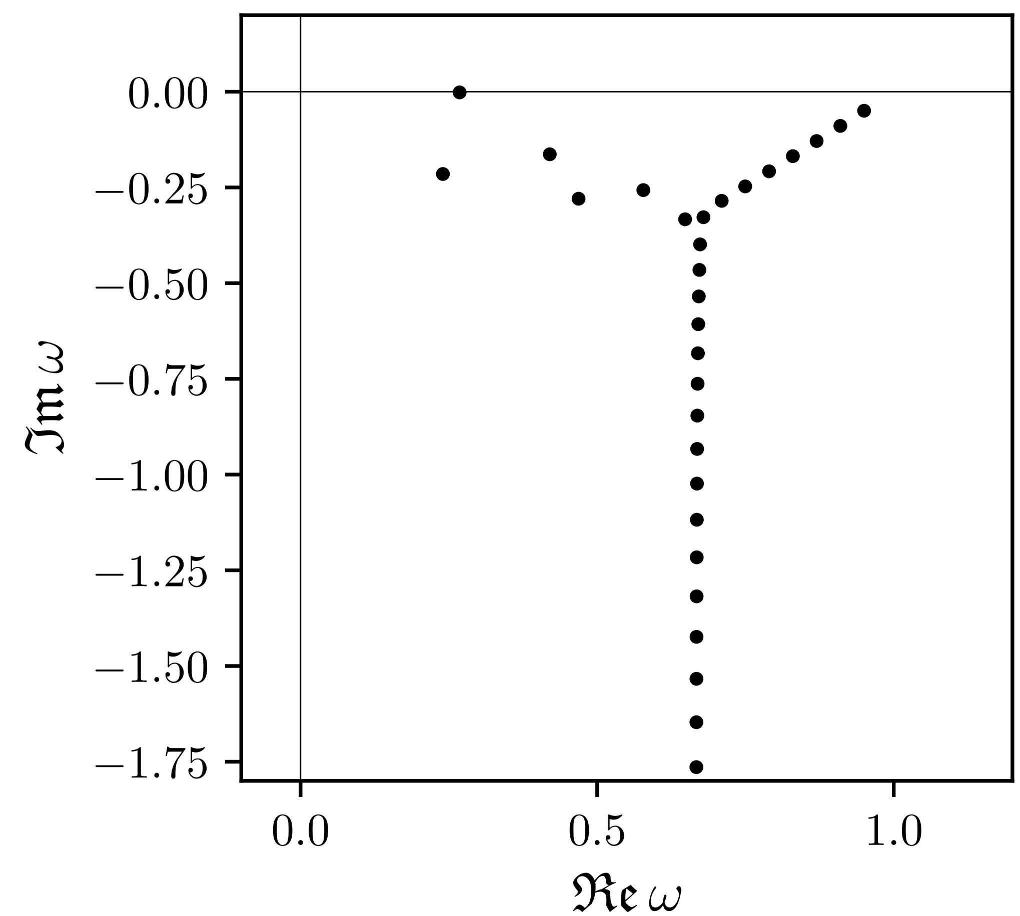
$$\vec{u} = \left(1 - y^2 + \partial_y \Phi, -\partial_x \Phi, 0\right)$$

Eigenvalue analysis

$$\Phi = \phi(y) e^{-i\omega t + i\alpha x}$$

$$\alpha \, O_{\mathrm{OS}} \, \phi(y) = \omega \phi(y)$$

Spectrum at Re = 5000



Critical Re = 5772.22 *[Orszag] (1971)*

Evolution of linear perturbations

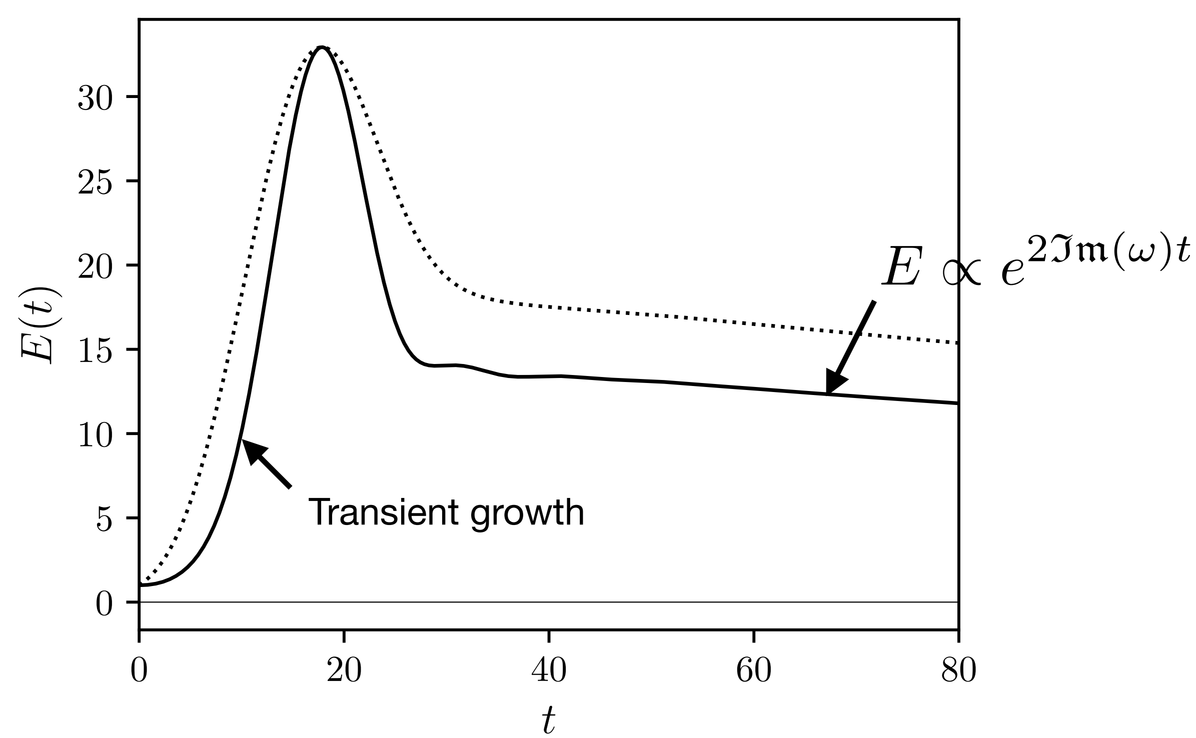
Kinetic energy $$E = \int_{-1}^{1} \left(|\partial_y \phi|^2 + \alpha^2 |\phi|^2 \right) dy$$

For a single mode:

$$E \propto e^{2\mathfrak{Im}(\omega)t}$$

$$\begin{cases} \mathrm{Re} < \mathrm{Re}_c & \text{decay} \\ \mathrm{Re} > \mathrm{Re}_c & \text{growth} \end{cases}$$

For a sum of modes at $\mathrm{Re} < \mathrm{Re}_c$:



[Reddy, Schmid, Henningson] (1993)

Why?

$\exists$ natural inner-product associated to E

$$\langle a, b \rangle = \int_{-1}^{1} \left(\partial_y a^* \partial_y b + \alpha^2 a^* b \right) dy$$

s.t. $\quad E[\phi] = \langle \phi, \phi \rangle$

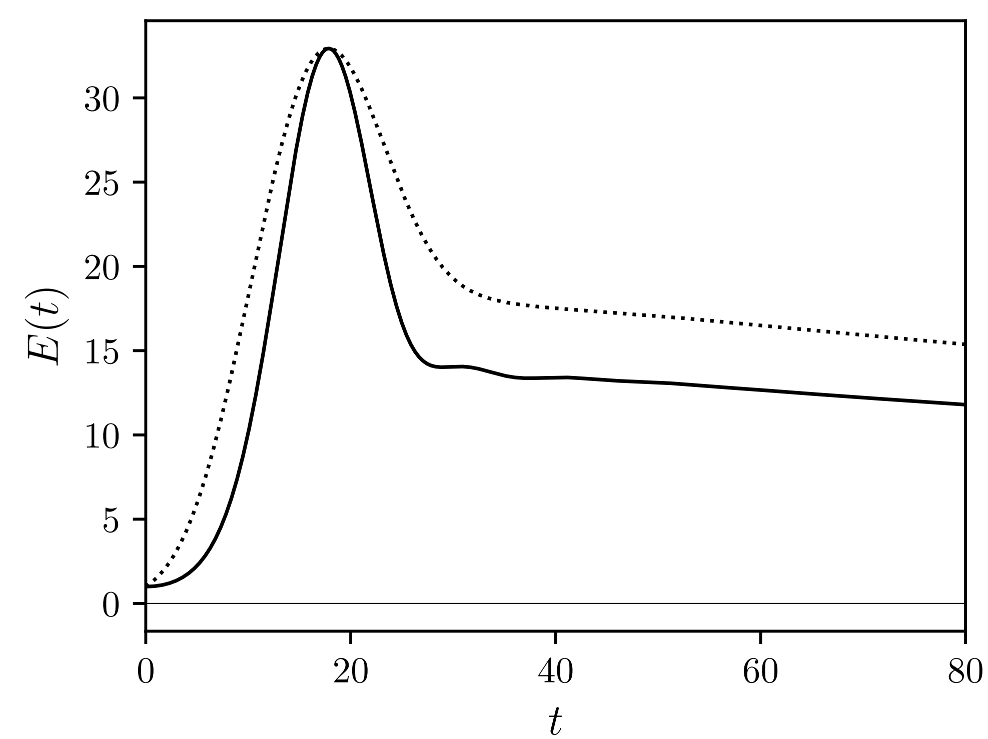
& the Hamiltonian O_{OS} is non-normal wrt to $\langle \cdot, \cdot \rangle$

In particular,

$$\langle \phi_n, \phi_m \rangle \neq 0$$

So that

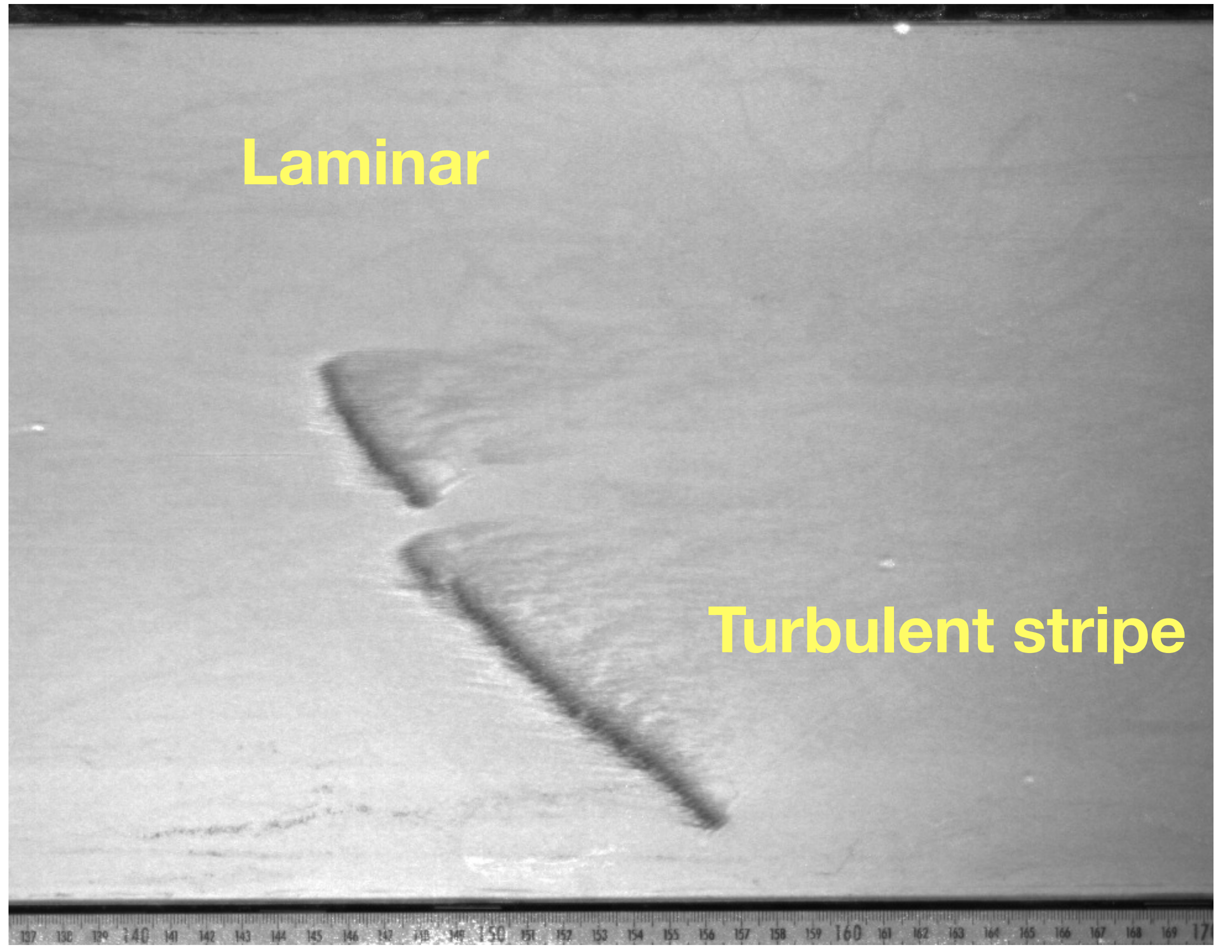
$$E\left[\sum_n c_n \phi_n\right] \neq \sum_n E\left[\phi_n\right]$$



Phenomenological importance

Linear perts ultimately decay

Nonlinearities



$\otimes\, y$

$\xrightarrow{} x$

From 'Onset of turbulence in plane Poiseuille flow'
C Paranjape, PhD Thesis (2019)

~ Sustained nonlinear structures can form,
provided suitably large amplitude initial data

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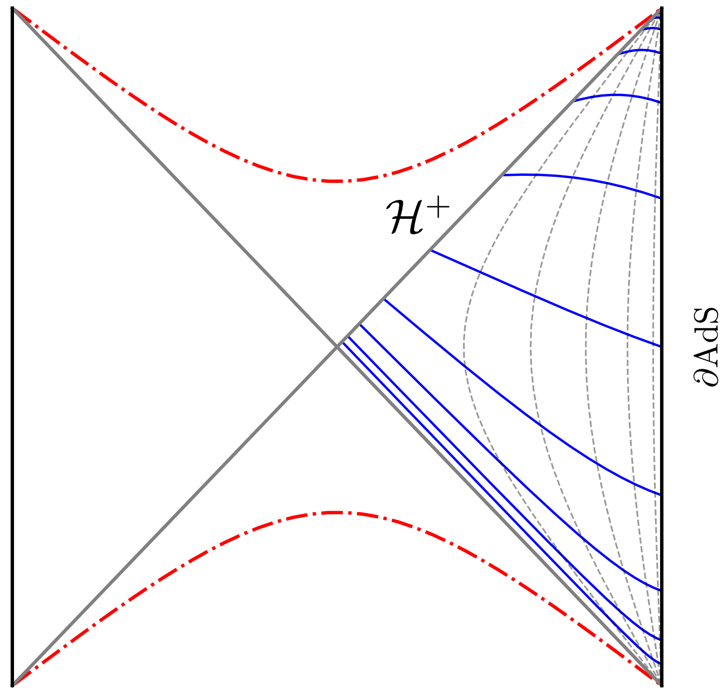


$\leftrightarrow$

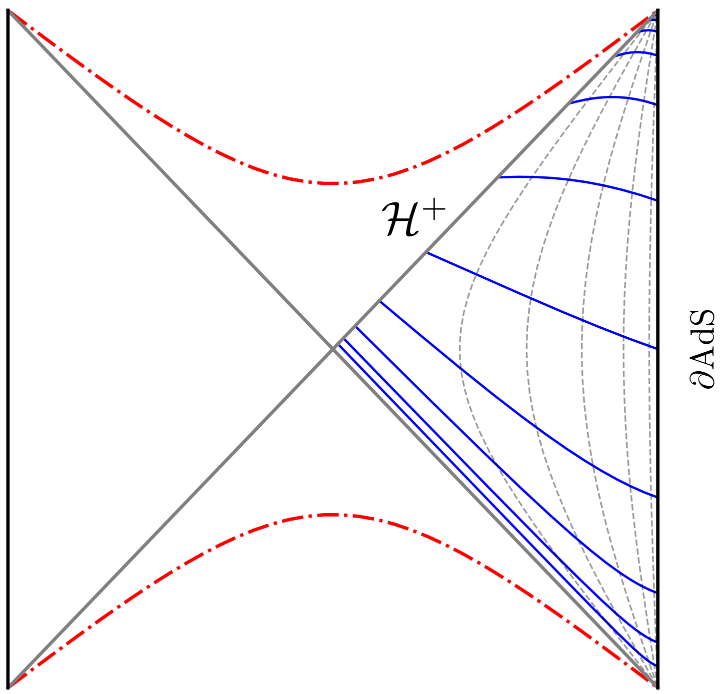
Holographic QFT at finite temperature

$$E_\psi = \int_{\Sigma_\tau} (T_\psi)^\mu{}_\tau n_\mu d\Sigma_\tau$$

$$\langle a, b \rangle = \dots$$

$\mathcal{H}$ is non-normal with respect to $\langle \cdot, \cdot \rangle$

Analytic example (BTZ)



$$\phi_n(\tau, z) = e^{-(2n+\Delta)\tau}(z+1)^{-n-\frac{\Delta}{2}}\,{}_2F_1\left(-n,-n;1-2n-\Delta;1-z^2\right)$$

$$\omega_n = -i(\Delta + 2n),$$

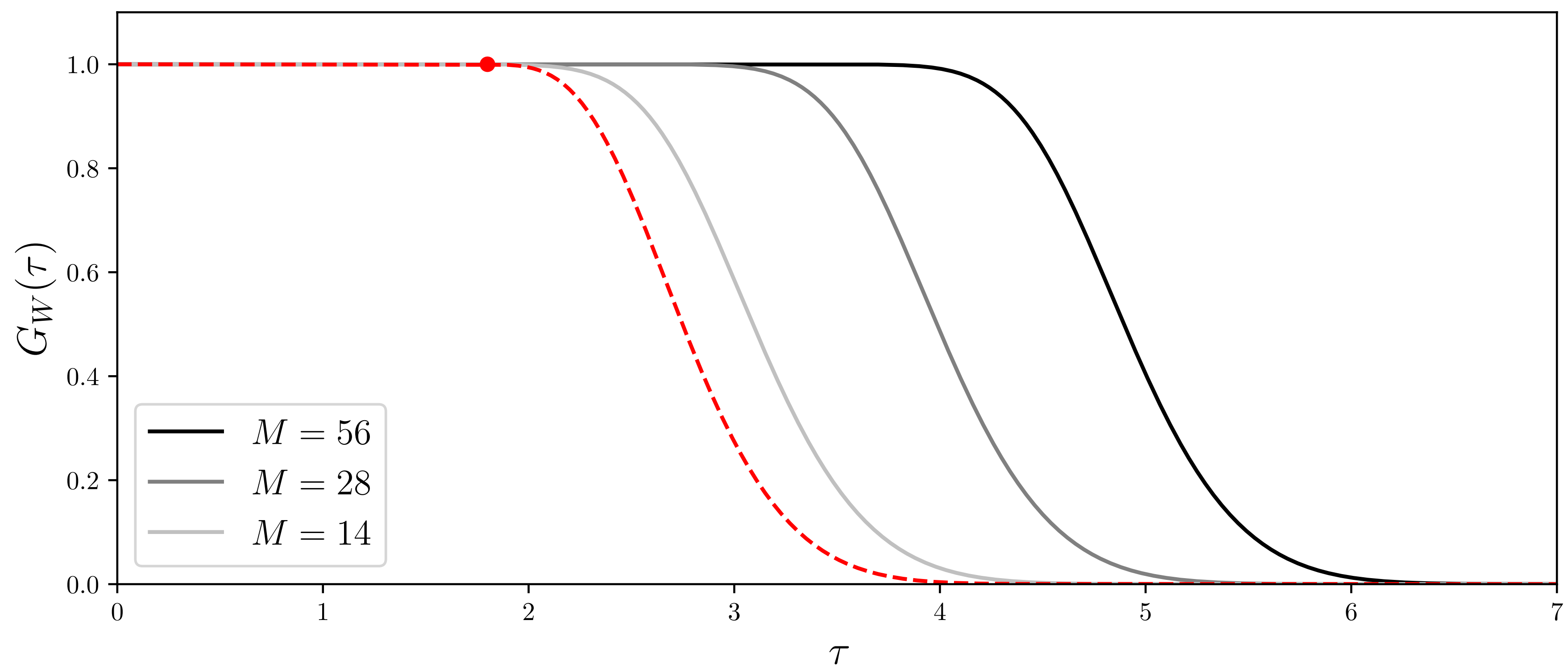
2 modes

$$\phi(\tau, z) = a_1\frac{e^{-2\tau}}{1+z} + a_2\frac{e^{-4\tau}(2+z^2)}{3(1+z)^2},$$

$$E(\tau) = \frac{1}{4}|a_1|^2 e^{-4\tau} + \frac{1}{6}(a_1^* a_2 + a_1 a_2^*)e^{-6\tau} + \frac{1}{8}|a_2|^2 e^{-8\tau}.$$

$$\frac{E(\tau)}{E(0)} = 1 - 24\frac{|a_1+a_2|^2}{6|a_1|^2 + 4(a_1^* a_2 + a_1 a_2^*) + 3|a_2|^2}\tau + O(\tau)^2$$

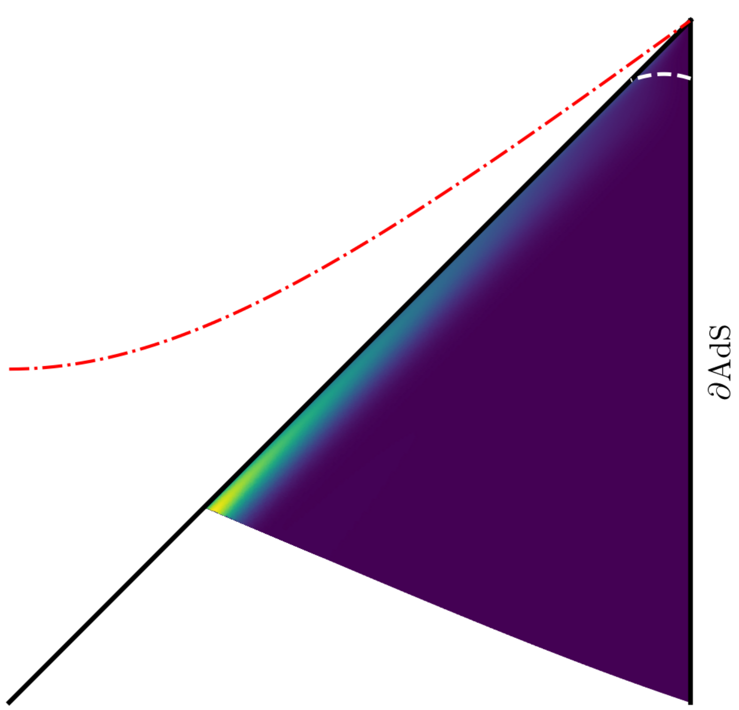
M modes

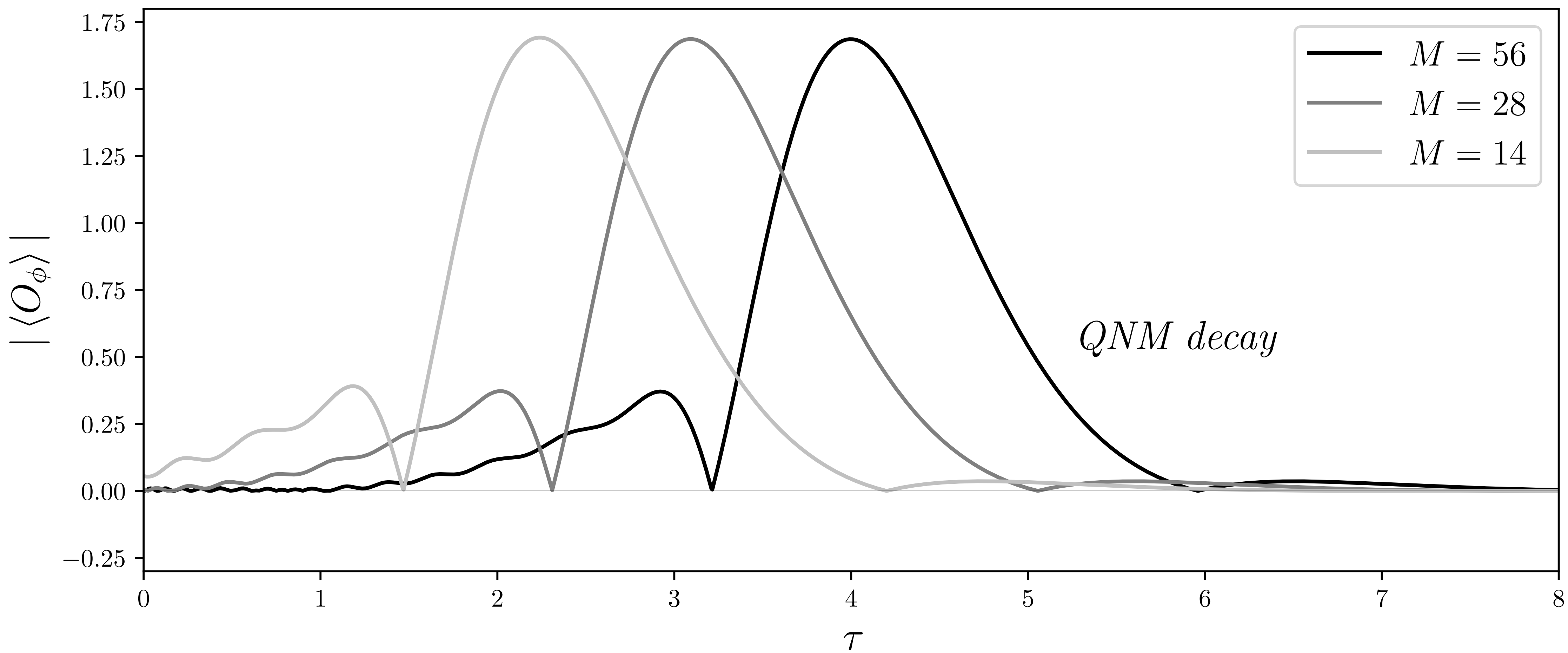


- More modes allows for longer periods of sustained E

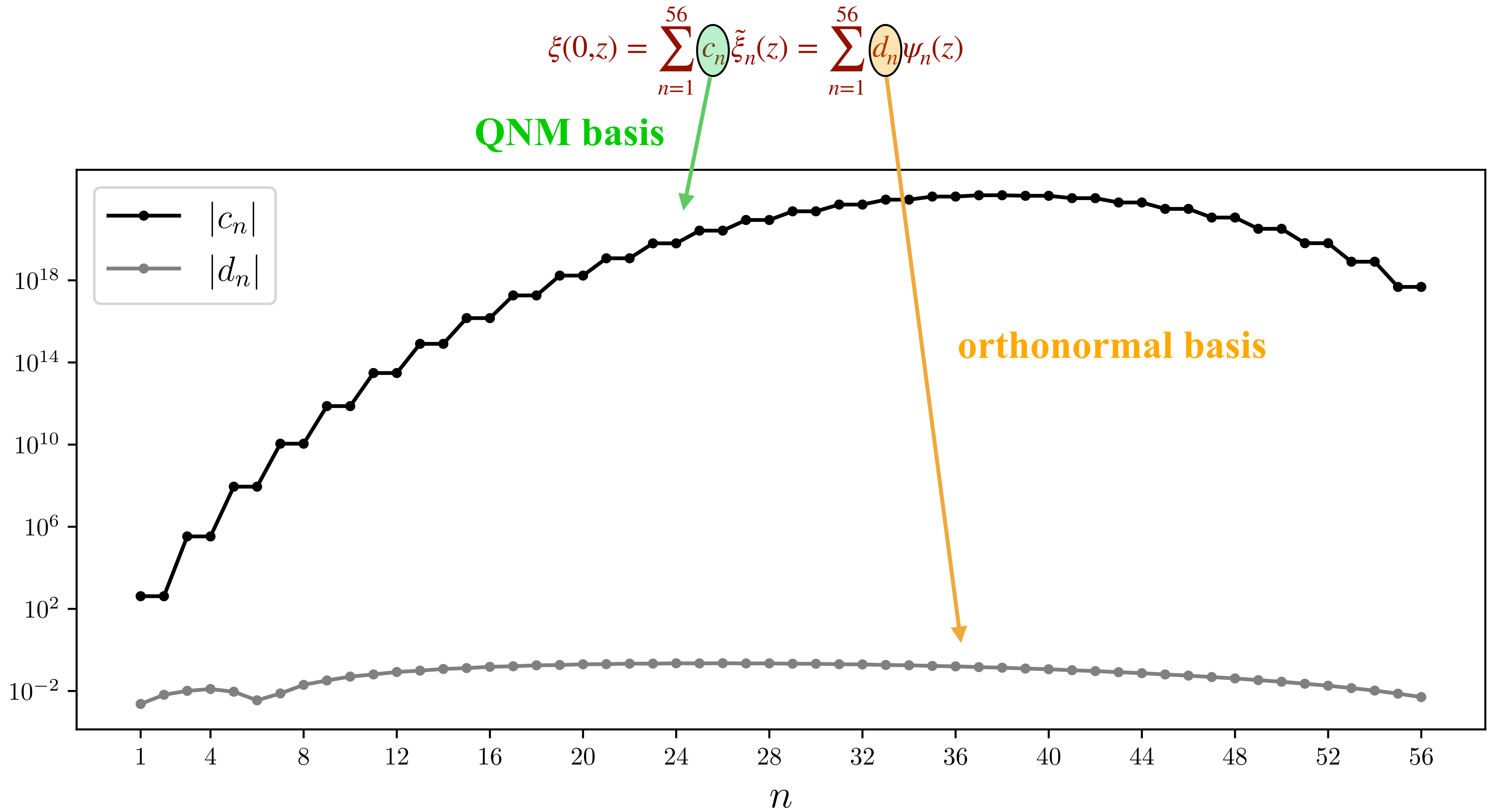
$$\tau \sim \log M$$

- States which take an arbitrarily long time to thermalise
- Non-hydrodynamic in nature
- Holds for Schwarzschild-AdS$_{d+1}$, dS$_{d+1}$, Schwarzschild





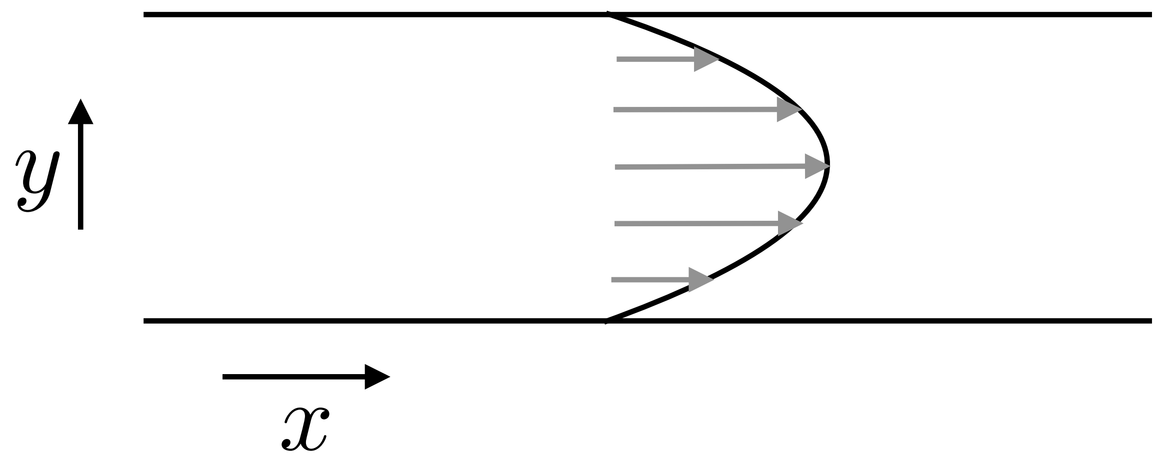
$|\langle O_\phi \rangle|$
τ
$M = 56$
$M = 28$
$M = 14$
QNM decay



$\xi(0,z) = \sum_{n=1}^{56} c_n \tilde{\xi}_n(z) = \sum_{n=1}^{56} d_n \psi_n(z)$
QNM basis
orthonormal basis
$|c_n|$
$|d_n|$
10^{18}
10^{14}
10^{10}
10^{6}
10^{2}
10^{-2}
1 4 8 12 16 20 24 28 32 36 40 44 48 52 56
n

But what about *growth*?

BH radial direction



U(1)

$$\vec{u} = (1 - y^2, 0, 0)$$

Radial electric field

$$\Phi = \phi(y) e^{-i\omega t + i\alpha x}$$

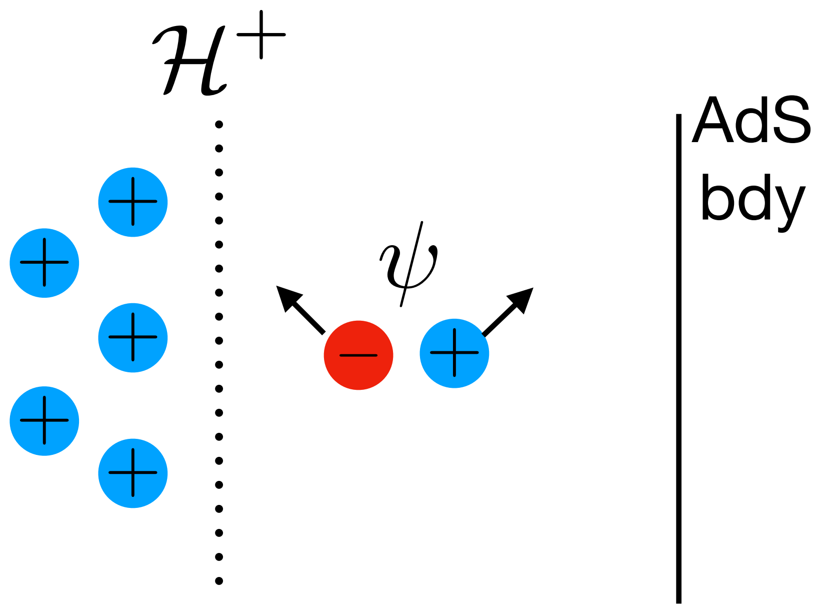
Charged scalar QNMs

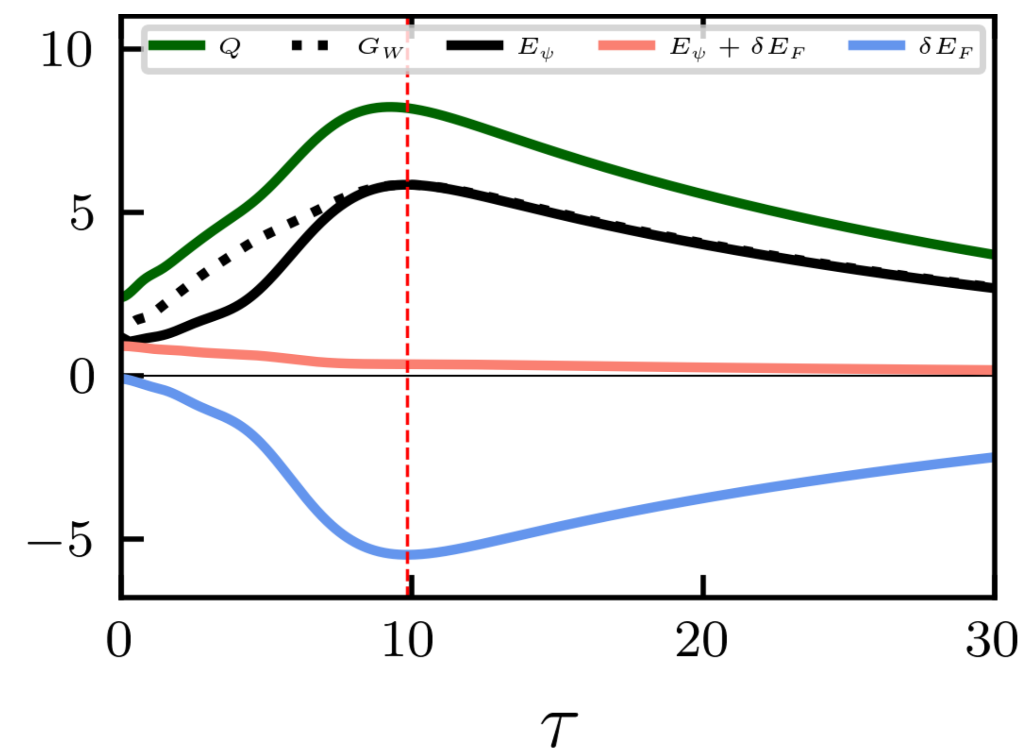
RN-AdS w/ charged scalar $\leftrightarrow$ Holographic superfluid

Energy growth from superradiance

Physical mechanism

Classical wave analogue of pair production



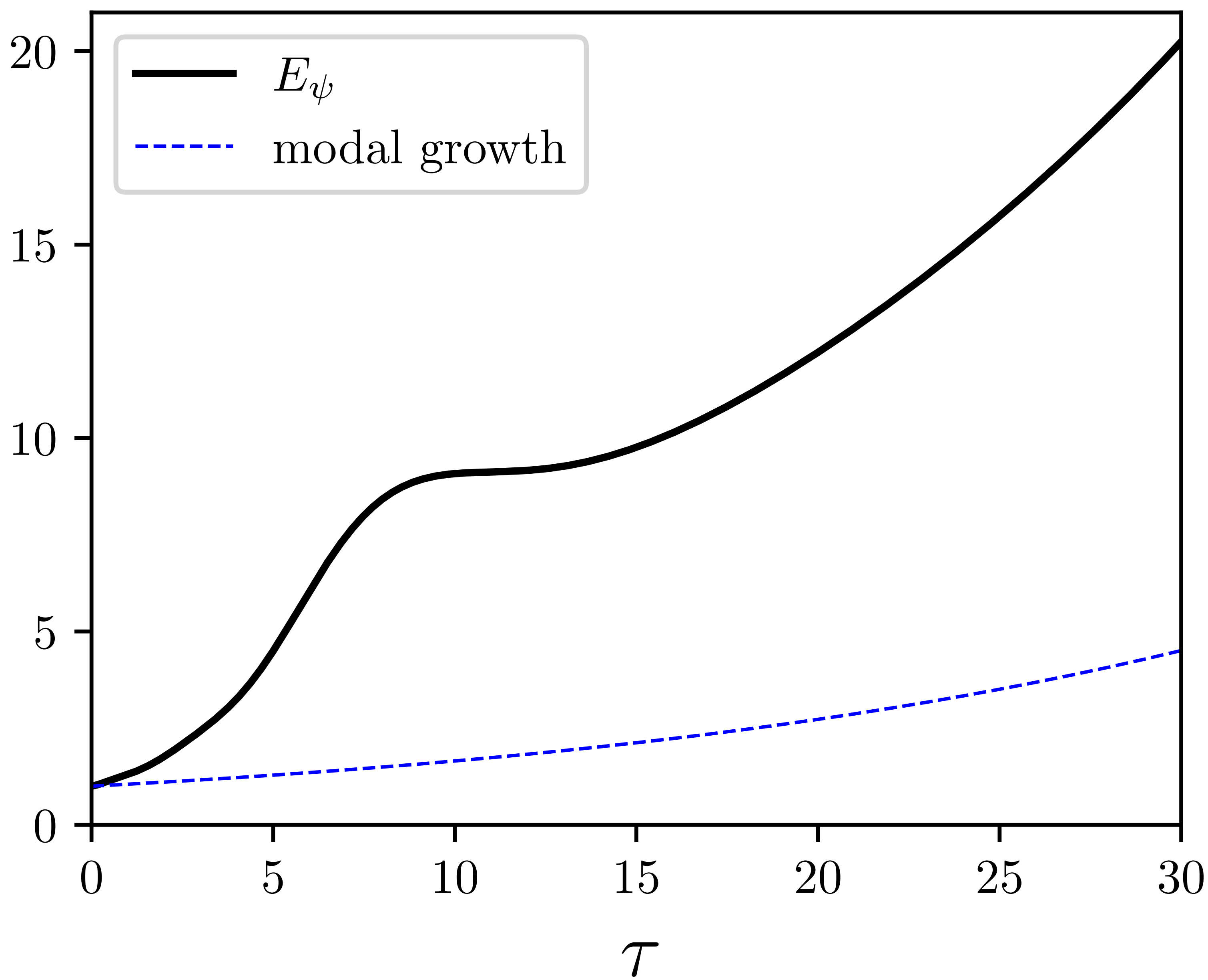


$T < T_c$ Runaway process (to broken phase)

$T > T_c$ Process still occurs, but transiently

"Transient superradiance"

Even when $T < T_c$ the non-modal instability is faster than the QNM



Non-modal Growth

Unstable QNM

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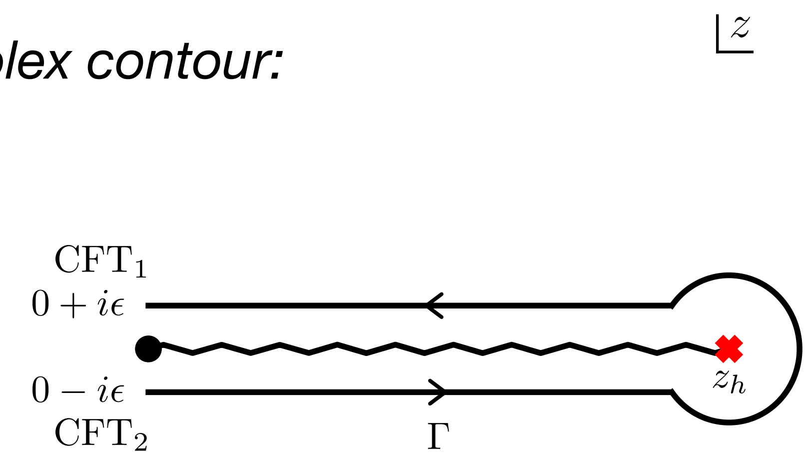
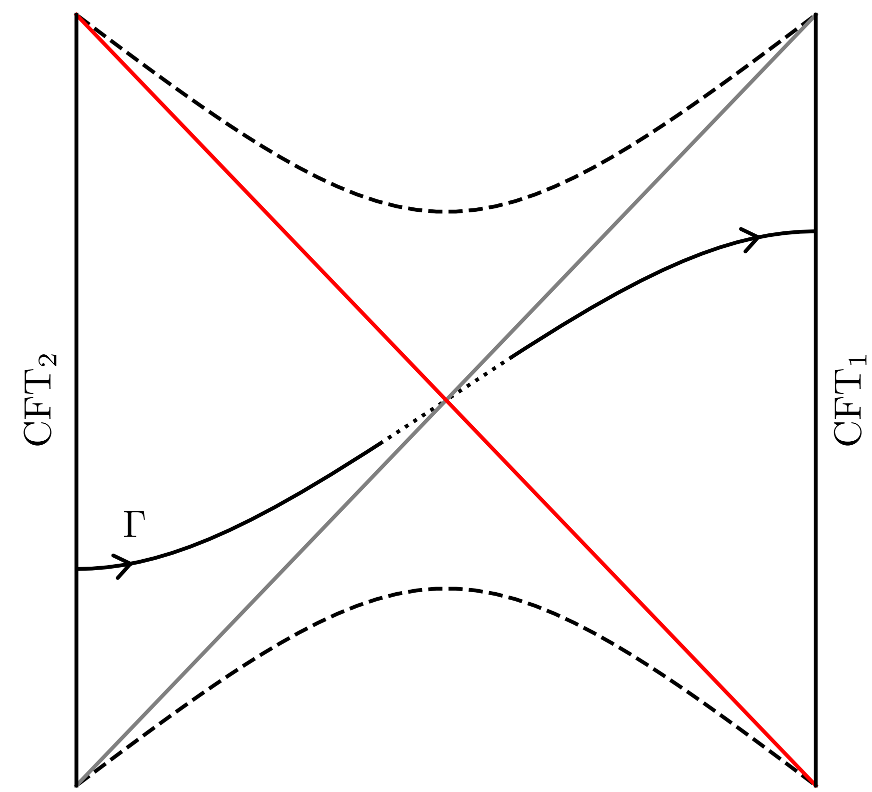
Aside: QNM orthogonality relations

Constructions: de Sitter *[Jafferis, Lupsasca, Lysov, Ng, Strominger] (2013)*
Kerr *[Green, Hollands, Sberna, Toomani, Zimmerman] (2022)*
AdS black holes *[Arnaudo, Carballo, BW] (2025)*

Main ingredients:

- *Complex contour:*





- *Discrete symmetry operators:* $\langle a, b \rangle_{\mathrm{ortho.}} \equiv \langle \mathcal{CPT} a, b \rangle_{\mathrm{KG}}$

Orthogonality… but with respect to a bilinear (not inner product)

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Summary

- In **non-normal** systems, eigenvalue analysis misses transient effects
- Perturbations of BHs are non-normal
- Govern perturbations of strongly coupled plasmas
- We identified transient effects:
    - States which take an arbitrarily long time to thermalise
    - Holographic superconductors are non-modally unstable for $T > T_c$ (just like transition to turbulence for water in a pipe)

Thank you for your attention!