

Advanced School on Precision Cosmology and Dark Matter,
GGI, Florence, 25-29.08.2025

Linear perturbation theory & CMB

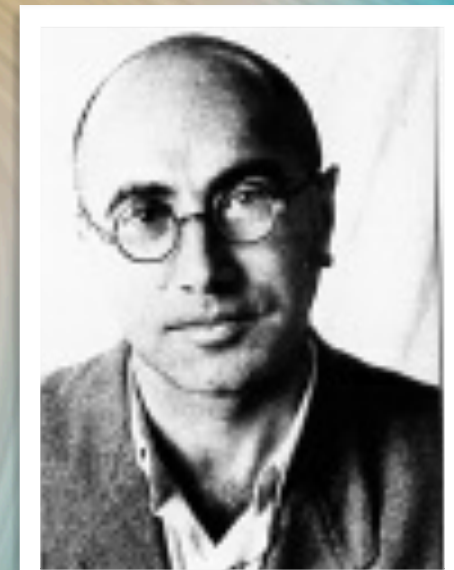
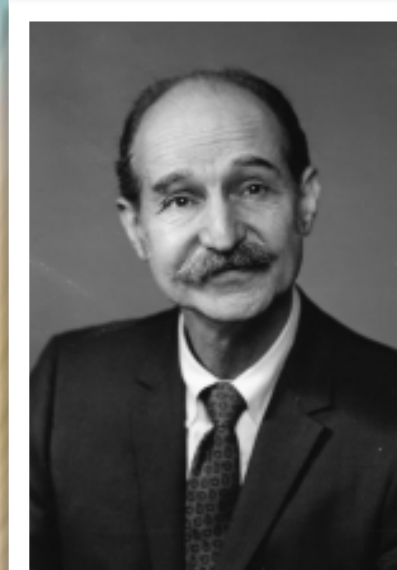
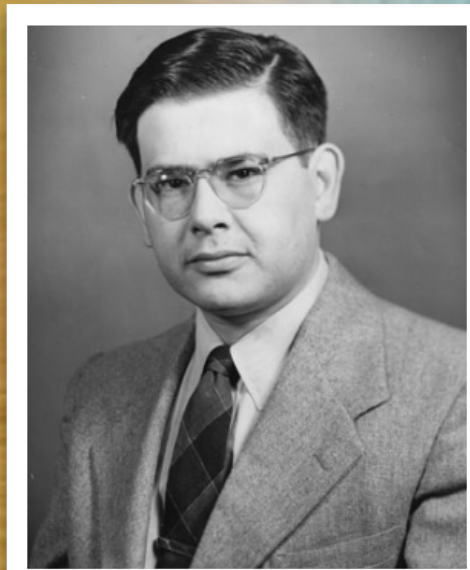
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Cosmic Microwave Background: an early prediction!

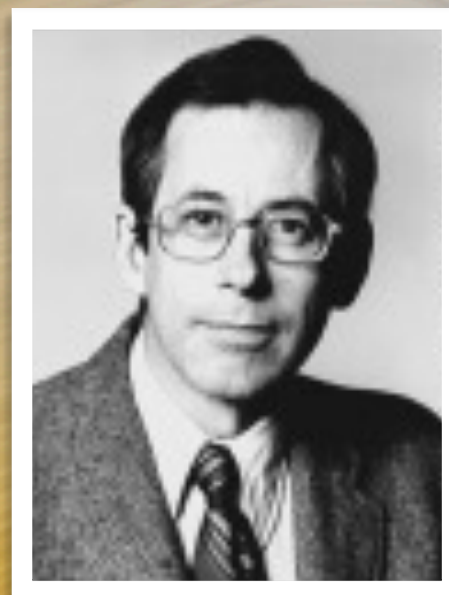
Cosmic photons with blackbody spectrum and $T \sim \mathcal{O}(10)K$ predicted on basis of models of Nucleosynthesis by

Gamow, Alpher, Hermann (40-50's),

Zel'dovitch (60's),

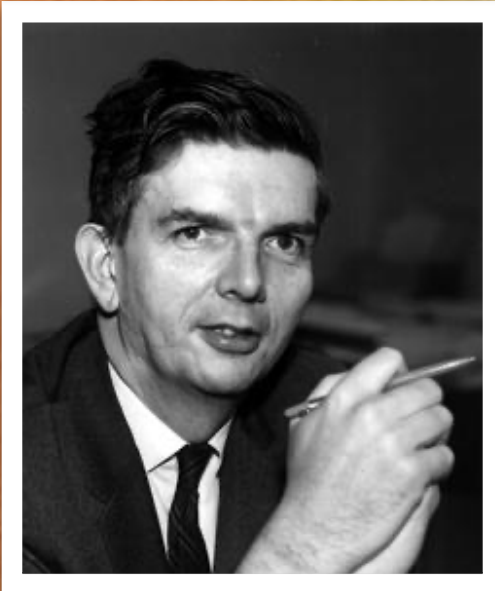


Dicke, Peebles (60's)...



The CMB is no pigeon shit!

Dicke, Peebles (60's)...



While discussing about
dedicated experiment...

... contacted thanks to
luck and coffee room
discussions by...



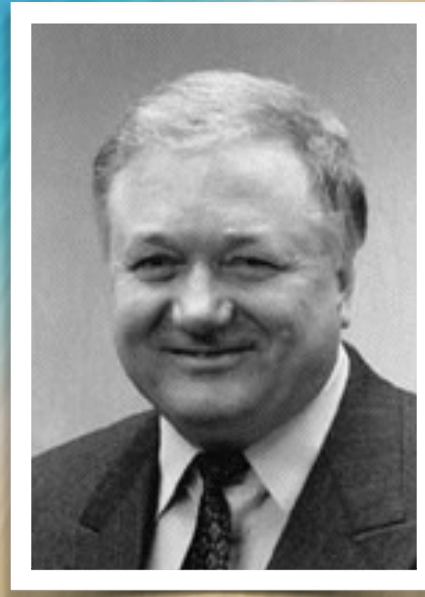
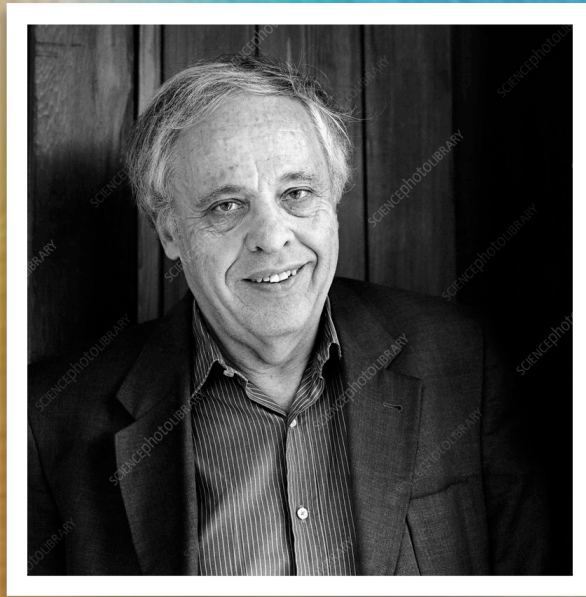
Penzias & Wilson (1964)



Nobel prize Penzias & Wilson (1978)

Prediction of non-trivial correlations in CMB anisotropies!

70-80's : Peebles, Silk, Sunyaev and respective collaborators (70-80's)...



...discuss information
contain in CMB
temperature spectrum and
acoustic oscillations !

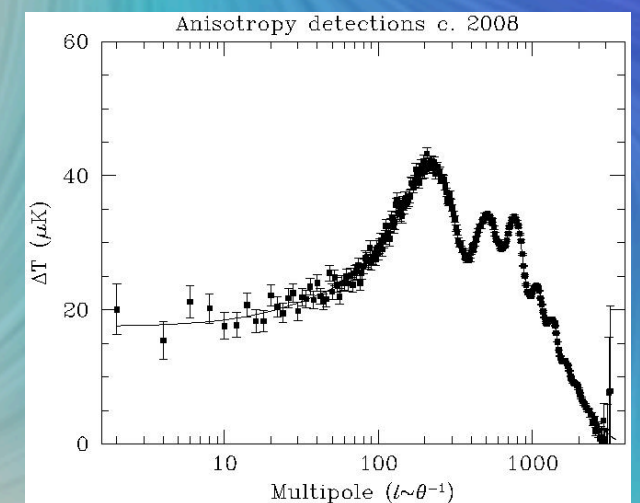
90's: precise prediction of CMB spectra:
Bertschinger, Hu, Kamionkowski, Ma, Seljak,
Sujiyama, White, Zaldarriga + many others...

Confirmation by COBE, Boomerang, WMAP, Planck ...

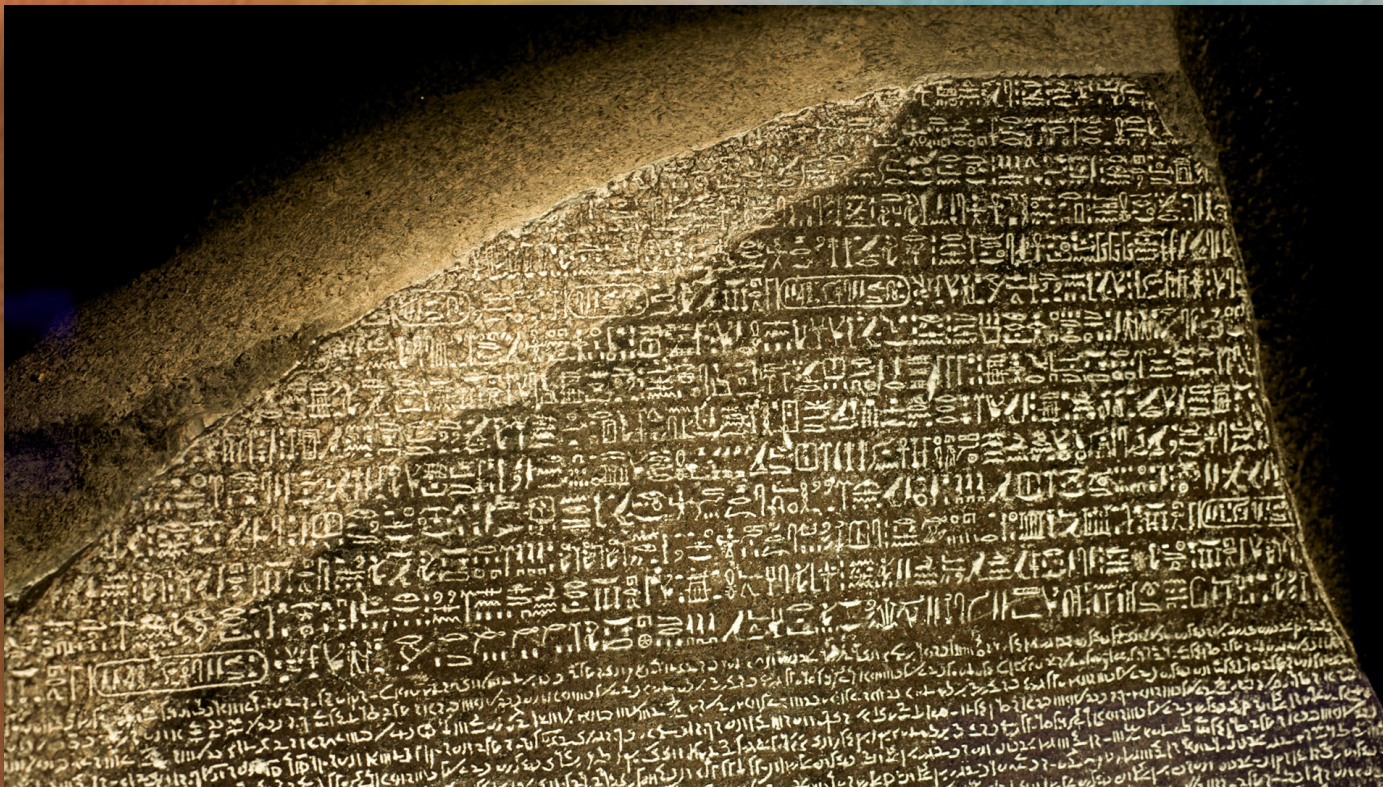


Nobel prize Mather & Smoot 2006

Nobel prize Peebles 2019



The CMB is the Rosetta Stone of cosmology



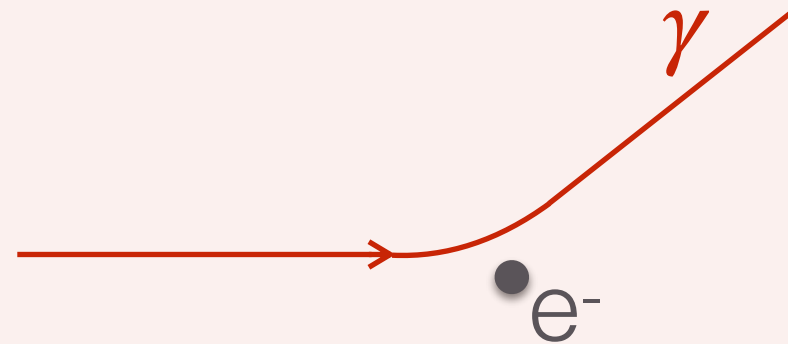
Plan

- Thomson scattering
- Linear theory of stochastic cosmological perturbations
- Spectrum of temperature anisotropies
- Why can we measure independently the Λ CDM parameters ?
- Polarisation
- *Tensors modes*
- *Spectral distortions*
- *CMB lensing*
- *Observations*

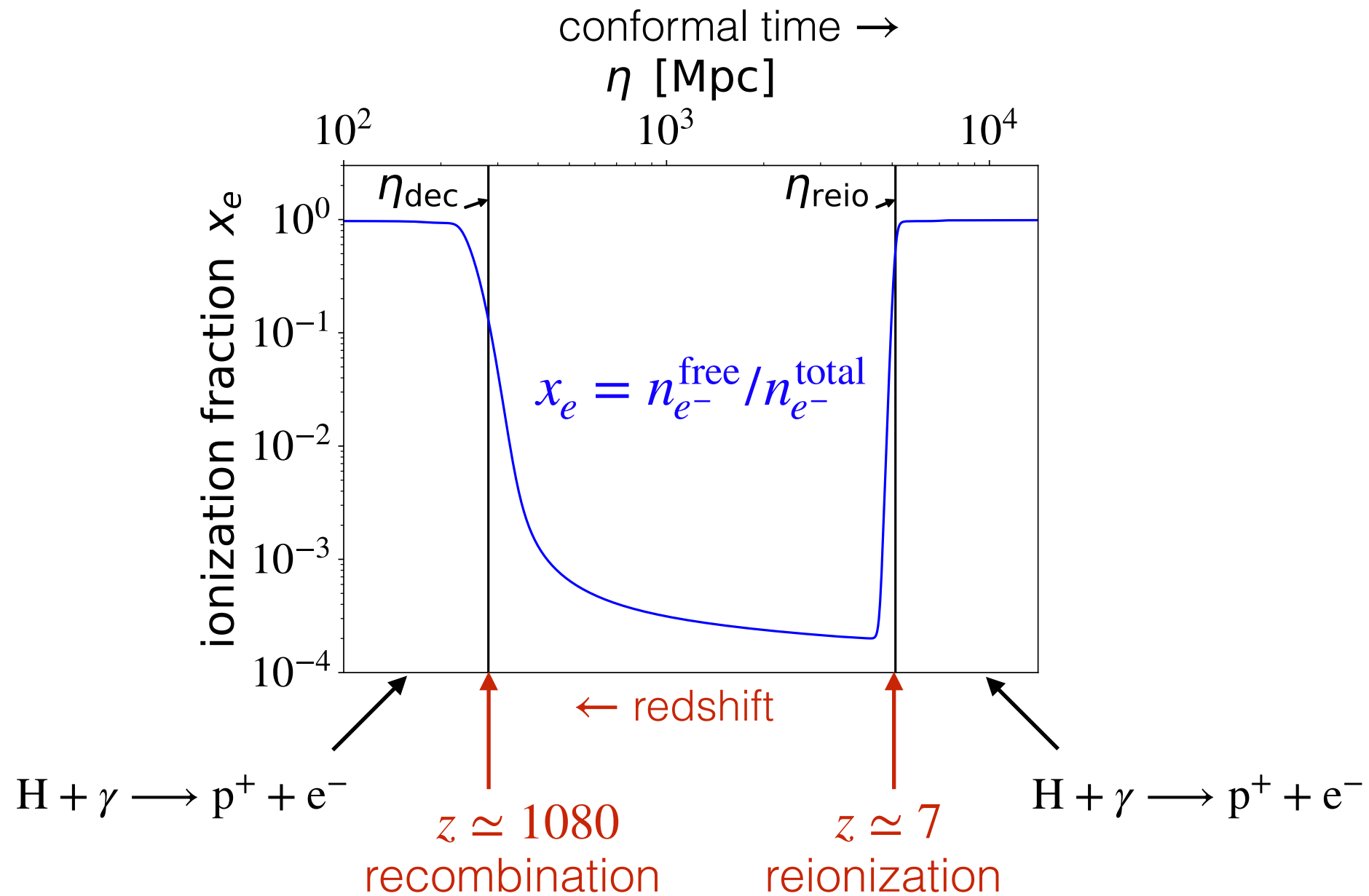
...following:

- The Young Universe: Primordial Cosmology,
edited by R. Taillet (John Wiley & Sons, 2022) ISBN : 1789450322
→ Chapter 2: CMB, by JL
- Chapter 5 of: Neutrino Cosmology, JL et al., CUP 2013
→ Chapter 5: CMB
- The Ingredients of the Universe, Master course at RWTH Aachen U.
Link on Indico

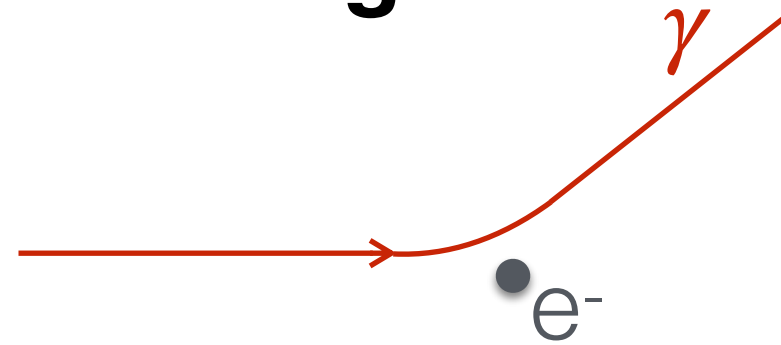
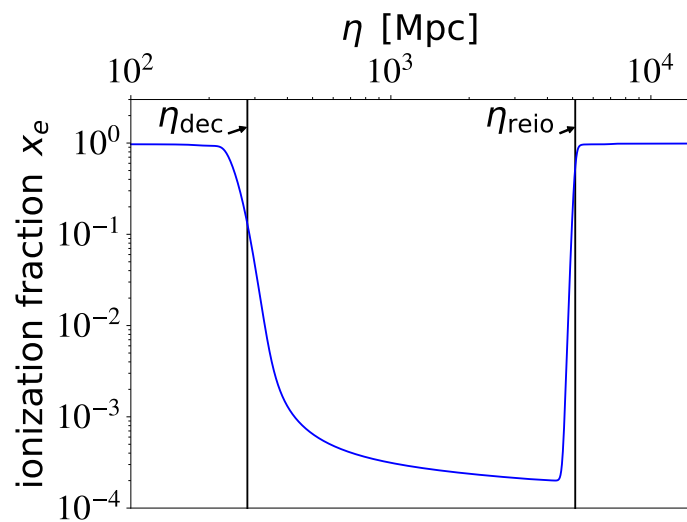
Thomson scattering



Ionisation fraction in the Universe



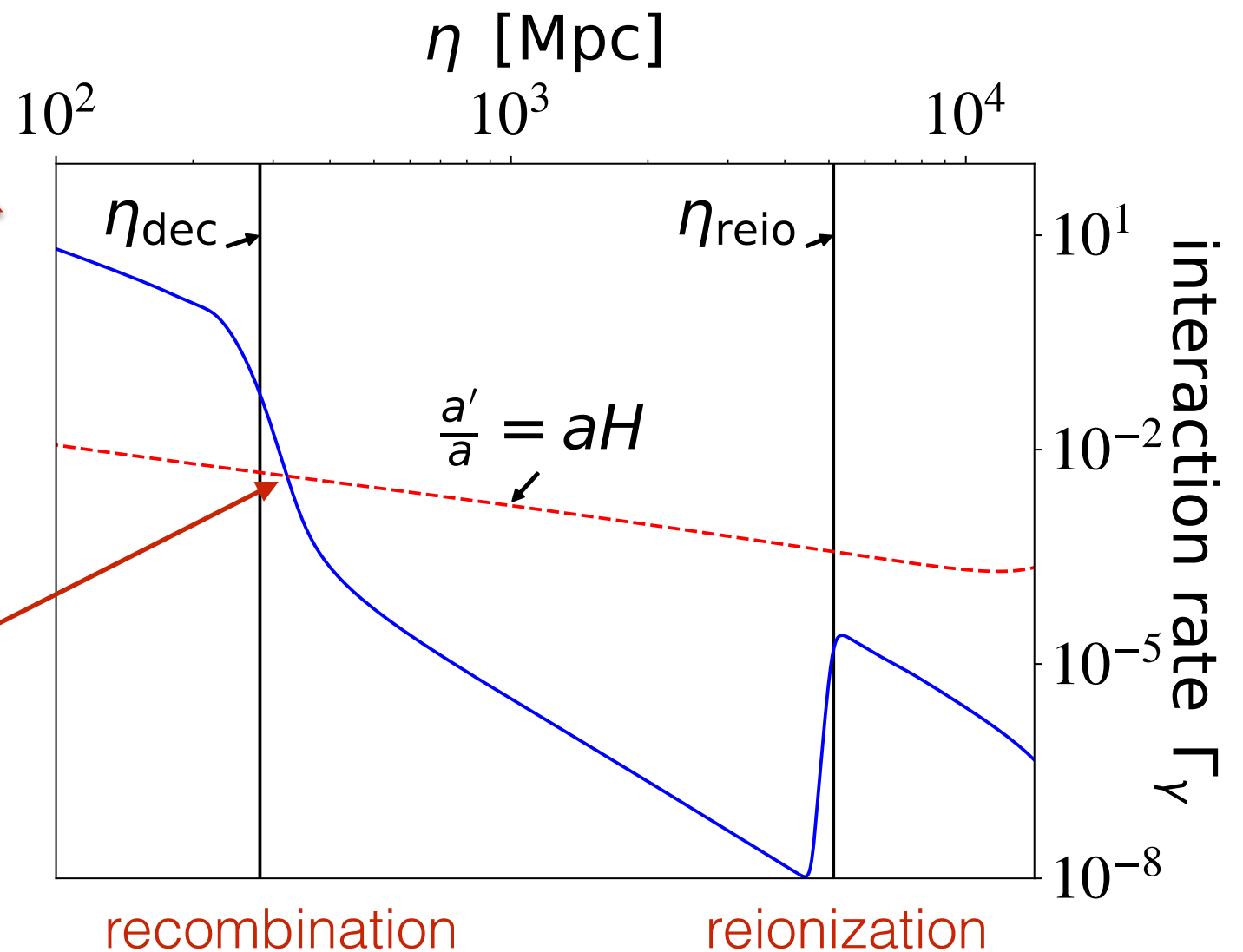
Thomson scattering rate



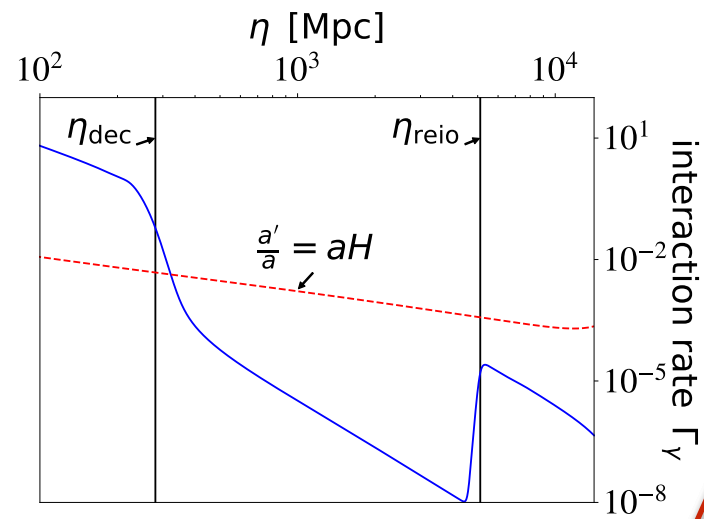
$$\Gamma_\gamma \equiv \frac{dN}{d\eta} = \sigma_T a x_e n_e$$

$$\propto a^{-2} x_e$$

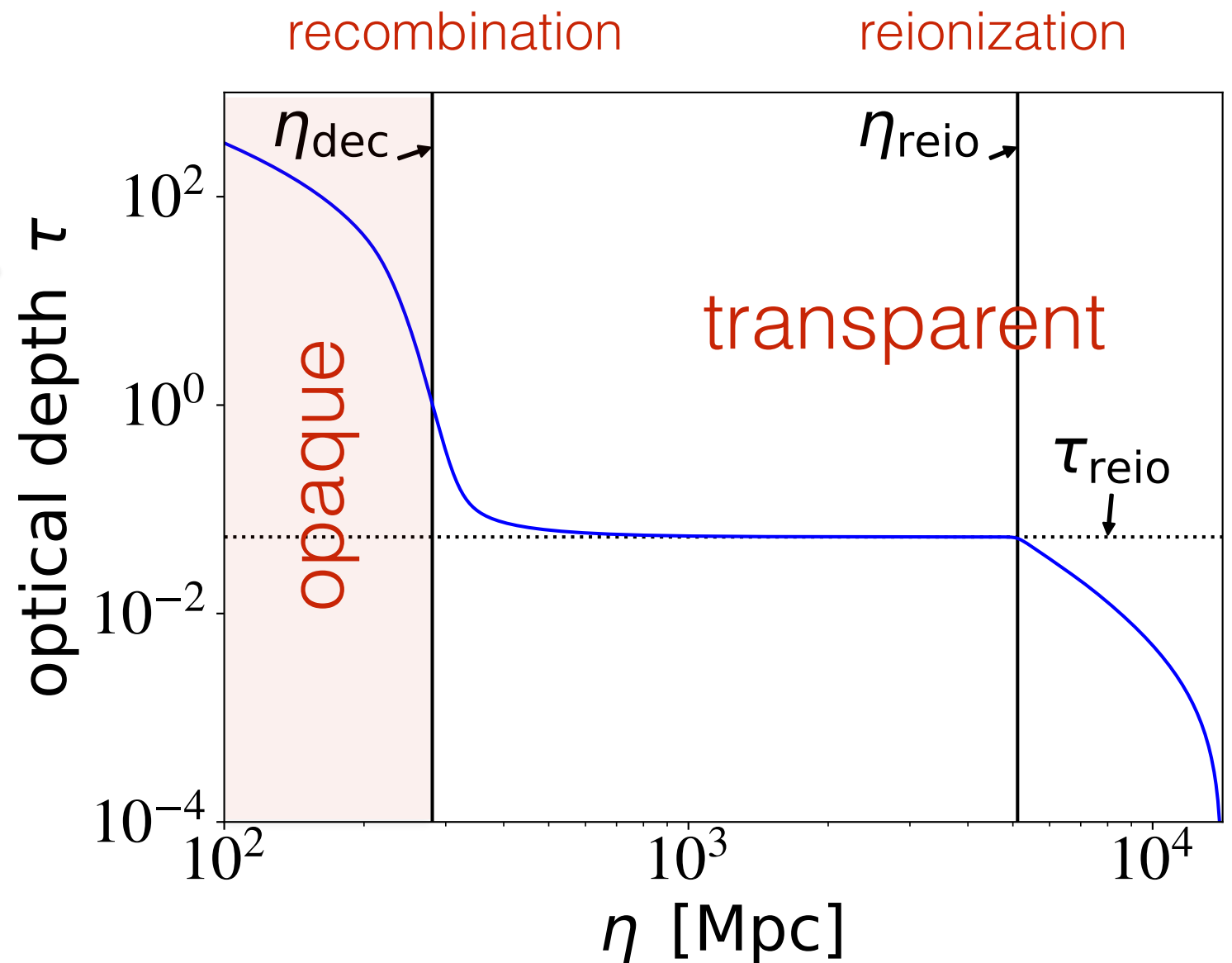
decoupling



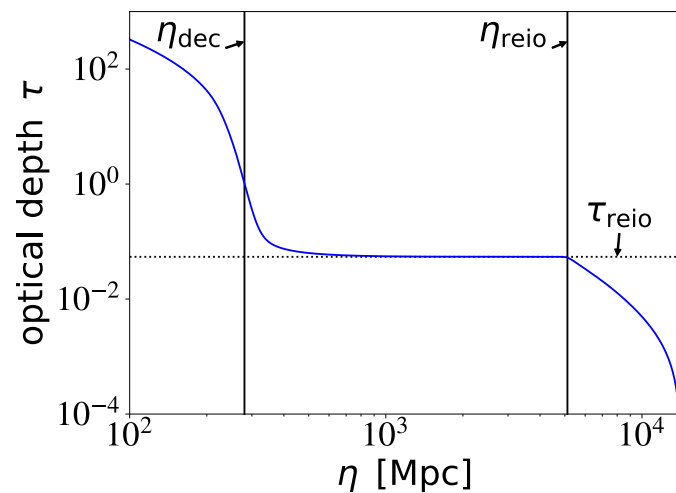
Optical depth of cosmic fog



$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' \Gamma_\gamma(\eta')$$

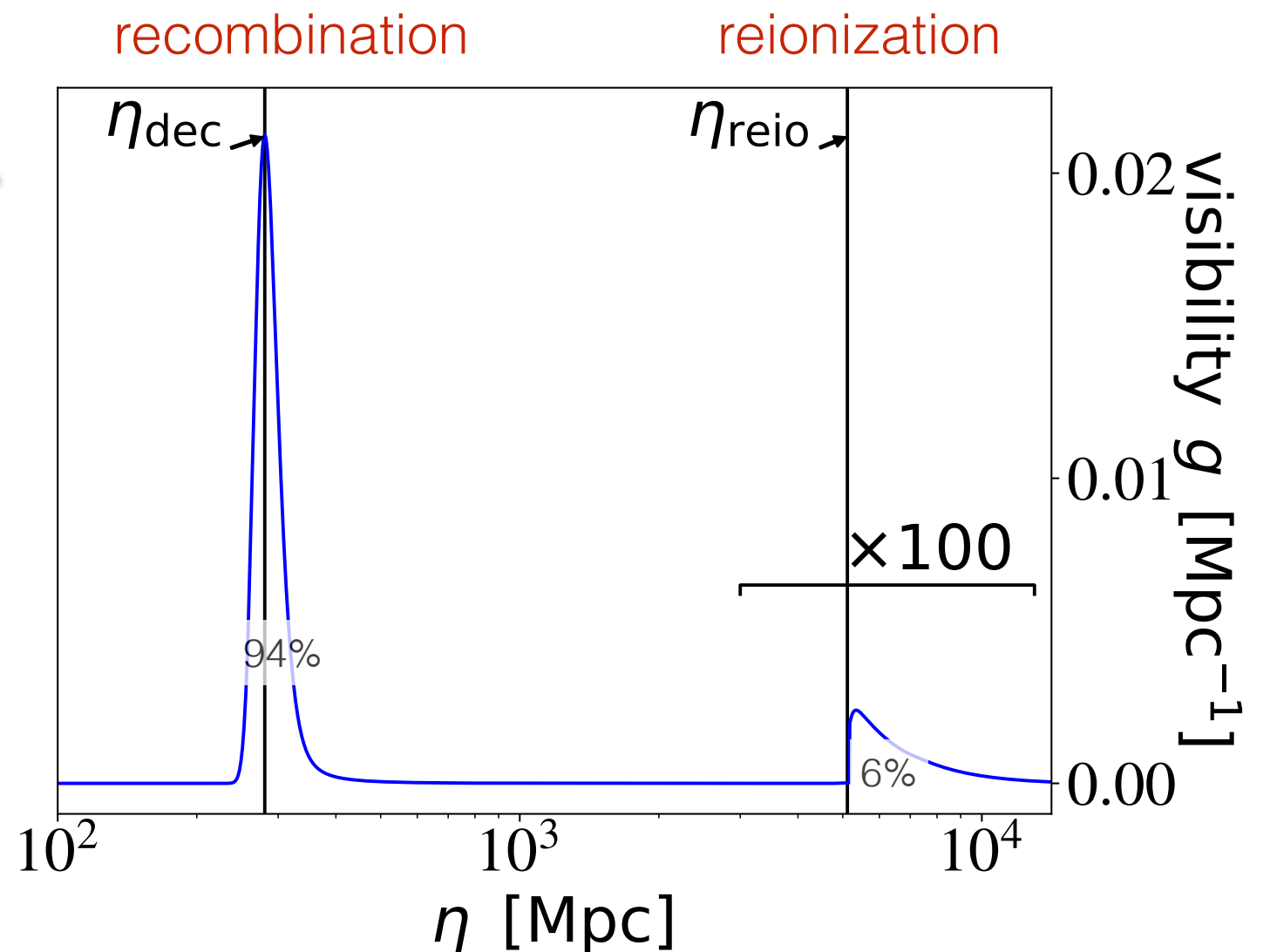


Visibility function



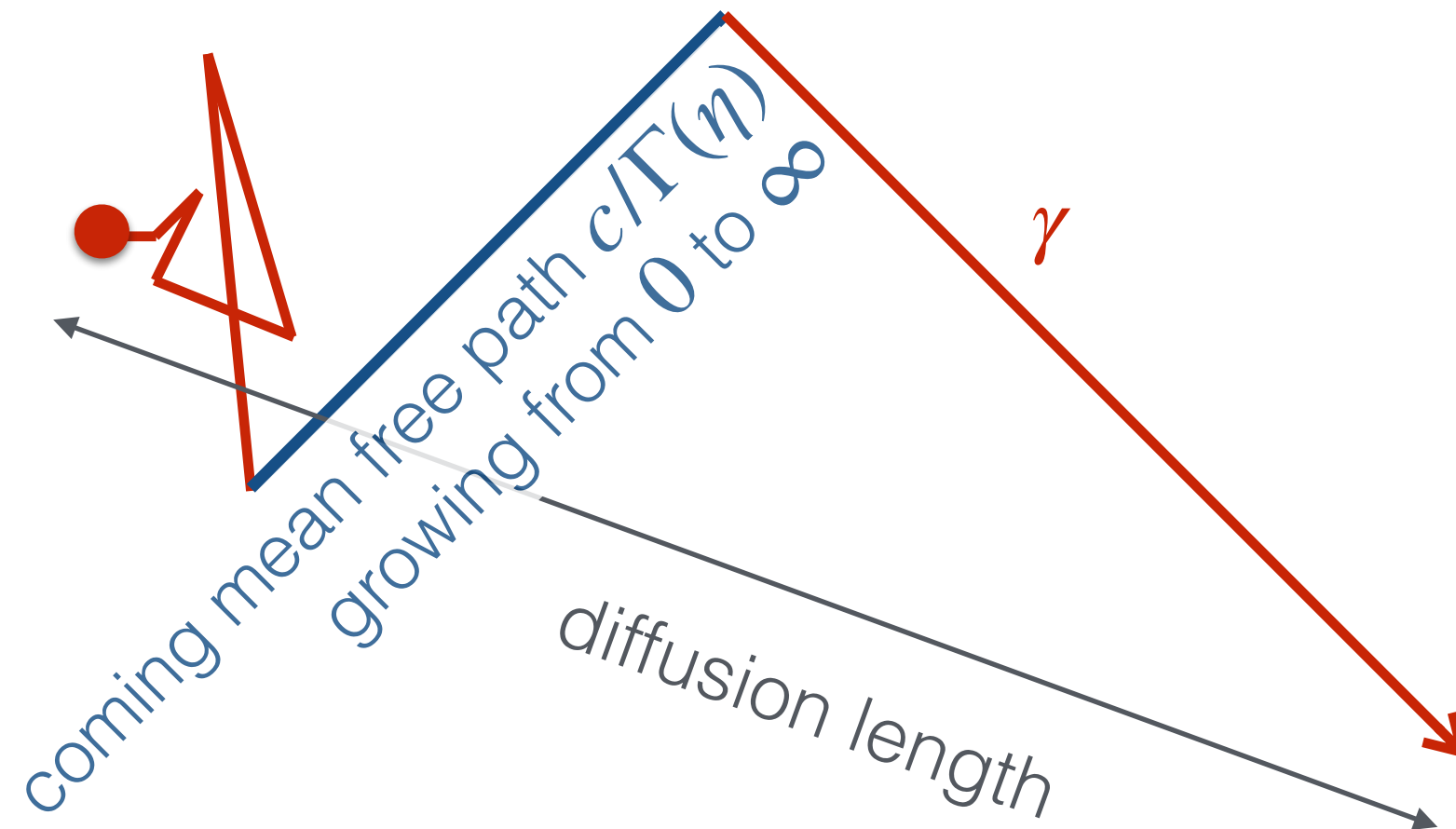
probability of last interaction at η
 = probability of interaction at η
 x (1-probability of interaction after η)

$$g(\eta) \equiv -\tau' e^{-\tau}$$



Mean free path and diffusion length

random walk



$$\lambda_d(\eta) = a(\eta) r_d(\eta) \simeq a(\eta) \left[\int_{\eta_{\text{ini}}}^{\eta} d\tilde{\eta} c^2 \Gamma_{\gamma}^{-1}(\tilde{\eta}) \right]^{1/2}$$

Linear cosmological perturbations

Bardeen decomposition

$$g_{\mu\nu}(\eta, \vec{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \vec{x}) \quad \text{and} \quad T^{\mu\nu}(\eta, \vec{x}) = \bar{T}^{\mu\nu}(\eta) + \delta T^{\mu\nu}(\eta, \vec{x})$$

background

perturbations
10 d.o.f

(for each species!)



James Bardeen 1986

FLRW background invariant under spatial rotations

⇒ irreducible representations of SO(3) → decoupled sectors

⇒ Bardeen

scalars (gravity forces) : 4 d.o.f.
vectors (gravito-magnetism) : 4 d.o.f.
tensors (gravitational waves) : 2 d.o.f.

Newtonian gauge

Gauge freedom: perturbations depends on choice of coordinates



Newtonian gauge: eliminate 2 scalar d.o.f. to stick to diagonal $\delta g_{\mu\nu}$

$$\begin{aligned} ds^2 &= -(1 + 2\psi)dt^2 + (1 - 2\phi)a^2(t)d\vec{x}^2 \\ &= a^2(\eta) [-(1 + 2\psi)d\eta^2 + (1 - 2\phi)d\vec{x}^2] \end{aligned}$$

Local distortion of time
= generalised
gravitational potential

Local distortion
of expansion rate

Matter perturbations

$\delta T_{\mu\nu,X}$: still 4 d.o.f. per species X



δ_X : relative fluctuation of **energy density**

θ_X : divergence of **bulk velocity**

δp_X : fluctuation of (isotropic) **pressure**

σ_X : **anisotropic stress** = quadrupole of (anisotropic) pressure

Local pressure relates to local density (e.g. $\delta p_X = w_X \delta \rho_X$)



only 3 d.o.f.

Perfect fluids (strong interactions)



Pressure is isotropic ($\sigma_X = 0$)



only 2 d.o.f.

Comoving Fourier space

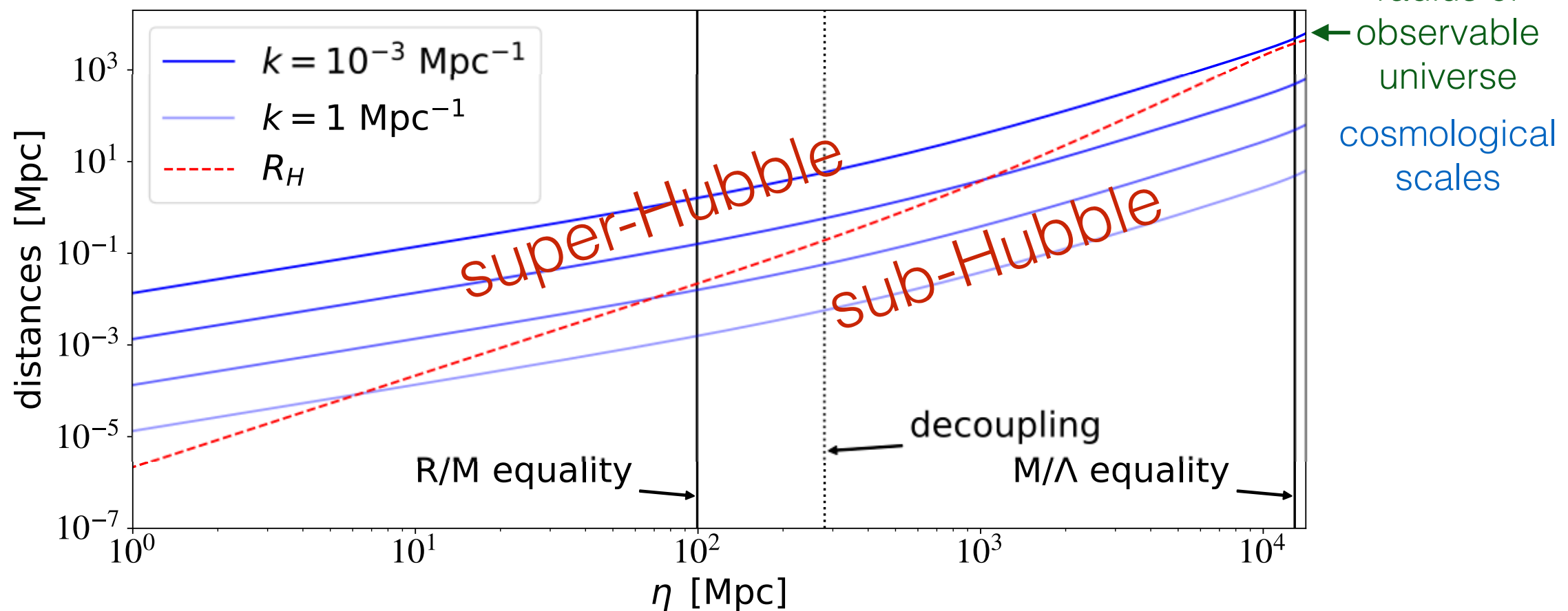
Thereafter: only flat cosmologies for simplicity

$$\delta_X(\eta, \vec{k}) = \int \frac{d^3 \vec{x}}{(2\pi)^{3/2}} \delta_X(\eta, \vec{x}) e^{-i \vec{k} \cdot \vec{x}}$$

coming wavevector
comoving coordinate

Wavelengths $\lambda(\eta) = a(\eta) 2\pi/k$

Decelerated expansion $\Rightarrow$ grow slower than Hubble radius $R_H(\eta) = c a(\eta)/\dot{a}(\eta)$



Condition for Hubble crossing : $k \sim aH \sim 1/\eta$

Linearised Einstein Equations

- from $G_0^0 = 8\pi G T_0^0$:
$$\frac{2}{a^2} \left[k^2 \phi + 3 \frac{a'}{a} \left(\phi' + \frac{a'}{a} \psi \right) \right] = -8\pi G \sum_X \bar{\rho}_X \delta_X$$

dominates on sub-Hubble



Poisson equation :

$$-\frac{k^2}{a^2} \phi = 4\pi G \bar{\rho}_{\text{total}} \delta_{\text{total}}$$

dominates on super-Hubble



Using Friedmann :

$$2\psi = -\delta_{\text{total}}$$

- from $G_j^i = 8\pi G T_j^i$:
$$\frac{2}{3} \frac{k^2}{a^2} (\phi - \psi) = 8\pi G \sum_X (\bar{\rho}_X + \bar{p}_X) \sigma_X$$

perfect fluids $\Rightarrow \sigma_X = 0 \Rightarrow \phi = \psi$

- other scalar equations redundant with upcoming equations of motion (Bianchi identity)

Equations of motion

Without details: each decoupled species fulfils energy/momentum conservation:

♦ continuity equation: $\delta'_X = -(1 + w_X)(\theta_X - 3\phi') - 3\frac{a'}{a}(c_s^2 X - w_X)\delta_X$

♦ Euler equation: $\theta'_X = -\frac{a'}{a}(1 - 3c_a^2 X)\theta_X + \frac{c_s^2 X}{1 + w_X}k^2\delta_X - k^2\sigma_X + k^2\psi$
(featuring sound speed c_s and adiabatic sound speed c_a)

In Λ CDM:

- **CDM** : negligible pressure/stress: closed system
- **e-/baryons**: negligible pressure/stress but Thomson scattering: $+ \frac{4}{3} \frac{\bar{\rho}_\gamma}{\bar{\rho}_b} \tau' (\theta_b - \theta_\gamma)$
- **photons, neutrinos**: when not strongly coupled, anisotropic stress

→ need Boltzmann equation

$$\frac{d}{d\eta} f_\gamma = C[f_\gamma, f_e] \qquad \frac{d}{d\eta} f_\nu = 0$$

Summary: perturbed degrees of freedom and equations of motion

Degrees of freedom	Equation of motion
Gravitational potential $\psi(\eta, \vec{x})$	Einstein 00: $\phi, \psi \leftrightarrow \delta$
Scale factor distortion $\phi(\eta, \vec{x})$	Einstein ij: $(\phi - \psi) \longrightarrow \sigma$
CDM density/velocity $\delta_c(\eta, \vec{x}), \theta_c(\eta, \vec{x})$	continuity $\delta'_c + \text{Euler } \theta'_c$
Baryon density/velocity $\delta_b(\eta, \vec{x}), \theta_b(\eta, \vec{x})$	Continuity $\delta'_b + \text{Euler } \theta'_b$ (incl. Thomson)
Photons $f_\gamma(\eta, \vec{x}, p, \hat{n})$	Boltzmann $\frac{d}{d\eta} f_\gamma = \text{Thomson}$
[Neutrinos $f_\nu(\eta, \vec{x}, p, \hat{n})$]	[Boltzmann $\frac{d}{d\eta} f_\nu = 0$]

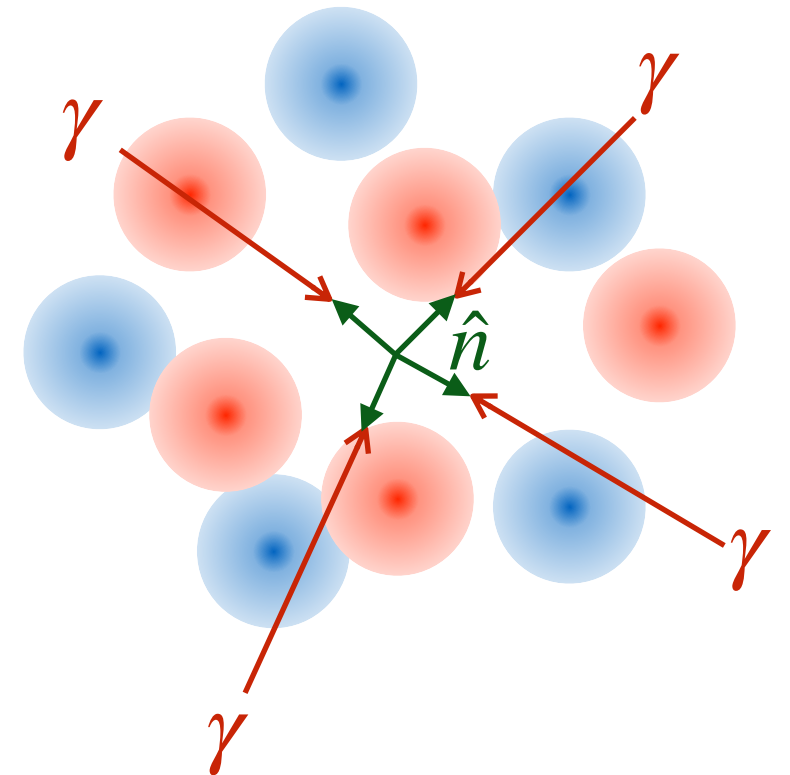
Photon phase-space distribution

Blackbody shape: $f_\gamma(\eta, \vec{x}, p, \hat{n}) = \frac{1}{e^{\frac{p}{T(\eta, \vec{x}, \hat{n})}} - 1}$

Up to very good approximation: **preserved** even when leaving thermal equilibrium, but becomes direction-dependent due to gravitational interactions:

redshifting along geodesics:

$$\frac{d \ln(a p)}{d\eta} = \underbrace{\phi'}_{\text{dilation}} - \underbrace{\hat{n} \cdot \vec{\nabla} \psi}_{\text{gravitational Doppler}}$$



Then: $T(\eta, \vec{x}, \hat{n}) = \bar{T}(\eta) \left(1 + \Theta(\eta, \vec{x}, \hat{n}) \right)$

Linearised Boltzmann equation

Temperature fluctuation: $T(\eta, \vec{x}, \hat{n}) = \bar{T}(\eta) (1 + \Theta(\eta, \vec{x}, \hat{n}))$

Monopole and dipole of Θ account for local density & bulk velocity:

$$\Theta(\eta, \vec{x}, \hat{n}) = \frac{1}{4}\delta_\gamma(\eta, \vec{x}) + \hat{n} \cdot \vec{v}_\gamma(\eta, \vec{x}) + \text{higher multipoles}$$

Linearised Boltzmann:

$$\Theta' + \hat{n} \cdot \vec{\nabla} \Theta - \phi' + \hat{n} \cdot \vec{\nabla} \psi = -\Gamma_\gamma \left(\hat{n} \cdot (\vec{v}_\gamma - \vec{v}_b) + \text{higher multipoles} \right)$$

dilation

gravitational Doppler

Thomson scattering

Thomson scattering wants to align velocity of photons vs. electron/baryons, and to wash out higher multipoles!

Boltzmann hierarchy

Start from linearised Boltzmann and perform:

1. Fourier transformation

2. Legendre expansion $\Theta(\eta, \vec{k}, \hat{n}) = \sum_l (-i)^l (2l+1) \Theta_l(\eta, \vec{k}) P_l(\hat{k} \cdot \hat{n})$

$$\delta'_\gamma + \frac{4}{3}\theta_\gamma - 4\phi' = 0$$

$$\theta'_\gamma + k^2 \left(-\frac{1}{4}\delta_\gamma + \sigma_\gamma \right) - k^2\psi = \tau'(\theta_\gamma - \theta_b)$$

$$\Theta'_l - \frac{kl}{2l+1} \Theta_{l-1} + \frac{k(l+1)}{2l+1} \Theta_{l+1} = \tau' \Theta_l \quad \forall l \geq 2$$

relates to Θ_2

⇒ Solved together with previous equations by Einstein-Boltzmann solvers
(CMBFAST, CAMB, CLASS...)