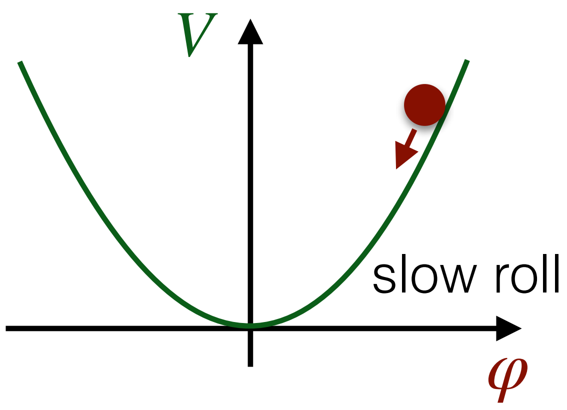
Stochastic theory of cosmological perturbations

Initial conditions



Canonical single-field inflation guarantees:

A. stochastic perturbations with independent Fourier modes

B. gaussian statistics for each Fourier mode / each d.o.f.

⇒ described by variance(wavenumber) = power spectrum

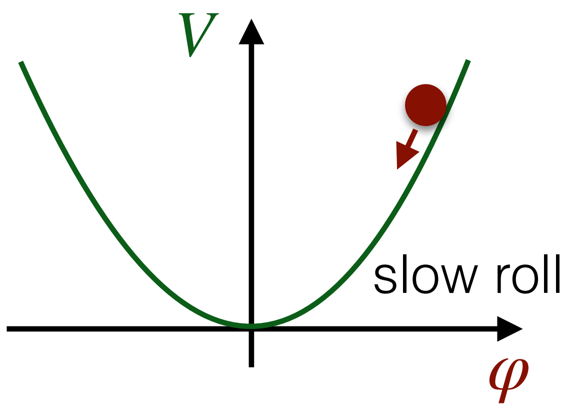
C. for each Fourier mode, all d.o.f. related to each other (fully correlated) on super-Hubble scales: "adiabatic initial conditions"

e.g. during RD:

$$-2\psi = -2\phi \underset{\substack{\uparrow \\ \text{Einstein eq.}}}{=} \boxed{\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_{\rm b} = \frac{4}{3}\delta_{\rm c}} \underset{\substack{\uparrow \\ \text{Einstein eq.}}}{=} \text{constant}$$

(Comes from $A(\eta,\vec{x}) = \bar{A}(\eta + \delta\eta(\vec{x})) = \bar{A}(\eta) + \underbrace{\bar{A}'(\eta)\,\delta\eta(\vec{x})}_{\substack{\text{perturbation } \delta A(\eta,\vec{x}) \\ \text{in adiabatic case}}}$)

Primordial power spectrum



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Velocity potential $\vec{v}_{\rm tot} = \vec{\nabla} v_{\rm tot}$

⇒ need power spectrum for single degree of freedom, e.g. curvature perturbation $\mathcal{R} \equiv \phi - \frac{a'}{a}\frac{v_{\rm tot.}}{a^2}$ in Newt. Gauge

⇒ Primordial spectrum: $\langle \mathscr{R}(\eta_i, \vec{k}) \mathscr{R}^*(\eta_i, \vec{k}') \rangle = \delta_D(\vec{k}' - \vec{k})\; P_{\mathscr{R}}(k)$

D. Power law, nearly scale-invariant spectrum: $P_{\mathscr{R}}(k) = \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_*}\right)^{n_s - 1}$

Transfer functions

For each Fourier mode $\vec{k}$:

- all perturbations → system of linear coupled differential equations
- adiabatic ICs → single constant of integration $\mathcal{R}(\eta_{\rm ini},\vec{k})$
- $\forall A \in \{\phi, \psi, \delta_X, \theta_X, \Theta_\ell, \dots\}$

$$\boxed{A(\eta,\vec{k})} = \boxed{T_A(\eta,k)}\,\boxed{\mathscr{R}(\eta_i,\vec{k})}$$

stochastic Fourier mode

stochastic IC

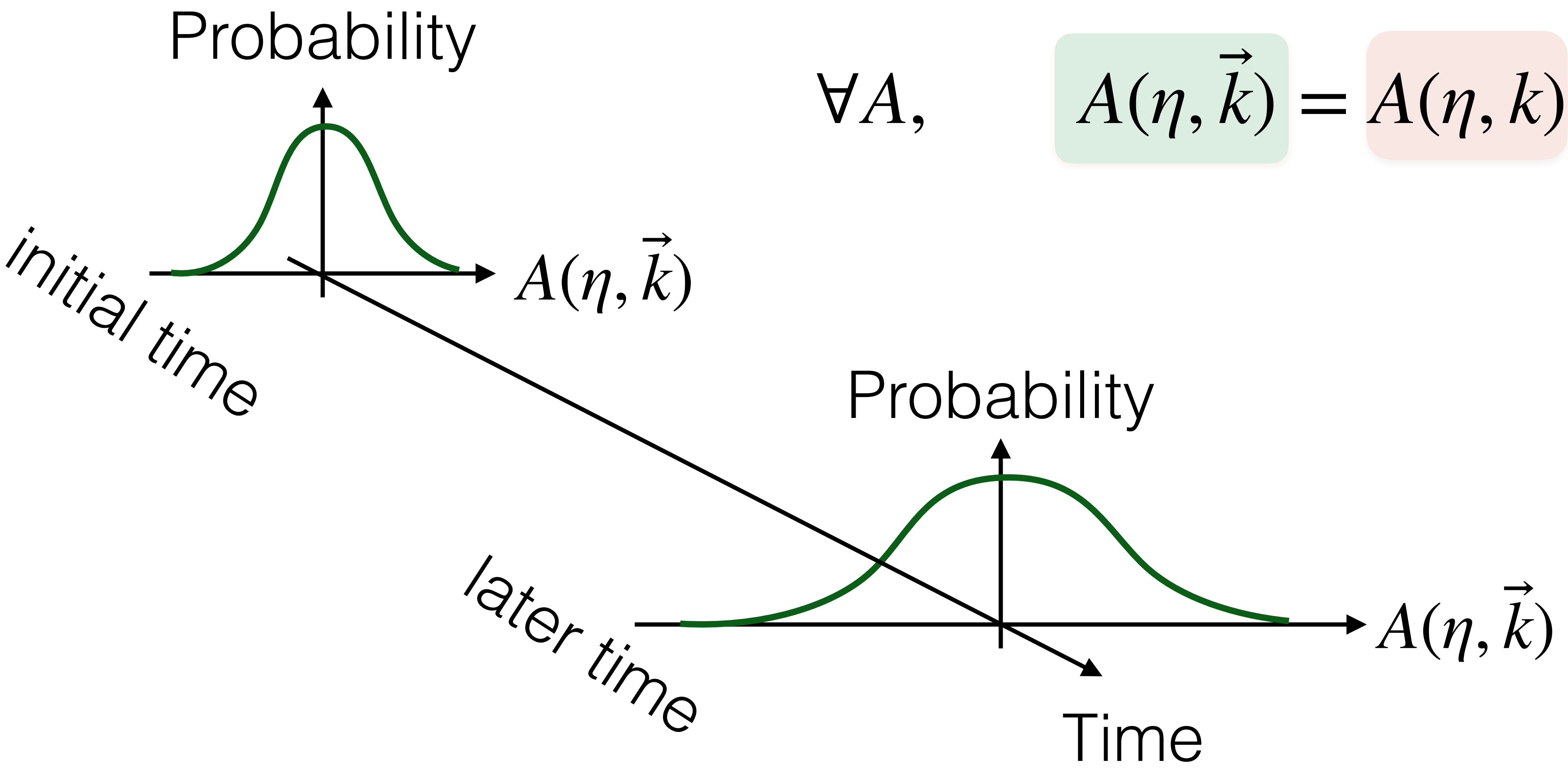
Deterministic solution of e.o.m. normalised to $\mathscr{R}$ =1

= transfer function of A

Isotropic background ⇒ depends only on k

⇒ denoted later as $A(t,k)$

Linear transport of probability



$$\forall A, \qquad A(\eta, \vec{k}) = A(\eta, k)\,\mathscr{R}(\eta_i, \vec{k})$$

Linearity of solutions ⇒ probability shape always preserved

(standard model: Gaussian)

⇒ variance evolves like square of transfer function

Power spectrum

Adiabatic initial conditions

⇒ for <u>any</u> perturbation at <u>any</u> time:

$$\langle A(\eta,\vec{k})\,A^*(\eta,\vec{k}')\rangle = A(\eta,k)\,A^*(\eta,k')\,\langle \mathscr{R}(\eta_{\rm i},\vec{k})\,\mathscr{R}^*(\eta_{\rm i},\vec{k}')\rangle$$

$$= \boxed{|A(\eta,k)|^2\,P_{\mathscr{R}}(k)}\;\delta_D(\vec{k}-\vec{k}')$$

transfer function of A

power spectrum $P_A(\eta,k)$ of A at η

primordial curvature spectrum

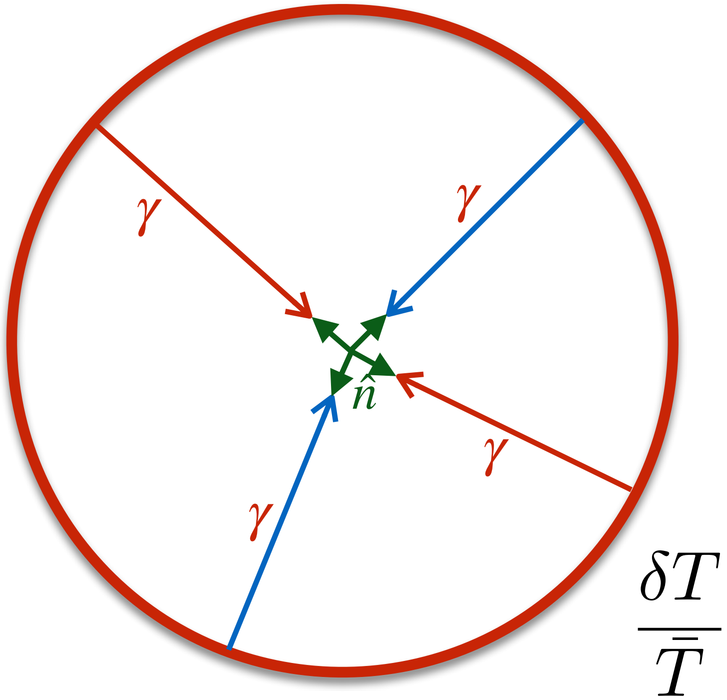
Spectrum of temperature anisotropies

Temperature multipoles

$g(\eta)$ very peaked at $\eta_{\rm dec}$

$\Downarrow$

last scattering sphere



$$\frac{\delta T}{\bar{T}}(\hat{n}) = \Theta(\eta_0,\vec{o}, -\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

inversion + Fourier + Legendre ⇒ $a_{lm} = (-i)^l \int \frac{d^3\vec{k}}{2\pi^2} Y_{lm}(\hat{k}) \Theta_l(\eta_0,\vec{k})$

stochastic, Gaussian ← stochastic, Gaussian

correlation/variance ⇒ $\langle a_{lm} a^*_{l'm'} \rangle = \delta^K_{ll'} \delta^K_{mm'} \left[\frac{2}{\pi} \int dk\, k^2 \Theta_l^2(\eta_0, k)\; P_{\mathscr{R}}(k) \right]$

photon transfer function

primordial spectrum

Temperature power spectrum

Defined as: $$C_l = \langle a_{lm} a_{lm}^* \rangle = \frac{2}{\pi} \int dk\, k^2 \Theta_l^2(\eta_0, k)\, P_{\mathscr{R}}(k)$$

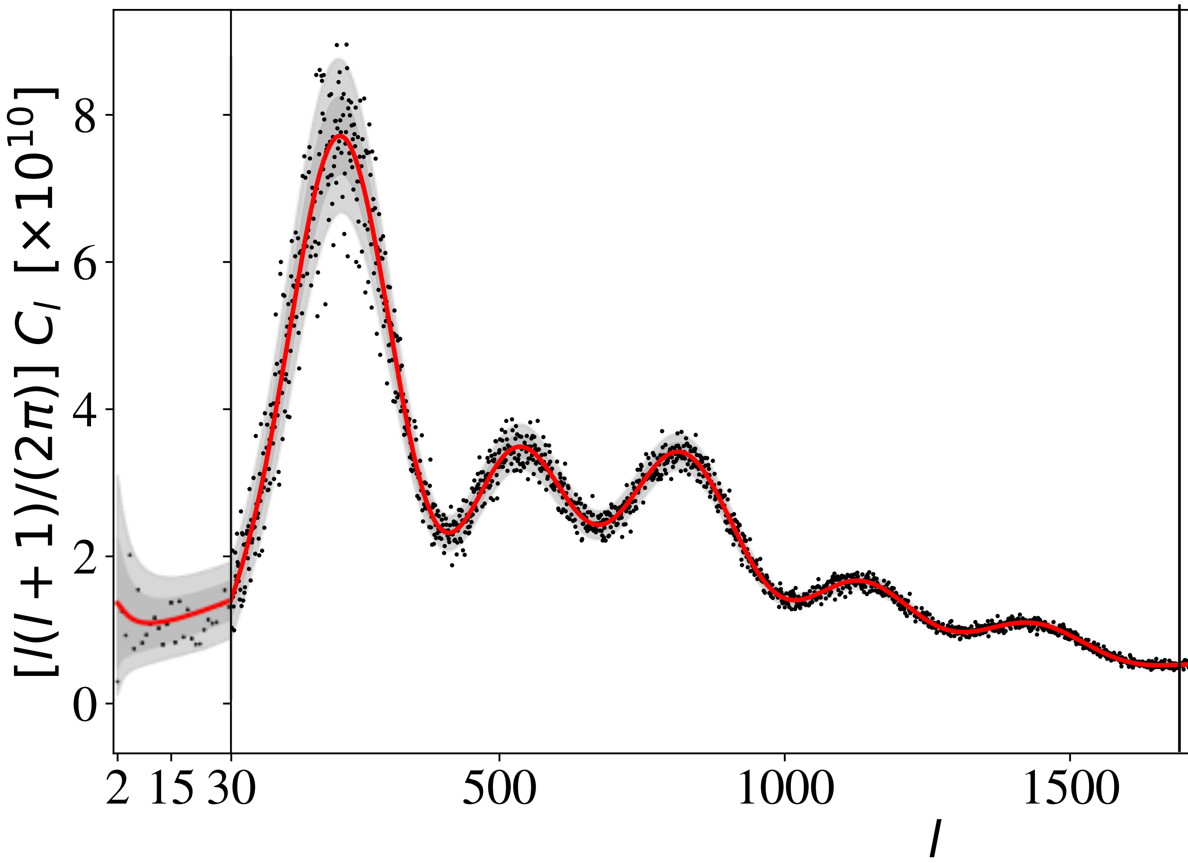
photon transfer function

primordial spectrum

theory ⟷ observations

Estimator: $$\hat{C}_l(a_{lm}) \equiv \frac{1}{2l+1} \sum_{-l \leq m \leq l} |a_{lm}|^2$$

Cosmic variance: $$\langle (\hat{C}_l - C_l)^2 \rangle = \frac{2}{2l+1} C_l^2$$



Physics of temperature anisotropies

“Line-of-sight” integral in Fourier space

Boltzmann hierarchy ⇒ formal solution Zaldarriaga & Harari astro-ph/9504085:

$$\Theta_l(\eta_0,\overset{(\rightarrow)}{k}) = \int_{\eta_{\mathrm{ini}}}^{\eta_0} d\eta\,\{g\,(\Theta_0+\psi)\;j_l(k(\eta_0-\eta))$$

$$+\,g\,k^{-1}\theta_{\mathrm{b}}\;j_l'(k(\eta_0-\eta))$$

$$+\,e^{-\tau}(\phi'+\psi')\;j_l(k(\eta_0-\eta))\}$$

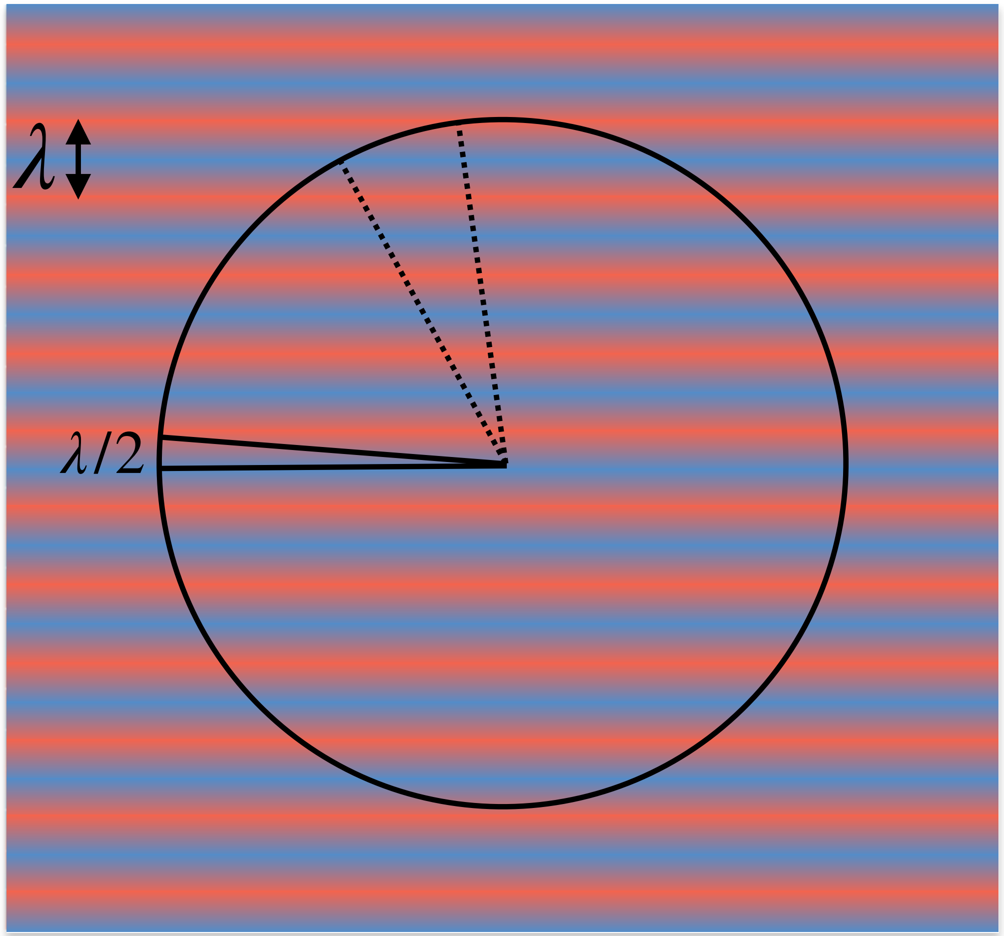
valid both for
single mode $\vec{k}$ or
transfer function with k

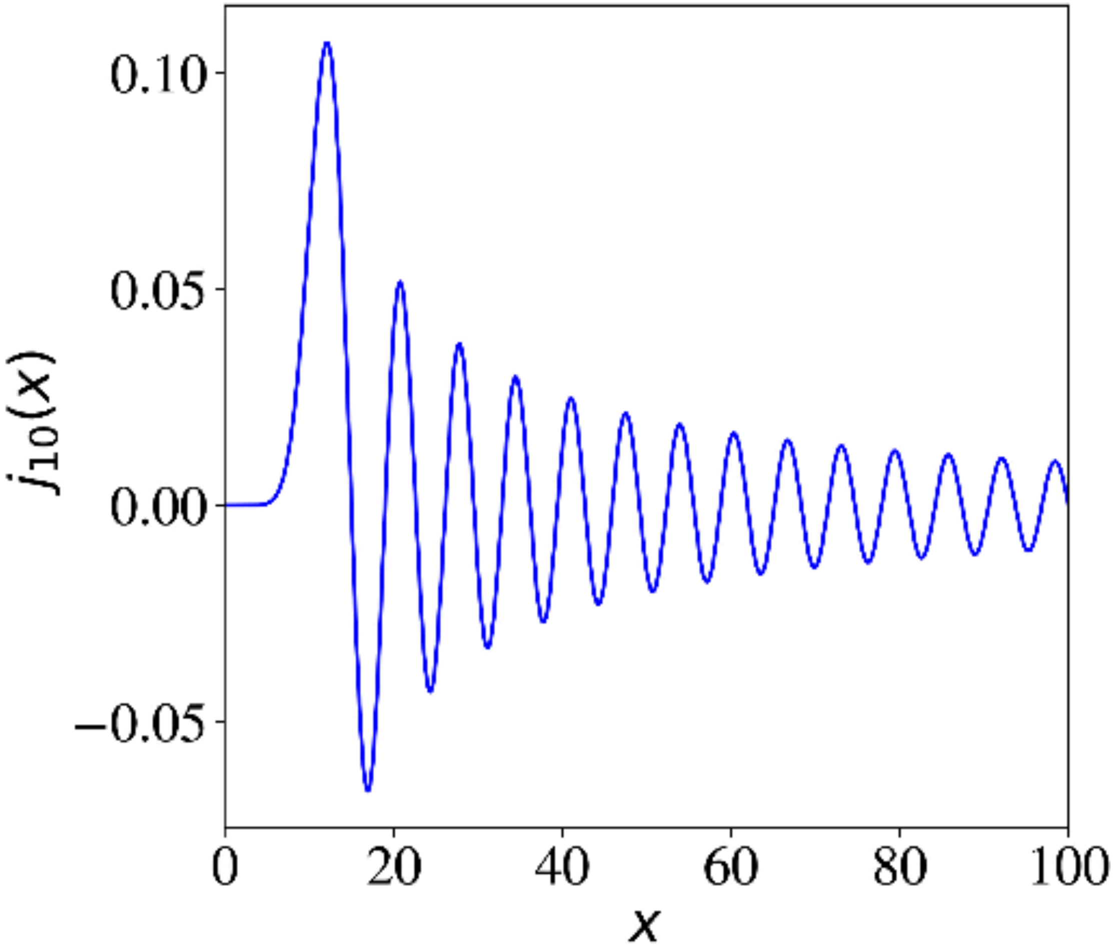
structure: $\int d\eta\; f(\eta)\; A(\eta,\vec{k})\; j_\ell(k(\eta_0-\eta))$

“Physical effects relevant at times described by $f(\eta)$
imprint CMB photon anisotropies described in Fourier space by $A(\eta,\vec{k})$,
that project to multipole space according to $j_\ell(k(\eta_0-\eta))$”

Angular projection of Fourier modes

Role of $j_\ell(k(\eta_0 - \eta))$?





Main contribution: $\theta = \frac{\pi}{l} = \frac{\lambda/2}{d_\mathrm{a}} = \frac{a(\eta)\,\pi/k}{a(\eta)\,(\eta_0 - \eta)} \quad \Leftrightarrow \quad l = k(\eta_0 - \eta)$

Other contributions: harmonics

Sachs-Wolfe term

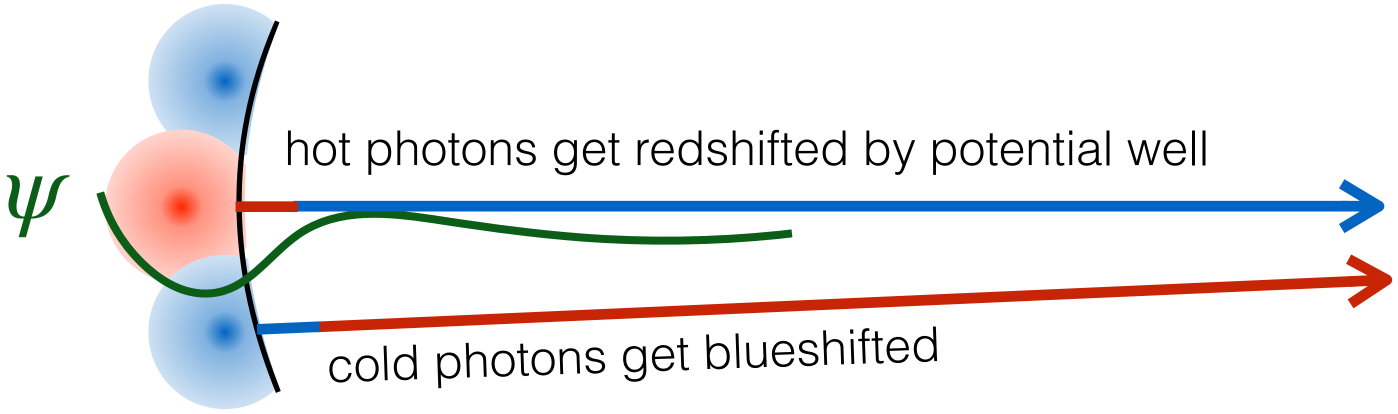
$$\begin{aligned}\Theta_l(\eta_0,\vec{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \, \{ & g \, (\Theta_0 + \psi) \; j_l(k(\eta_0 - \eta)) \\ & + \, g \, k^{-1} \theta_{\rm b} \; j_l'(k(\eta_0 - \eta)) \\ & + \, e^{-\tau} (\phi' + \psi') \, j_l(k(\eta_0 - \eta)) \}\end{aligned}$$

Neglecting reionization: $g(\eta)$ very peaked at $\eta_{\rm dec}$

$\Rightarrow$ effect takes place only on last scattering sphere

$\Rightarrow$ mode k project to $\ell = k(\eta_0 - \eta_{\rm dec})$

$\Theta_0(\eta_{\rm dec}, \vec{k}) + \psi(\eta_{\rm dec}, \vec{k})$ = intrinsic fluctuation + gravitational Doppler shift



(super-Hubble modes with
adiabatic IC: $\psi = -2\Theta_0$,
Sachs-Wolfe effect wins,
negative picture of last
scattering sphere !)

Doppler term

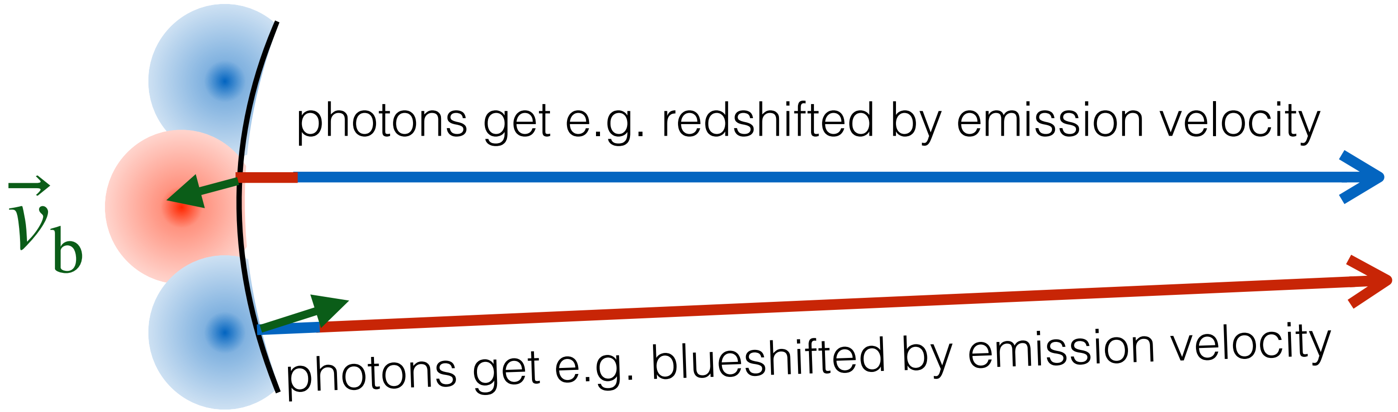
$$\Theta_l(\eta_0,\vec{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \left\{ g\,(\Theta_0+\psi)\; j_l(k(\eta_0-\eta)) \right.$$
$$+\; g\, k^{-1}\theta_{\rm b}\; j_l'(k(\eta_0-\eta))$$
$$\left. +\; e^{-\tau}(\phi'+\psi')\, j_l(k(\eta_0-\eta)) \right\}$$

Neglecting reionization: $g(\eta)$ very peaked at $\eta_{\rm dec}$

$\Rightarrow$ effect takes place only on last scattering sphere

$\Rightarrow$ mode k project to $\ell = k(\eta_0 - \eta_{\rm dec})$

$\hat{n}\cdot\vec{v}_{\rm b}^{\rm scalar} \rightarrow k^{-1}\theta_{\rm b}$ = velocity Doppler shift (${j}_\ell'$ from a gradient)



photons get e.g. redshifted by emission velocity

photons get e.g. blueshifted by emission velocity

Integrated Sachs-Wolfe (ISW) term

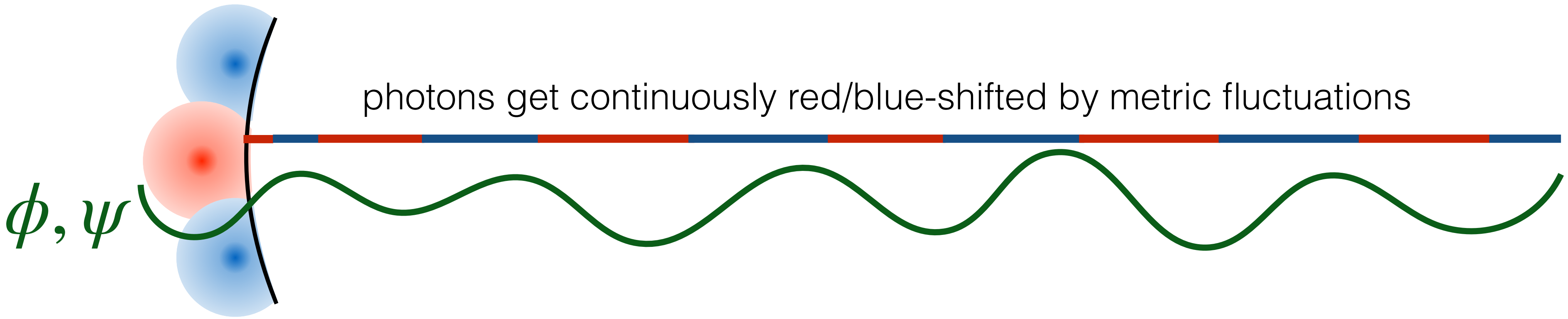
$$\Theta_l(\eta_0,\vec{k}) = \;\dots\; + e^{-\tau}(\phi' + \psi')\, j_l(k(\eta_0 - \eta))$$

Neglecting reionization: $e^{-\tau}$ negligible before $\eta_{\rm dec}$, $\simeq 1$after

$\Rightarrow$ effect takes place at all times $\eta > \eta_{\rm dec}$ along each line of sight

$\Rightarrow$ mode k projects from each sphere to $\ell = k(\eta_0 - \eta)$

$\partial_\eta\{\phi(\eta,\vec{k}) + \psi(\eta,\vec{k})\}$ comes from dilation + gravitational Doppler effects



- ϕ, ψ static: no dilation, gravitational Doppler effect is conservative: only $(\psi_{\rm dec} - \psi_{\rm obs})$
- ϕ, ψ time-dependent: net effect (e.g. net redshift when crosses deepening potential wells)

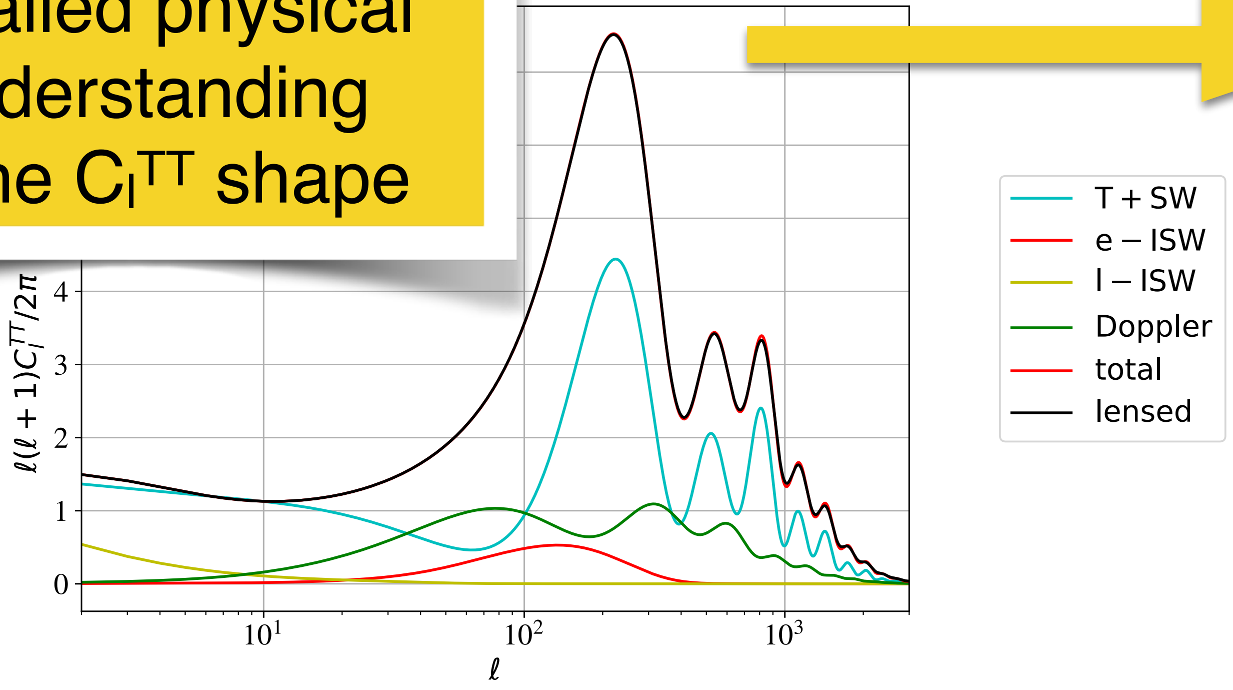
Summary

Final goal: compute $C_\ell = \langle a_{lm} a_{lm}^* \rangle = \frac{2}{\pi} \int dk\, k^2 \Theta_\ell^2(\eta_0, k)\ \ P_{\mathcal{R}}(k)$

with transfer functions

$$\Theta_l(\eta_0,k) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta\, \{ g\, (\Theta_0 + \psi)\ j_l(k(\eta_0 - \eta)) \\ + g\, k^{-1} \theta_{\rm b}\ j_l'(k(\eta_0 - \eta)) \\ + \ e^{-\tau} (\phi' + \psi')\, j_l(k(\eta_0 - \eta)) \}$$

Detailed physical understanding of the C_l^{TT} shape



behaviour of

$\Theta_0(\eta_{\rm dec}, k)$

$\theta_{\rm b}(\eta_{\rm dec}, k)$

$\psi(\eta \geq \eta_{\rm dec}, k) \simeq \phi$

Tight-Coupling Approximation (TCA)

When $\Gamma_\gamma \gg \dfrac{a'}{a}$:
tightly-coupled baryon-photon fluid: $\left\{ \begin{array}{l} \Theta_0 = \frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_b \\ 3k\Theta_1 = \theta_\gamma = \theta_b \\ \Theta_{l\geq 2} = 0 \end{array} \right.$

$\Theta_0 = \frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_b$ → from thermal equilibrium

$3k\Theta_1 = \theta_\gamma = \theta_b$, $\Theta_{l\geq 2} = 0$ → from efficient Thomson scattering

⇒ photon Boltzmann hierarchy + baryon fluid equations —> single TCA equation:

$$\Theta_0'' + \underbrace{\frac{R}{1+R}\frac{a'}{a}\Theta_0'}_{\text{baryon damping}} + \underbrace{k^2 c_s^2 \Theta_0}_{\text{pressure force}} = \underbrace{-\frac{k^2}{3}\psi}_{\text{gravity force}} + \underbrace{\frac{R}{1+R}\frac{a'}{a}\phi'}_{\text{local baryon damping}} + \underbrace{\phi''}_{\text{dilation}}$$

Squared sound speed / baryon-to-photon ratio: $c_s^2 = \dfrac{1}{3(1+R)}$, $R \equiv \dfrac{3\bar{\rho}_b}{4\bar{\rho}_\gamma} \propto a$

Tight-coupling equation

$$\Theta_0'' + \frac{R}{1+R}\frac{a'}{a}\Theta_0' + k^2 c_{\rm s}^2 \Theta_0 = -\frac{k^2}{3}\psi + \frac{R}{1+R}\frac{a'}{a}\phi' + \phi''$$

baryon damping

pressure force

gravity force

local baryon damping

dilation

Squared sound speed / baryon-to-photon ratio: $c_{\rm s}^2 = \frac{1}{3(1+R)}$, $R \equiv \frac{4\bar{\rho}_{\rm b}}{3\bar{\rho}_\gamma} \propto a$

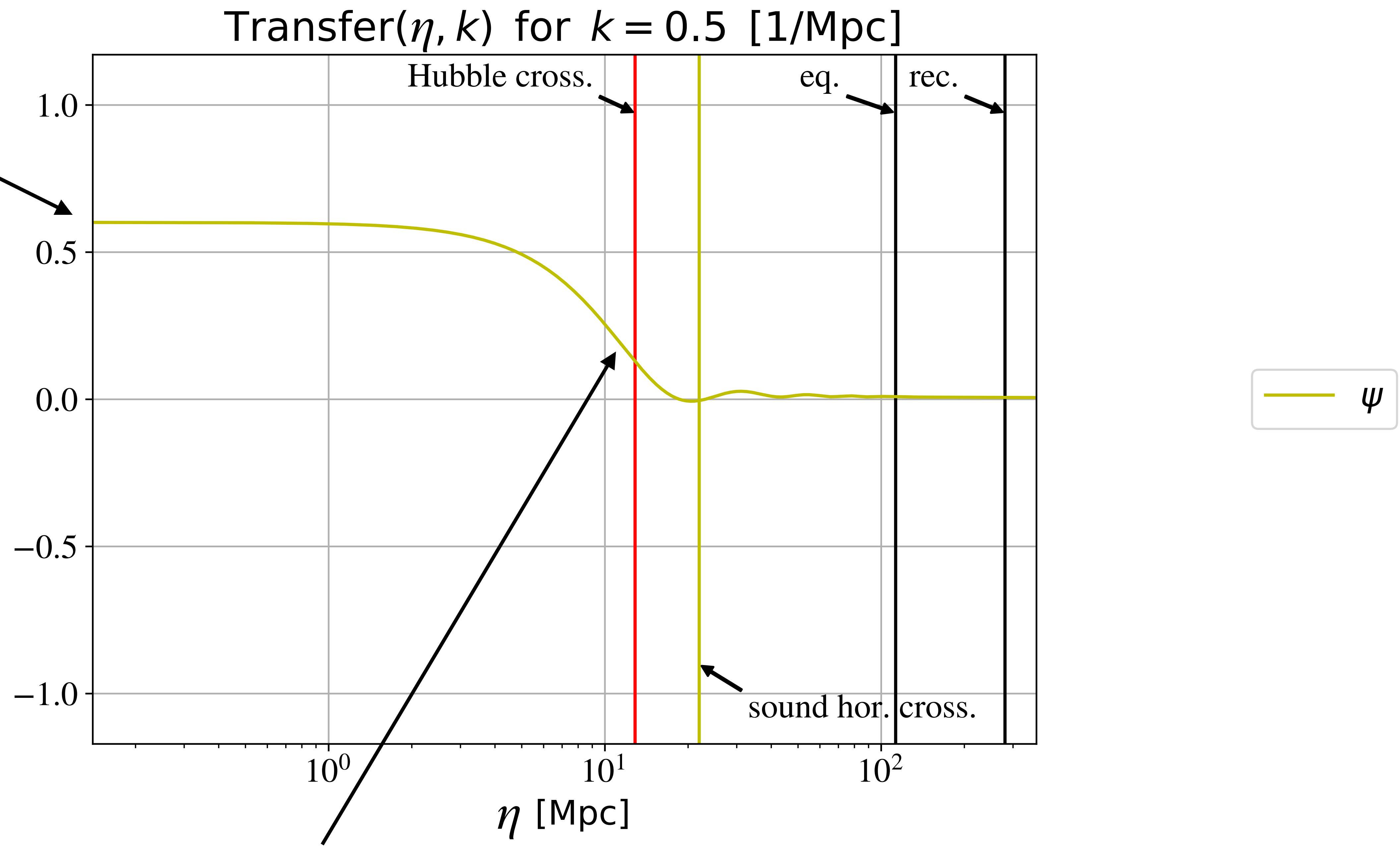
Equilibrium point neglecting metric time derivatives: $\Theta_0^{\rm equi.} = -\frac{1}{3c_{\rm s}^2}\psi = -(1+R)\psi$

WKB TCA solution “ “ “ : $\Theta_0 = A(1+R)^{-1/4}\cos\left(k \int c_{\rm s}(\eta)d\eta\right) - (1+R)\psi$

Very good approximation up to gravity boost + (Silk) damping/diffusion effects

Evolution for one mode with given k

Adiabatic IC
($\mathcal{R}$=1, ψ= 2/3)



Metric damped near Hubble crossing during RD

—> photon pressure, Poisson: $-k^2\phi = 4\pi G\, a^2\, \delta\rho_r \propto a^2\rho_r\, \delta_r \sim a^{2-4+0} \sim a^{-2}$

—> very different from MD: $-k^2\phi = 4\pi G\, a^2\, \delta\rho_m \propto a^2\rho_m\, \delta_m \sim a^{2-3+1} \sim \text{constant}$