Light-Cone Approach to Cosmological Observables beyond Linear Order

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[based on PB, G. Fanizza, G. Marozzi and M. Medeiros, to appear soon]

New Physics from Galaxy Clustering at GGI

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The Era of Precision Cosmology

- An unprecedented amount of high-precision data from large-scale cosmological surveys (*Vera Rubin, Euclid, Roman*,…) is about to be released.

- To take advantage of these **new opportunities**, theoretical predictions should be as accurate as the level of precision of cosmic surveys.

- In **structure formation** and **gravitational collapses**, many non-linear effects come into play and can be detected in **galaxy surveys**:
    - Couplings of linear modes and relativistic effects (light-cone distorsions, RSD,…).

Impact on Large Scale Structure Cosmological Observables

GLC Coordinates: Physical Motivation

Light-rays propagate on the past light-cone of an observer.

Cosmological observations are made on the past light-cone of the observer.

Search for **physically motivated/observations oriented** set of coordinates.

GLC Coordinates: Physical Interpretation

GLC Coordinates: Formal Definition

- The **GLC coordinates** are [Gasperini, Marozzi, Nugier, Veneziano, JCAP, 1107 (2011) 008]

$$x^\mu = (\tau, w, \tilde{\theta}^a)\,, \quad a = 1{,}2$$

$\tau =$ const. $\leftrightarrow$ geodesic obs.

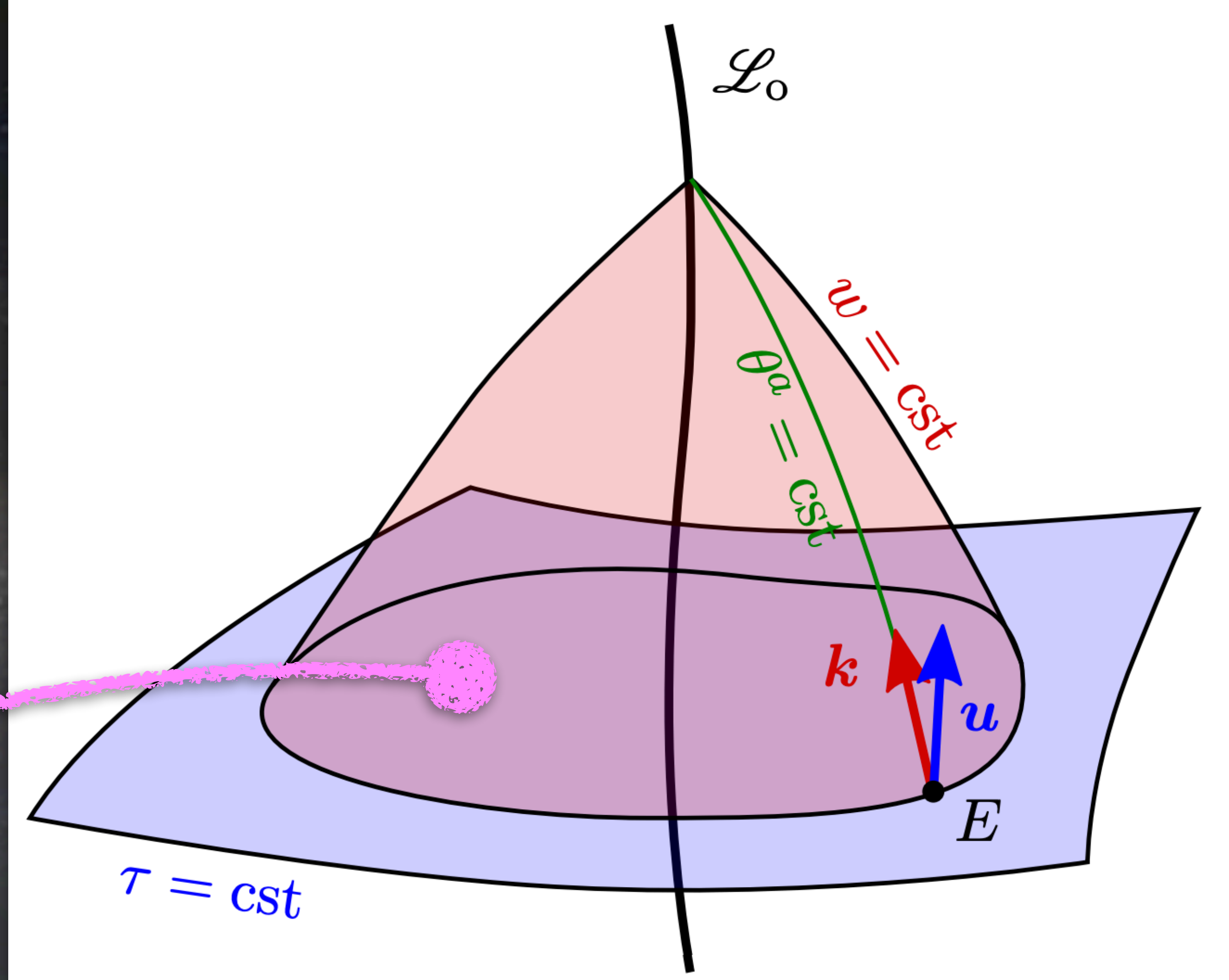
$w =$ const. $\leftrightarrow$ past LC

Angular directions in the sky

- The **GLC gauge** is

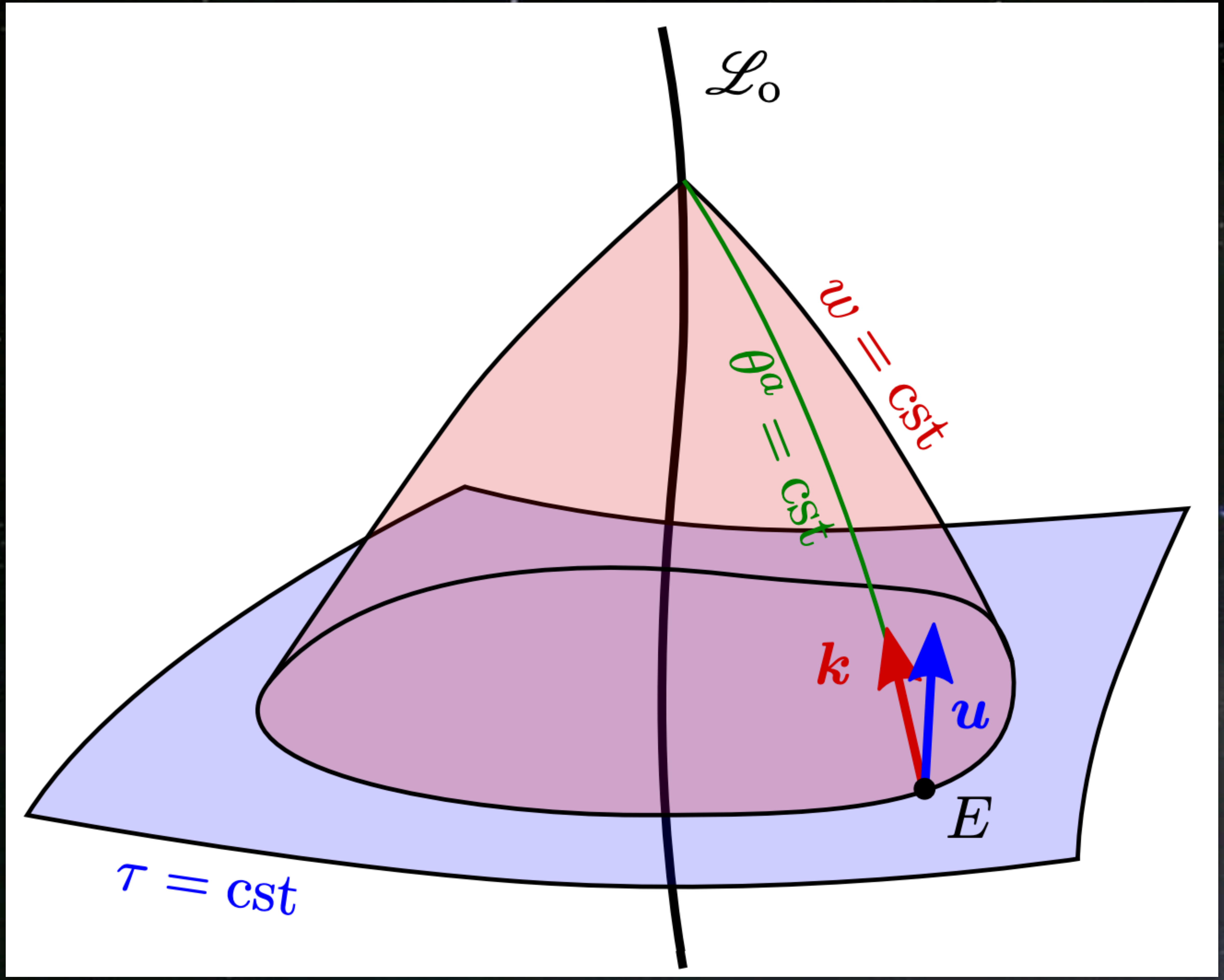
$$\mathrm{d}s^2 = -2\Upsilon \mathrm{d}\tau\, \mathrm{d}w + \Upsilon^2 \mathrm{d}w^2 + \gamma_{ab}(\mathrm{d}\tilde{\theta}^a - \mathcal{U}^a \mathrm{d}w)(\mathrm{d}\tilde{\theta}^b - \mathcal{U}^b \mathrm{d}w)$$

Induced metric on $\mathbb{S}^2 \ni \tilde{\theta}^a$



[Fleury, Fanizza, Nugier, JCAP, 06 (2016) 008]

GLC Coordinates: Physical Meaning



- **Light-like geodesics** are *exactly* solved by

$$\tilde{\theta}^a = \text{const.}$$

- A given **source** is identified with

$$\left(z, \tilde{\theta}^a_{\text{s}}\right)$$

$$1 + z = \frac{\Upsilon\left(\tau_{\text{o}}, w, \tilde{\theta}^a_{\text{o}}\right)}{\Upsilon\left(\tau_{\text{s}}, w, \tilde{\theta}^a_{\text{s}}\right)} \qquad \tilde{\theta}^a_{\text{s}} = \tilde{\theta}^a_{\text{o}}$$

Perturbation Theory on the Light-Cone

- It is possible to connect

$$\text{SPT}: \begin{cases} y^\mu = (\eta, r, \theta^a) \\ g_{\mu\nu} = \bar{g}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} \end{cases} \longrightarrow \text{perturbed LC}: \begin{cases} y^\mu = (\tau, r, \tilde{\theta}^a) \\ f_{\mu\nu} = \bar{f}_{\mu\nu} + f^{(1)}_{\mu\nu} + f^{(2)}_{\mu\nu} \end{cases}$$

- At the **background**

$$\mathrm{d}\eta = \frac{\mathrm{d}\tau}{a(\tau)}\,, \qquad r = w - \eta(\tau)\,, \qquad \theta^a = \tilde{\theta}^a$$

- The **perturbed** LC metric is

$$\mathrm{d}s^2 = a(\tau)^2 \Big[\left(L^{(1)} + L^{(2)}\right) \mathrm{d}\tau^2 - \frac{2}{a}\left(1 - a\left(M^{(1)} + M^{(2)}\right)\right) \mathrm{d}\tau\,\mathrm{d}w + 2\left(V^{(1)}_a + V^{(2)}_a\right) \mathrm{d}\tau\,\mathrm{d}\tilde{\theta}^a$$
$$+ \left(1 + N^{(1)} + N^{(2)}\right) \mathrm{d}w^2 + 2\left(U^{(1)}_a + U^{(2)}_a\right) \mathrm{d}w\,\mathrm{d}\tilde{\theta}^a + \left(\bar{\gamma}_{ab} + \gamma^{(1)}_{ab} + \gamma^{(2)}_{ab}\right) \mathrm{d}\tilde{\theta}^a\,\mathrm{d}\tilde{\theta}^b \Big]$$

Map between Perturbed FLRW and LC Metrics

- The following diagram **commutes**:

bkg.

$$f_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} g_{\rho\sigma}$$

bkg.+
perturbations

SPT	$\xrightarrow{\text{gauge}}$	$\widetilde{\text{SPT}}$
diff. $\downarrow$		$\downarrow$ diff.
GLC	$\xrightarrow{\text{gauge}}$	$\widetilde{\text{GLC}}$

$$y^\mu \to \tilde{y}^\mu = y^\mu + \epsilon^\mu_{(1)} + \frac{1}{2}\left(\epsilon^\nu_{(1)}\partial_\nu \epsilon^\mu_{(1)} + \epsilon^\mu_{(2)}\right)$$

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu_{(1)} + \frac{1}{2}\left(\xi^\nu_{(1)}\partial_\nu \xi^\mu_{(1)} + \xi^\mu_{(2)}\right)$$

Key Advantages of the GLC Gauge

Non-Perturbative Framework

↓

Fully non-linear expressions for light-like cosmological observables

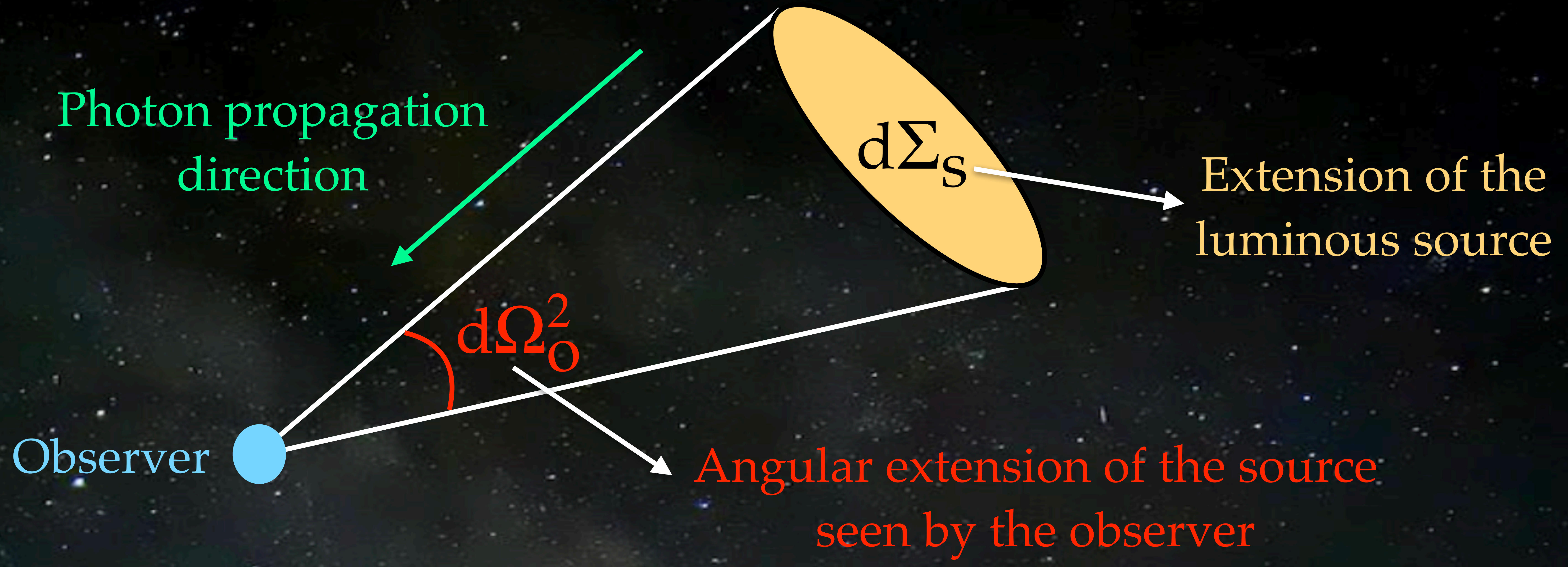
Perturbation Theory

↓

Observables **factorized** as products of perturbations at the **source's and observer's position**

Observer and source are connected through $\tilde{\theta}^a = \text{const.}$ **geodesics** on a $w = \text{const.}$ **past light-cone** of a **free-falling obs**.

Example: The Angular Distance - Redshift Relation



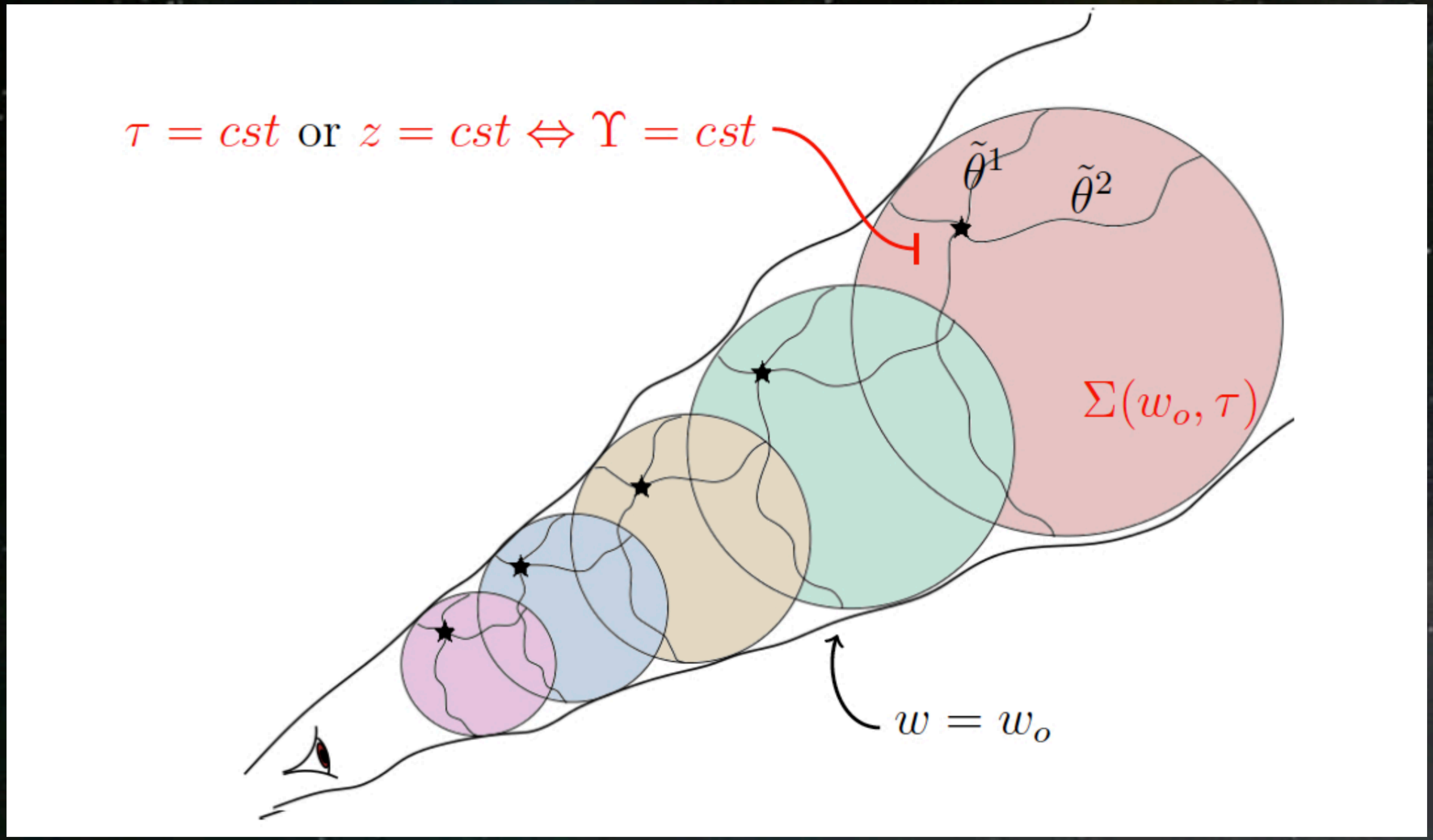
In the GLC gauge
[Fanizza, Gasperini, Marozzi, Veneziano, JCAP, 1311 (2013) 019]

$$d_A^2 \equiv \frac{\mathrm{d}\Sigma_s}{\mathrm{d}\Omega_o^2} = \det[J] = \frac{\sqrt{\gamma}}{\left(\frac{\det[\dot{\gamma}_{ab}]}{4\sqrt{\gamma}}\right)_o}$$

Jacobi Map

Gauge Invariant Approach

- We express the results in terms of gauge invariant variables (the values perturbations acquire in the **GLC gauge**).



[Nugier, 1508.07464]

Gauge invariant expressions for light-like cosmological observables

↓ observations by a free-falling observer

↓ GR effects at first and second order

The Observational Synchronous Gauge

- With a **complete gauge fixing** at the **observer's position**:

IR divergences $\sim 1/r^n$ at the observer's position are eliminated in a **model independent way**

Observational Synchronous Gauge = standard counterpart of the GLC gauge

[Fanizza, Marozzi, Medeiros, Schiaffino, JCAP, 02 (2021) 014]

$\tilde{\theta}^a$ are directions of observations in the sky

The observer is free-falling

Angular Distance at First Order

- In terms of **Bardeen potentials**:

$$
\begin{aligned}
d_{\mathrm{A}}^{(1)}(z) = &-(\Psi_z^{\mathrm{I}} + \Psi_z^{\mathrm{A}}) + \left(1 - \frac{1}{r_z \mathcal{H}_z}\right)\left[\Psi_{\mathrm{o}}^{\mathrm{I}} - \Psi_{\mathrm{o}}^{\mathrm{A}} - (\Psi_z^{\mathrm{I}} - \Psi_z^{\mathrm{A}})\right] \\
&- \frac{1}{r_z}\int_{\eta_z}^{\eta_{\mathrm{o}}} \mathrm{d}\eta\, \frac{\eta - \eta_z}{\eta_{\mathrm{o}} - \eta} D^2 \Psi^{\mathrm{I}} + \frac{2}{r_z}\int_{\eta_z}^{\eta_{\mathrm{o}}} \mathrm{d}\eta\, \Psi^{\mathrm{I}} \\
&- \left(1 - \frac{1}{r_z \mathcal{H}_z}\right)\left(2\int_{\eta_z}^{\eta_{\mathrm{o}}} \mathrm{d}\eta\, \partial_\eta \Psi^{\mathrm{I}} + v_{||z}\right) - \frac{1}{r_z \mathcal{H}_z} v_{||\mathrm{o}} \\
&- \left(\mathcal{H}_{\mathrm{o}} - \frac{\mathcal{H}_{\mathrm{o}}}{r_z \mathcal{H}_z} + \frac{1}{r_z}\right)\frac{1}{a_{\mathrm{o}}}\int_{\eta_{\mathrm{in}}}^{\eta_{\mathrm{o}}} \mathrm{d}\eta\, a\Phi
\end{aligned}
$$

extra terms at the obs. such that $d_{\mathrm{A}}(z)$ is the one **measured by a free-falling obs.**

Summary and Outlook

Summary

The **full control** of observer terms at second order provides the gauge invariant formula for $d_{\mathrm{A}}(z)$ with GR effects and as seen by a free-falling observer (**new terms** not present in the literature).

Outlook

These **new tools** can be conveniently applied to compute other cosmological observables on the light-cone up to second order (*e.g.* the redshift drift).

The **cross-checking** of our results with the ones present in the literature provides a **validation** of the formalism.

Thanks for your attention!

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Backup Slides

More on the GLC Angles

- The **shift of direction** of received radiation wrt the angular direction of the source is expressed as [Fanizza, Gasperini, Marozzi, Veneziano, JCAP, 08 (2015) 020]

$$\theta^a_\mathrm{s} = \theta^a_\mathrm{s}(\theta^b_\mathrm{o}) , \quad a, b = 1,2$$

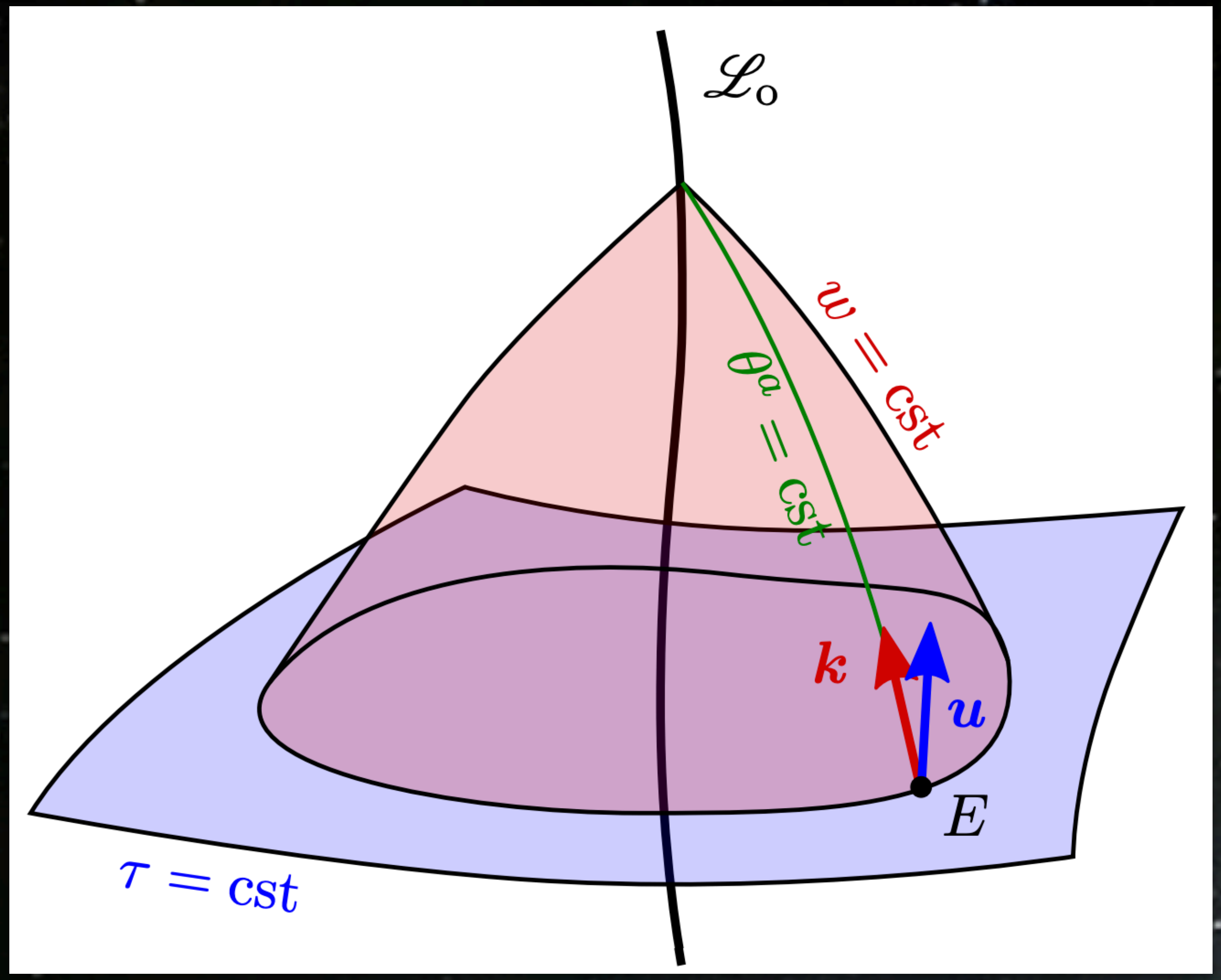
- This is a non-trivial relation because of **geometrical deflection effects**: the null light-cone hyper-surface has a distorted shape.

- It depends on the **geometry** and on the **gauge**.

- We can connect the **Poisson Gauge** and the **GLC one** via (the angles in the 2 gauges are demanded to be equal at the observer's position):

$$\theta^a_\mathrm{s} = \mathrm{GT}(\tilde{\theta}^a_\mathrm{s}) = \mathrm{GT}(\tilde{\theta}^a_\mathrm{o}) \equiv \mathrm{GT}(\theta^a_\mathrm{o})$$

Observed Redshift in GLC



- The **4-velocity** of a geodesic observer is

$$u^\mu = -\,\partial^\mu \tau\,, \qquad u^\mu u_\mu = -\,1$$

- The **wave-vector** of a in incoming photon is

$$k^\mu \propto \partial^\mu w\,, \qquad \partial^\mu w\,\partial_\mu w = 0$$

$$1 + z = \frac{\left(k^\mu u_\mu\right)_{\mathrm{S}}}{\left(k^\mu u_\mu\right)_{\mathrm{O}}} = \frac{\Upsilon\left(\tau_{\mathrm{O}}, w, \tilde{\theta}^a_{\mathrm{O}}\right)}{\Upsilon\left(\tau_{\mathrm{S}}, w, \tilde{\theta}^a_{\mathrm{S}}\right)}$$

$w_{\mathrm{S}} = w_{\mathrm{O}} \equiv w$

Scalar-PseudoScalar Decomposition

- Define the operators [Fanizza, Marozzi, Medeiros, Schiaffino, JCAP, 02 (2021) 014; Mitsou, Fanizza, Grimm, Yoo, Class. Quantum. Grav. 38 (2021) no. 5 055011]

$$D_{ab} \equiv D_{(a}D_{b)} - \frac{q_{ab}}{2}D^2 \,, \qquad \tilde{D}_{ab} \equiv \tilde{D}_{(a}D_{b)} \,, \qquad \tilde{D}_a \equiv \epsilon_a^{\ b} D_b$$

- Perturbations are decomposed according the their transformation properties under SO(2):

$$V_a^{(n)} = r^2\left[D_a v^{(n)} + \tilde{D}_a \hat{v}^{(n)}\right]$$
$$U_a^{(n)} = r^2\left[D_a u^{(n)} + \tilde{D}_a \hat{u}^{(n)}\right]$$
$$\gamma_{ab}^{(n)} = 2r^2\left[q_{ab}\nu^{(n)} + D_{ab}\mu^{(n)} + \tilde{D}_{ab}\hat{\mu}^{(n)}\right]$$

with scalar and pseudo-scalar variables ($D_a \hat{u} = 0$)

SPS-SVT Dictionary

- Use the **fully non-linear relations** to connect SVT perturbations to SPS ones:

$$\begin{cases} a^2 L = -2\left(\phi - \frac{1}{2}\mathcal{C}_{rr} - \mathcal{B}_r\right) \\ aM = -\mathcal{B}_{rr} - \mathcal{C}_{rr} \\ N = \mathcal{C}_{rr} \\ aV_a = -\mathcal{B}_a - \mathcal{C}_{ra} \\ U_a = \mathcal{C}_{ra} \\ \delta\gamma_{ab} = \mathcal{C}_{ab} \end{cases} \Rightarrow \begin{cases} \phi = -\frac{1}{2}(a^2 L + N + 2aM) \\ \mathcal{B}_r = -N - aM \\ \mathcal{C}_{rr} = N \\ \mathcal{B}_a = -U_a - aV_a \\ \mathcal{C}_{ra} = U_a \\ \mathcal{C}_{ab} = \delta\gamma_{ab} \end{cases}$$

- Example:

$$\tilde{\phi}^{(1)} = -\frac{1}{2}(a^2\tilde{L}^{(1)} + \tilde{N}^{(1)} + 2a\tilde{M}^{(1)}) = -\frac{1}{2}(a^2 L^{(1)} + N^{(1)} + 2aM^{(1)}) - \left(\partial_\tau + \frac{\partial_w}{a}\right)\xi^0_{(1)} = \phi^{(1)} - \frac{\partial_\eta(a\epsilon^\eta_{(1)})}{a}$$

The Jacobi Map (I)

- Take the geodesic deviation equation (with λ affine parameter along the curve):

$$\nabla_\lambda^2 \xi^\mu = R^\mu_{\ \alpha\beta\nu} k^\alpha k^\nu \xi^\beta \,, \qquad \nabla_\lambda \equiv k^\alpha \nabla_\alpha$$

- Project (for $A = 1,2$):

$$\xi^\mu = \xi^A s^\mu_A \,, \qquad \xi^A = \xi^\mu s_\mu^A = g_{\mu\nu} \xi^\mu s^\nu_A$$

where

$$g_{\mu\nu} s^\mu_A s^\nu_B = \delta_{AB} \,, \qquad s^\mu_A u_\mu = 0 = s^\mu_A k_\mu = \Pi^\mu_\nu \nabla_\lambda s^\nu_A$$

$$\Pi^\mu_\nu \equiv \delta^\mu_\nu - \frac{k^\mu k_\nu}{(u^\rho k_\rho)^2} - \frac{k^\mu u_\nu + u^\mu k_\nu}{u^\rho k_\rho}$$

The Jacobi Map (II)

- We obtain

$$\frac{d^2\xi^A}{d\lambda^2} = R^A_B \xi^B , \qquad R^A_B \equiv R_{\alpha\beta\mu\nu} k^\alpha k^\nu s^\beta_B s^\mu_A$$

where

$$\frac{d}{d\lambda} \equiv k^\mu \partial_\mu \boxed{= k^\tau \partial_\tau}$$

in the GLC gauge

- The **Jacobi Map** connects an observer to a source and it is the solution to the above equation written as

$$\xi^A(\lambda_S) = J^A_B(\lambda_O, \lambda_S) \left(\frac{k^\mu \partial_\mu \xi^B}{k^\nu u_\nu} \right)_O$$

Fixing the GLC Gauge

- Decompose the angular gauge mode in terms of SPS gauge modes:

$$\xi^a_{(1,2)} = q^{ab}\left(D_b\chi_{(1,2)} + \tilde{D}_b\hat{\chi}_{(1,2)}\right) , \qquad D_b\hat{\chi}_{(1,2)} = 0$$

- **Fix the GLC gauge on the light-cone** order by order in perturbation theory:

$$\tilde{L}^{(1)} = 0 = \tilde{v}^{(1)} = \tilde{\hat{v}}^{(1)} = \tilde{N}^{(1)} + 2a\tilde{M}^{(1)}$$

$$\tilde{L}^{(2)} = 0 = \tilde{v}^{(2)} = \tilde{\hat{v}}^{(2)} = 4\tilde{N}^{(2)} - (\tilde{N}^{(1)})^2 + 8a\tilde{M}^{(2)} - 4(\tilde{U}^{(1)})^2$$

First Order Gauge Modes

$$\xi^\tau_{(1)} = -\frac{1}{2}\int_{\tau_{\rm in}}^{\tau} \mathrm{d}\tau' \left(a^2 L^{(1)} + N^{(1)} + 2aM^{(1)}\right)\left(\tau', w - \eta(\tau) + \eta(\tau')\right) \; ,$$

$$\xi^w_{(1)} = \frac{1}{2}\int_{\tau}^{\tau_o} \mathrm{d}\tau' a L^{(1)} + w_0^{(1)}(w, \tilde\theta^a) ,$$

$$\chi_{(1)} = -\int_{\tau}^{\tau_o} \mathrm{d}\tau' \left(v^{(1)} + \frac{1}{2ar^2}\int_{\tau'}^{\tau_0} \mathrm{d}\tau'' a L^{(1)} + \frac{w_0^{(1)}}{ar^2}\right) + \chi_0^{(1)}(w, \tilde\theta^a) ,$$

$$\hat\chi_{(1)} = -\int_{\tau}^{\tau_o} \mathrm{d}\tau' \, \hat v^{(1)} + \hat\chi_0^{(1)}(w, \tilde\theta^a)$$

$$w \to w' = w'(w)$$
$$\tilde\theta^a \to \tilde\theta^{a'} = \tilde\theta^{a'}(w, \tilde\theta^{a'})$$

residual gauge freedom of the GLC gauge at the observer

Gauge Fixing at the Observer's Position

$r_o = w_o - \eta(\tau_o) = 0$ ⟷ observer sit at the center of the polar frame

Preserve the "observational gauge"

$$\xi^w_{(1,2)} \equiv w_0^{(1,2)} = \left(\epsilon^\eta_{(1,2)}\right)_o = \frac{\left(\xi^\tau_{(1,2)}\right)_o}{a_o}$$

No angular dependence at the observer's position

$$\chi_0^{(1,2)} = \chi_0^{(1,2)}(w)\,, \qquad \hat{\chi}_0^{(1,2)} = \hat{\chi}_0^{(1,2)}(w)$$