Lattice QCD and its Phenomenological Applications

Extending the Range and Improving the Precision of Flavour Physics

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GGI 20th Anniversary

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1. Introduction

- In this talk I will discuss a number of important topics in Lattice QCD Phenomenology to illustrate the improvement in precision and the extension in the range of physical quantities which can now be studied over the last ~ 20 years.

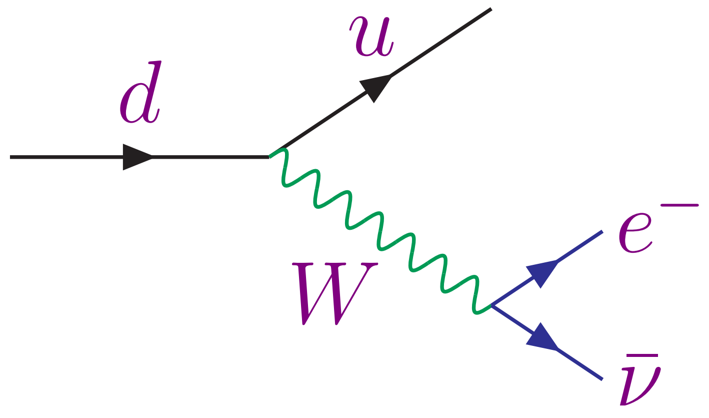
<u>Plan of the Talk</u>

1. Introduction
2. QED Corrections to Weak Decay Amplitudes
3. $K \to \pi\pi$ decays ($\Delta I = 1/2$ rule and ϵ'/ϵ)
4. The anomalous magnetic moment of the muon
5. Spectral Density Methods
6. Conclusions

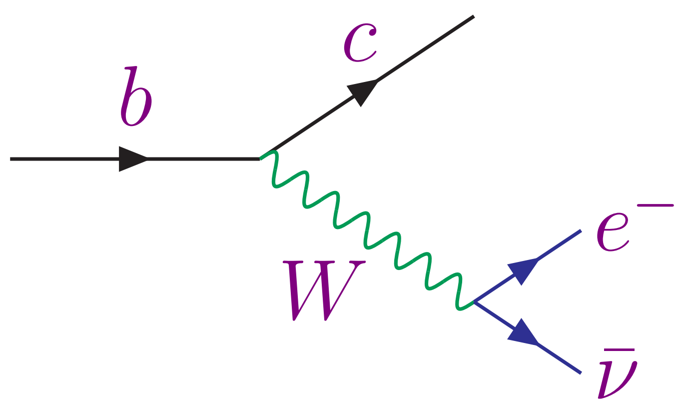
- There are, of course, a huge number of other topics, e.g. in spectroscopy, hadronic structure or even other flavour physics quantities which might have been discussed.

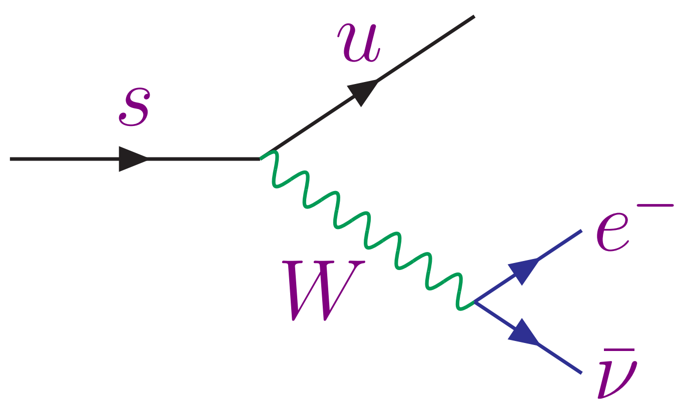
Brief introduction to Flavour Physics - Generalised β-decays

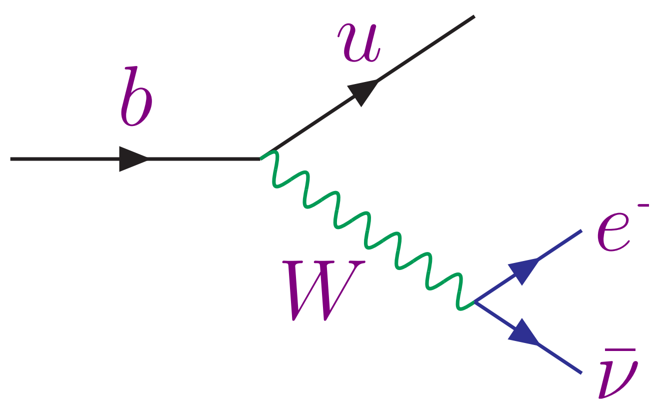
- At the level of quarks we understand nuclear β-decay in terms of the fundamental process:



- With the 3 generations of quarks and leptons in the standard model this is generalised to other *charged current* processes, e.g.:



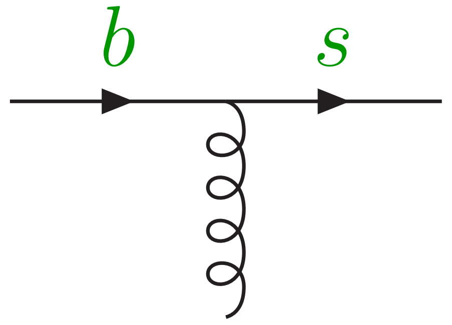


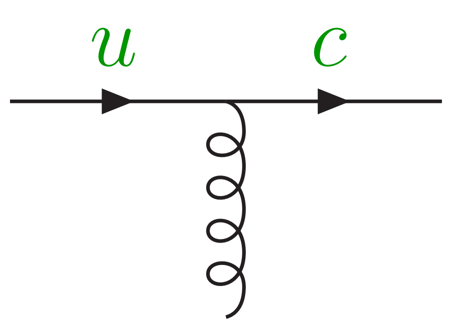


- Weak interaction eigenstates $\neq$ mass eigenstates.
- The weak charged current is of the form: $$J_\mu^+ = (\bar{u}, \bar{c}, \bar{t})_L \gamma_\mu V_{\mathrm{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \equiv (\bar{u}, \bar{c}, \bar{t})_L \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$
- CKM matrix has 4 independent parameters $\Rightarrow$ existence of a CP-violating phase.
- There are > 100 processes with which to overdetermine the 4 parameters and to check for consistency.

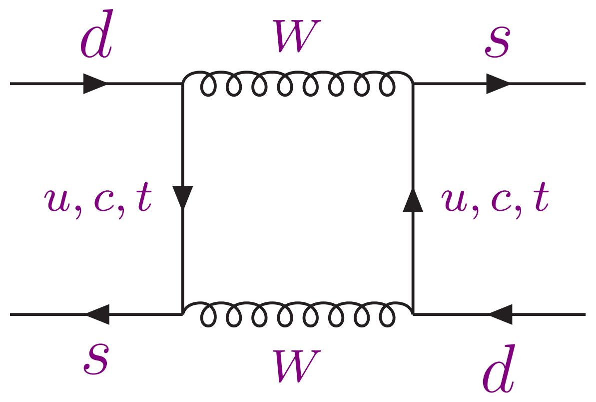
Flavour-Changing Neutral Currents (FCNC)

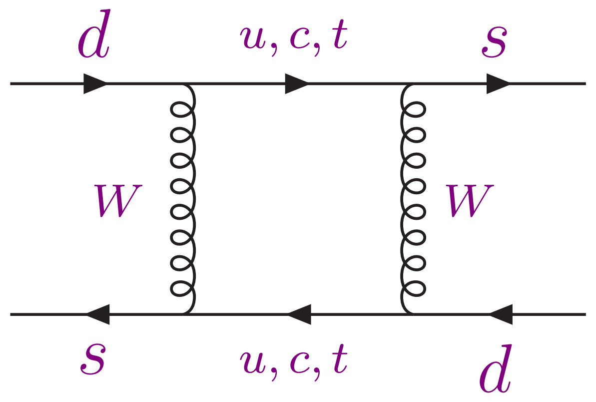
- In the SM, there are no FCNC reactions at tree level, i.e. no vertices such as:



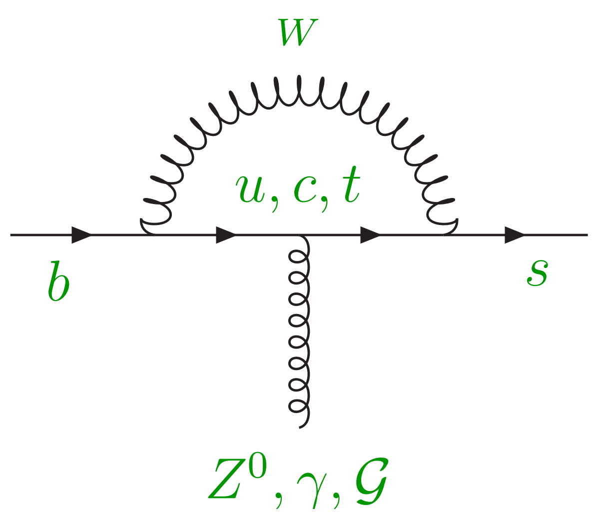


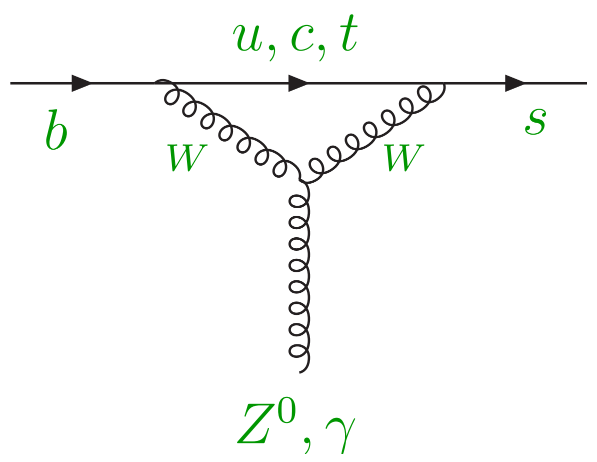
- Quantum loops, however, can generate FCNC reactions, through *box* or *penguin* diagrams, e.g. for K^0 - $\bar{K}^0$ mixing:





- Examples of penguin diagrams relevant for $b \to s$ decays:





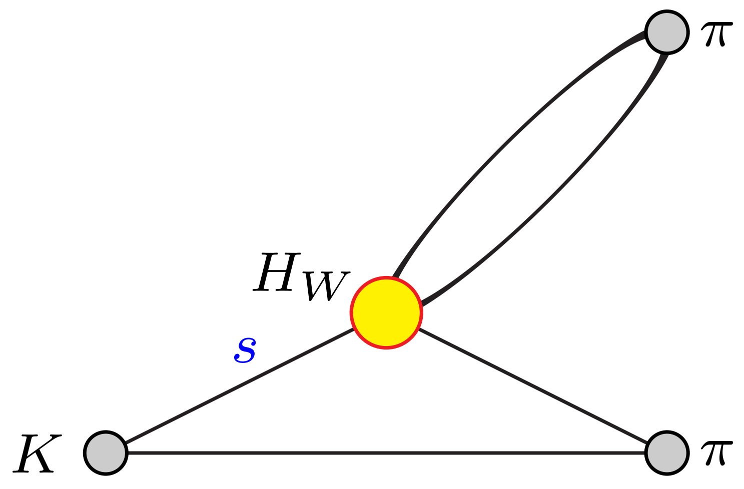
- Since FCNC processes are *rare* in the SM, they provide an excellent laboratory for searches for new physics.

Precision Flavour Physics

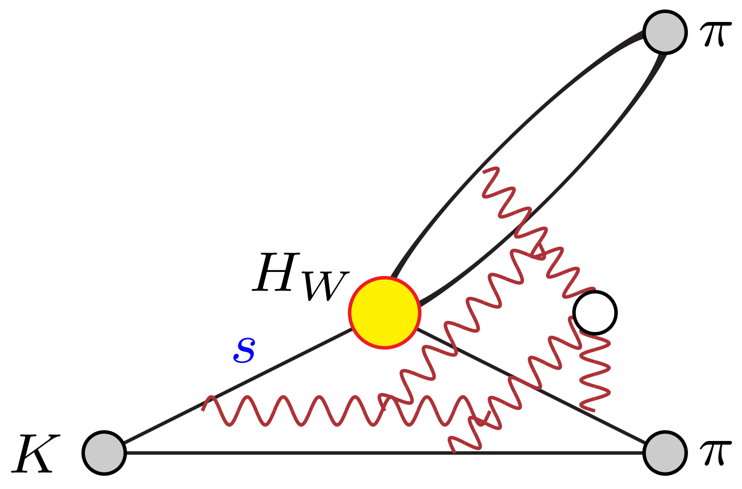
- Flavour physics, weak interaction processes in which the flavour (u, d, s, c, b, t) quantum number changes, is a key tool in exploring the limits of the Standard Model of Particle Physics and in searches for new physics.

- It is complementary to high-energy experiments (most notably the LHC).
  - Should the LHC experiments discover new elementary particles BSM, then precision flavour physics will be necessary to understand the underlying framework.
  - The discovery potential of precision flavour physics should also not be underestimated. (In principle, the reach is about two-orders of magnitude deeper than the LHC!

- Precision flavour physics requires control of hadronic effects for which lattice QCD simulations are essential.

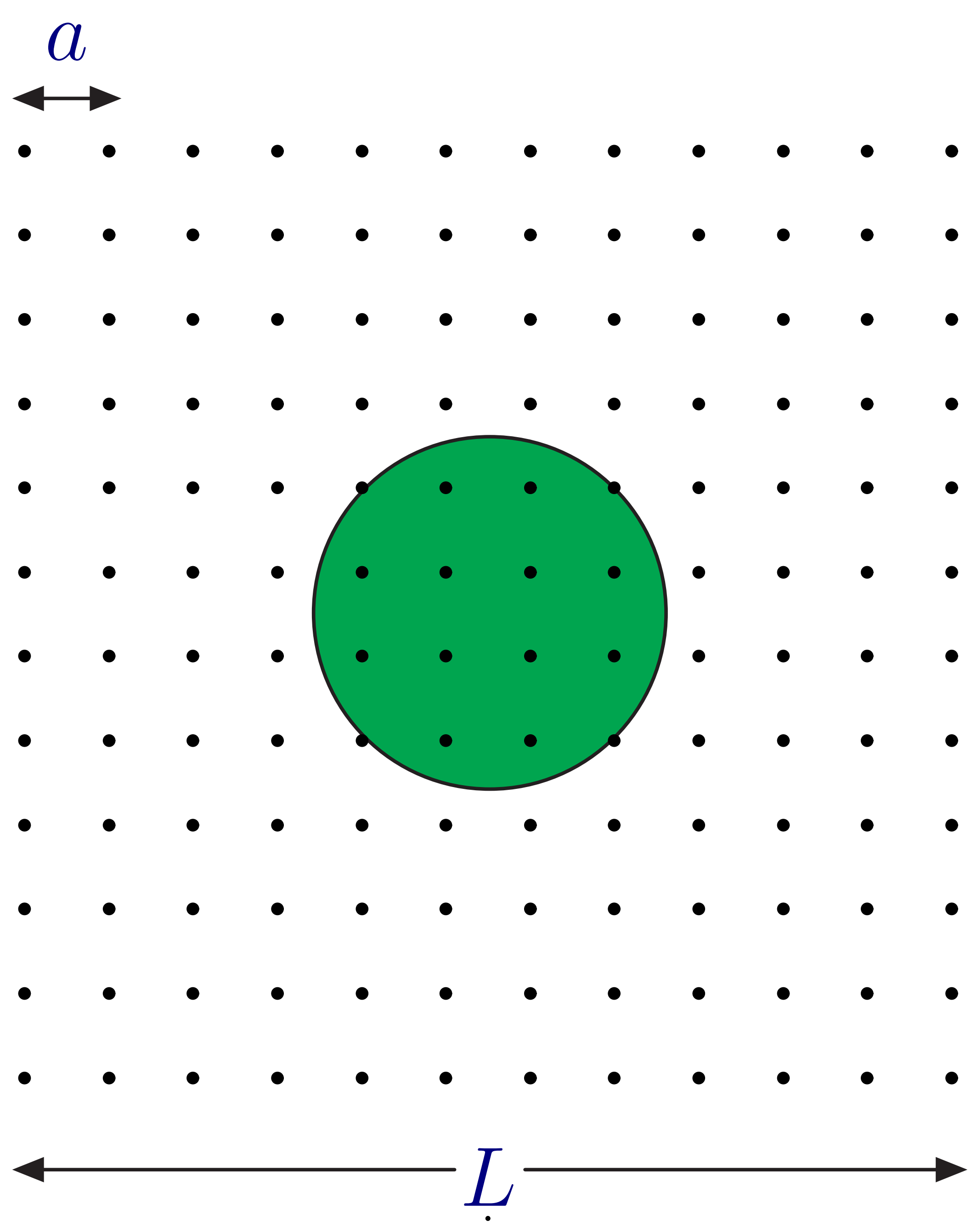


means



- While no unambiguous inconsistencies have appeared to date, there are a number of intriguing tensions!

Introduction to Lattice QCD



- Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$< 0 \,|\, O(x_1, x_2, \cdots, x_n) \,|\, 0 > = \frac{1}{Z} \int [dA_\mu]\,[d\psi]\,[d\bar{\psi}]\, e^{-S}\, O(x_1, x_2, \cdots, x_n)$$

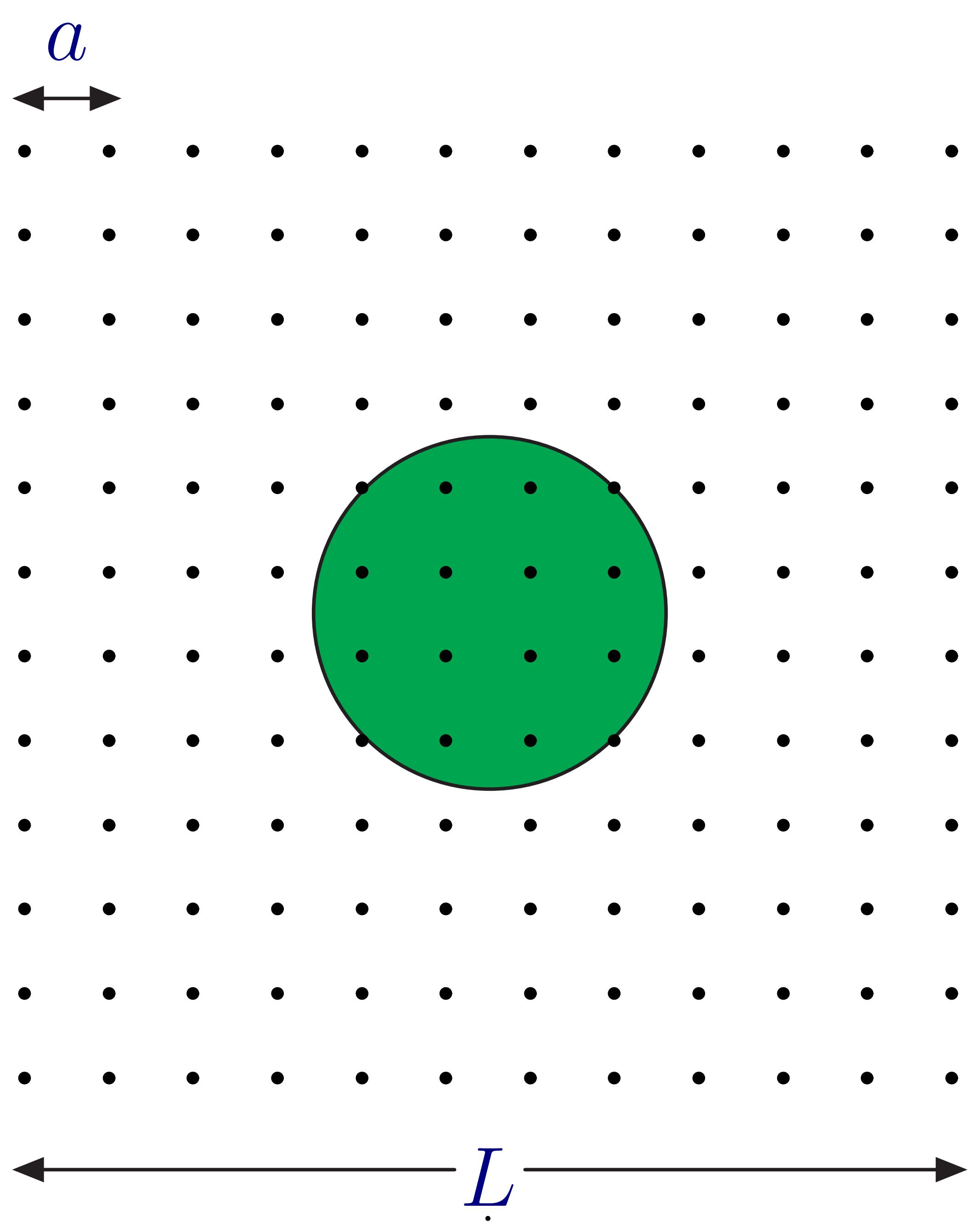
where $O(x_1, x_2, \cdots, x_n)$ is a multifocal operator composed of quark and gluon fields and Z is the partition function.

- The physics which can be studied depends on the multilocal operator $O(x_1, x_2, \cdots, x_n)$.

- The functional integral is performed by discretising Euclidean space-time and using Montecarlo integration.

- For the purposes of this talk, I assume that a suitable discretisation has been chosen and that the correlation functions can be computed.

Introduction to Lattice QCD (cont.)



- Lattice QCD is a general *first-principles* technique used to compute non-perturbative QCD effects in a huge variety of applications.

- In principle the systematic errors are controllable, and can be progressively reduced.

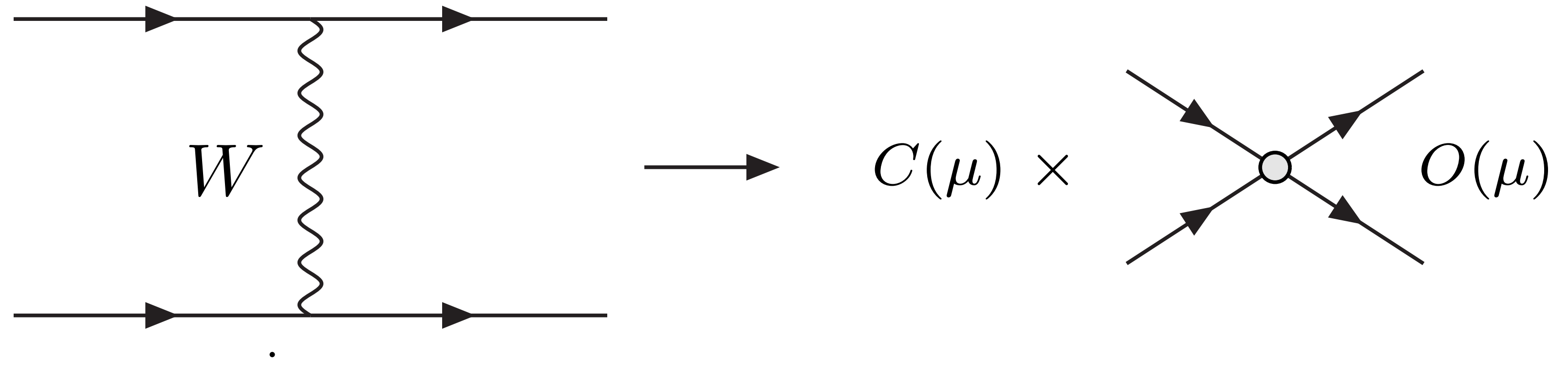
i) Continuum extrapolation $a \to 0$.

ii) Extrapolation to infinite-volume $L \to \infty$.

iii) Minkowski → Euclidean continuation.

- For some simple quantities in spectroscopy and flavour physics, the M→E continuation is not an issue, the discretisation and finite-volume effects are under control and results can be obtained with a precision at the sub-percent level.

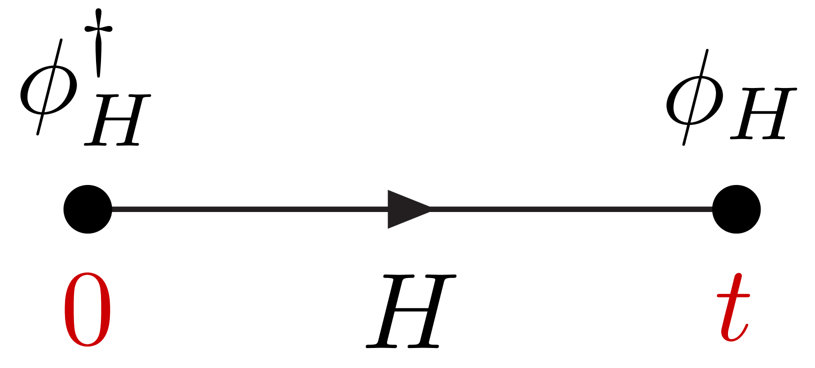
- The lattice spacing a (typically $0.05 - 0.1$ fm) is far too large to allow for propagating W,Z - bosons ⇒ use the Operator Product Expansion.



$C(\mu)$ - perturbative

Matrix element of $O(\mu)$ non-perturbative

Correlation functions



$$C_2(t) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\, \langle 0|\phi_H(t,\vec{x}\,)\,\phi_H^\dagger(0)\,|0\rangle\; = \sum_n \int d^3x\; e^{i\vec{p}\cdot\vec{x}} \langle 0|\phi_H(t,\vec{x}\,)|n\rangle\, \langle n|\phi_H^\dagger(0)|0\rangle$$

$$= \frac{1}{2E}\; e^{-iEt}\langle 0|\phi_H(0)|H(\vec{p}\,)\rangle\, \langle H(\vec{p}\,)|\phi_H^\dagger(0)|0\rangle + \cdots \qquad \left(E = \sqrt{m_H^2 + \vec{p}^{\,2}}\,\right)$$

$$\Rightarrow \frac{1}{2E}\; e^{-Et}\langle 0|\phi_H(0)|H(\vec{p}\,)\rangle\, \langle H(\vec{p}\,)|\phi_H^\dagger(0)|0\rangle + \cdots \qquad \text{in Euclidean space}$$

- H is the lightest state which can be created by $\phi_H^\dagger$ and the ellipses represent the contributions from heavier states.
- By fitting $C_2(t)$ as a function of t, both E (or m_H if $\vec{p} = 0$) and $|< 0\,|\,\phi_H(0)\,|\,H >|$ can be determined.
- For example: if $\phi_H = \bar{s}\gamma^\mu\gamma^5 u$, then f_K can be evaluated, $|\langle 0\,|\,\bar{s}\gamma^\mu\gamma^5 u\,|\,K^+\rangle\,| = f_K p^\mu$.
- For many current studies in flavour physics correlation functions of 3 or more operators need to be computed.

Calibration

- $N_f = 2 + 1 + 1$ isosymmetric QCD has four parameters, the three independent quark masses, $m_u = m_d, m_s, m_c$, and the coupling constant g.
    - 4 predictions therefore have to be sacrificed to determine the "physical" values of these parameters.
    - For the coupling constant we use "dimensional transmutation" to trade $g(a)$ for the lattice spacing a.
    - For example, one might require that $m_{\pi^0}, m_{K^0}, m_{D^+}$ and m_Ω take their physical values.
        - Of course, we only know how well we have done a posteriori.
        - We now have considerable experience to be able to tune the parameters with good precision.
- Once these parameters have been tuned, the evaluation of all other quantities constitute "predictions".
- The quark masses determined in this way, are "bare" masses corresponding to the lattice discretisation being used. They need to be renormalised.
- Given the remarkable precision we will see in the FLAG table, the evaluation of isospin-breaking corrections, including electromagnetic contributions, is currently an important area of investigation.

Some Recent Results from FLAG

FLAG = Flavour Physics Lattice Averaging Group

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	$N_F = 2+1$
m_{ud}[MeV]	4.1.1	3.427(51)	[7–9]	3.387(39)	3.43(11)
m_s[MeV]	4.1.1	93.46(58)	[7–9, 17, 18]	92.4(1.0)	94(3)
m_s/m_{ud}	4.1.2	27.227(81)	[7, 8, 20, 21]	27.42(12)	27.4(4)
m_u[MeV]	4.1.3	2.14(8)	[9, 23]	2.27(9)	2.19(15)
m_d[MeV]	4.1.3	4.70(5)	[9, 23]	4.67(9)	4.67(20)
m_u/m_d	4.1.3	0.465(24)	[23, 25]	0.485(19)	0.47(4)
$\overline{m}_c$(3 GeV)[GeV]	4.2.2	0.989(10)	[7–9, 18, 26, 27]	0.991(6)	
m_c/m_s	4.2.3	11.766(30)	[7–9, 18]	11.82(16)	
$\overline{m}_b(\overline{m}_b)$[GeV]	4.3	4.200(14)	[9, 34–37]	4.171(20)	
$f_+(0)$	5.3	0.9698(17)	[38, 39]	0.9677(27)	0.956(8)
$f_{K^\pm}/f_{\pi^\pm}$	5.3	1.1934(19)	[20, 42–45]	1.1917(37)	1.193(5)
$f_{\pi^\pm}$[MeV]	5.6			130.2(8)	
$f_{K^\pm}$[MeV]	5.6	155.7(3)	[21, 42, 43]	155.7(7)	
$\mathrm{Re}(A_2)$[GeV]	6.2			$1.50(4)(14) \times 10^{-8}$	
$\mathrm{Im}(A_2)$[GeV]	6.2			$-8.34(1.03) \times 10^{-13}$	
$\hat{B}_K$	6.3	0.717(18)(16)	[52]	0.7533(91)	0.738(20)
B_2	6.4	0.46(1)(3)	[52]	0.488(15)	
B_3	6.4	0.79(6)	[52]	0.757(27)	
B_4	6.4	0.78(2)(4)	[52]	0.903(14)	
B_5	6.4	0.49(3)(3)	[52]	0.691(14)	

Additional information needed since computations were performed with $m_u = m_d$

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268

1st FLAG Review , G.Colangelo et al., arXiv:1011.4408

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Additional information needed since computations were performed with $m_u = m_d$

- Between 2002 and 2018, together with A.Manohar, I was the reviewer on quark masses for the Particle Data Group.

  In 2002 I quoted:

  $$m_{ud} = 4.2 \pm 1.0\,\text{MeV}, \quad m_s = 105 \pm 25\,\text{MeV}\,.$$

  PDG(2002) Review 2024, K.Hagiwara et al., PRD 66(2002) 010001

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268 (31 Authors)

1st FLAG Review, G.Colangelo et al., arXiv:1011.4408 (12 authors)

2. QED corrections to Weak Decay Amplitudes

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_{ud}[MeV]	4.1.1	3.427(51)	[7–9]	3.387(39)	[10–16]		
m_s[MeV]	4.1.1	93.46(58)	[7–9, 17, 18]	92.4(1.0)	[11–15, 19]		
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m_u[MeV]	4.1.3	2.14(8)	[9, 23]	2.27(9)	[24]		
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m_u/m_d	4.1.3	0.465(24)	[23, 25]	0.485(19)	[24]		
$\overline{m}_c$(3 GeV)[GeV]	4.2.2	0.989(10)	[7–9, 18, 26, 27]	0.991(6)	[15, 28–32]		
m_c/m_s	4.2.3	11.766(30)	[7–9, 18]	11.82(16)	[29, 33]		
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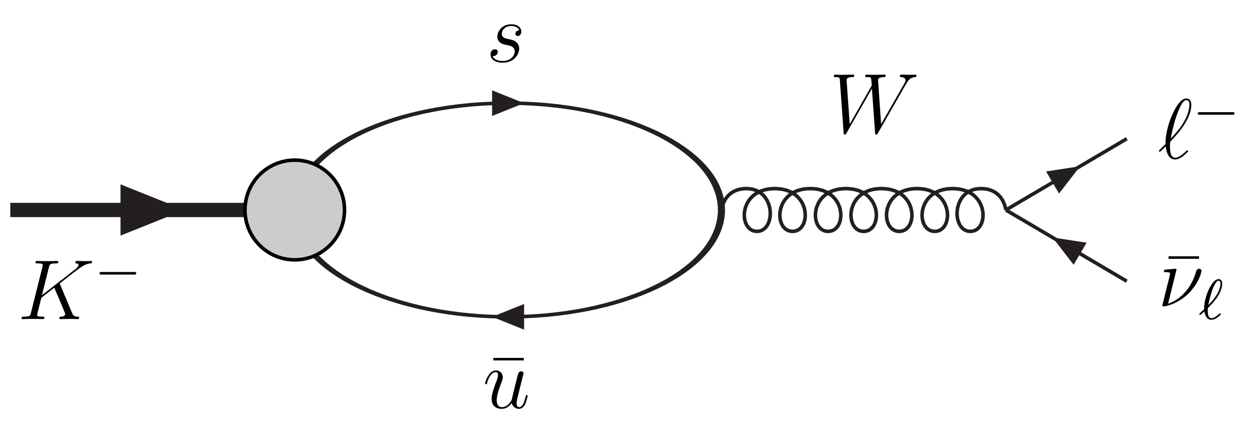
B_4	6.4	0.78(2)(4)	[52]	0.903(14)	[55, 56, 58]	0.76(2)(2)	[57]
B_5	6.4	0.49(3)(3)	[52]	0.691(14)	[55, 56, 58]	0.58(2)(2)	[57]

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268

- Lattice QCD results for some physical quantities are now so precise (sub percent) that QED corrections need to be included to make further progress.
- A good example is:

$$f_K = 155.7(3)\ \text{MeV}$$



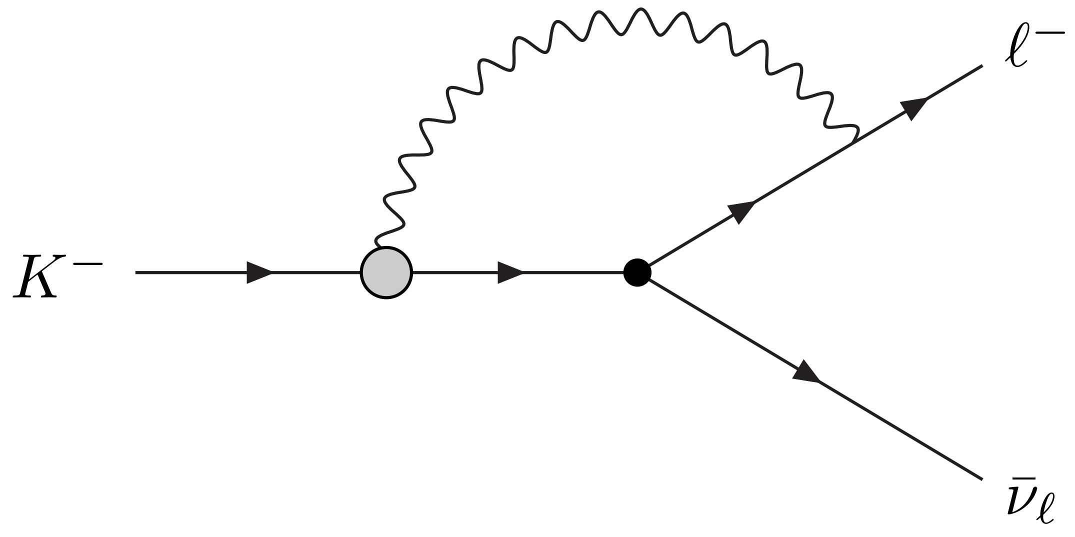
$$\langle 0|A_\mu|K(p)\rangle = f_K p_\mu\,, \qquad \Gamma^{(0)} = \frac{G_F^2|V_{us}|^2 f_K^2}{8\pi} m_K^3 r_\ell^2 \left(1-r_\ell^2\right)^2 \qquad r_\ell = m_\ell/m_K$$

Computing QED Corrections to Weak Decay Amplitudes - The Framework

QED Corrections to Hadronic Processes in Lattice QCD
N.Carrasco, V.Lubicz, G.Martinelli, CTS, N.Tantalo, C.Tarantino and M.Testa
arXiv:1502.00257

- Our aim is to calculate Γ including $O(\alpha_{\rm em})$ corrections.

- f_K no longer contains all the QCD effects.



- Calculating electromagnetic corrections to decay amplitudes has the major complication, not present in computations of the spectrum,
  the presence of infrared divergences

- This implies that when studying such processes , the physical observable must include soft photons in the final state.
  F.Bloch and A.Nordsieck, PR 52 (1937) 54

$$\Gamma(K^- \to \ell^-\bar\nu_\ell(\gamma)) = \Gamma(K^- \to \ell^-\bar\nu_\ell) + \Gamma(K^- \to \ell^-\bar\nu_\ell\gamma) \equiv \Gamma_0 + \Gamma_1\,.$$

- The generic question is how best to combine this understanding with lattice calculations of non-perturbative hadronic effects.

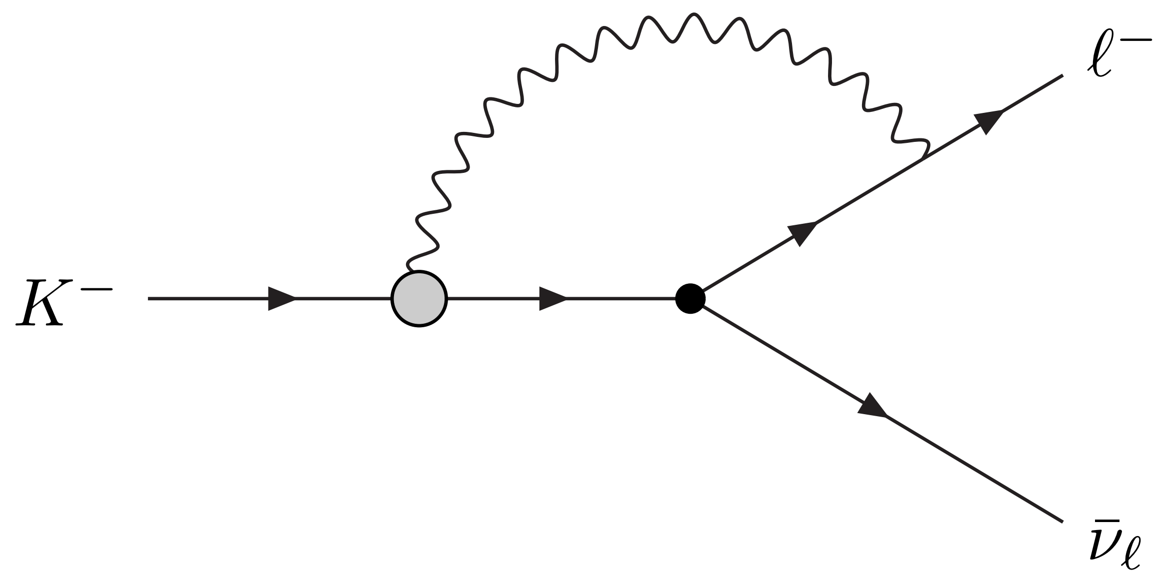
The Framework (cont.)

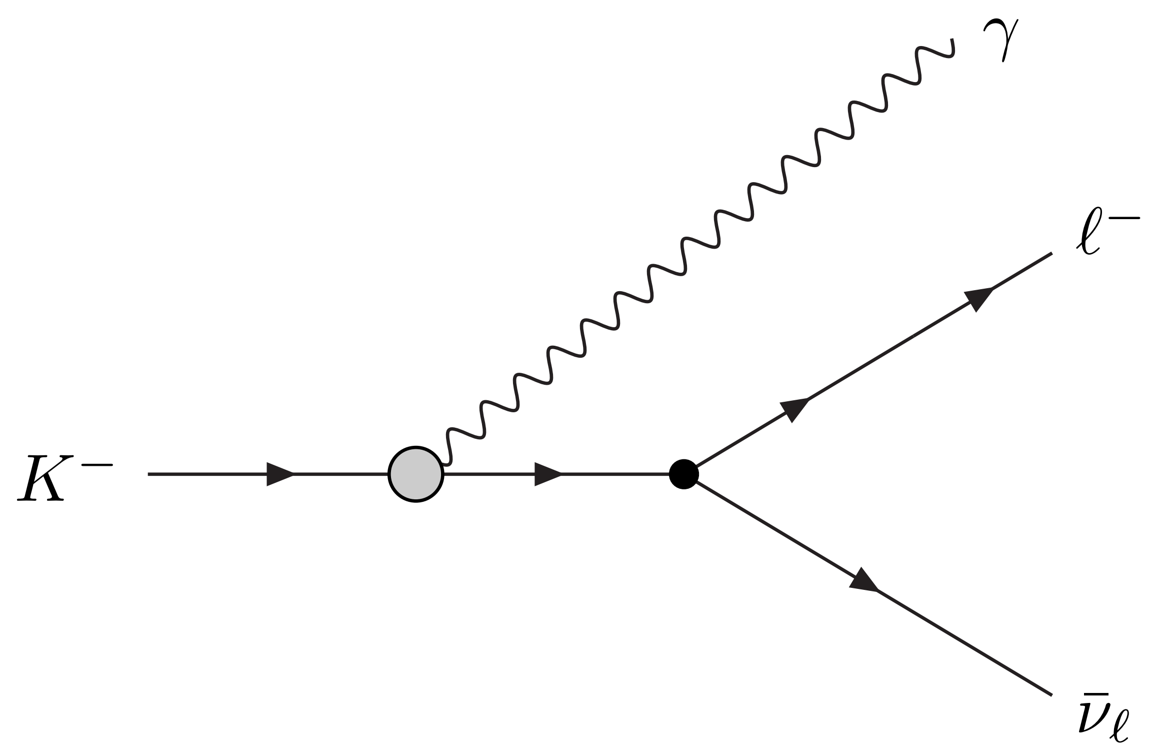
- Our proposal is to separate $\Gamma_0 + \Gamma_1$ into terms each of which is infrared convergent

$$\Gamma(\Delta E_\gamma) = \Gamma_0 + \Gamma_1(\Delta E_\gamma) = \Gamma_0 + \int_0^{2\Delta E_\gamma/m_P} dx_\gamma \, \frac{d\Gamma_1}{dx_\gamma}$$

$$= \lim_{L\to\infty} \left[\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right] + \lim_{\mu_\gamma\to 0} \left[\Gamma_0^{\mathrm{pt}}(\mu_\gamma) + \Gamma_1^{\mathrm{pt}}(\Delta E_\gamma, \mu_\gamma)\right] + \Gamma_1^{\mathrm{SD}}(\Delta E_\gamma) + \Gamma_1^{\mathrm{INT}}(\Delta E_\gamma) \, .$$

- $x_\gamma = 2E_\gamma/m_K$ in the rest frame of the kaon
- pt = "point like", SD = "Structure Dependent" and "INT" is the interference between pt and SD .
- "pt" contributions can be calculated in perturbation theory, whereas $\Gamma_0(L)$ and (for large ΔE_γ) Γ_1^{SD} and Γ_1^{INT} need to be computed non perturbatively.





Issues not discussed here

- When including QED, questions such as "**What is QCD?**" or equivalently "**How large are the electromagnetic corrections?**" are convention dependent due to the electromagnetic shift in the quark masses.

Light-meson leptonic decay rates in lattice QCD+QED
M.Di Carlo, D Giusti, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo , S.Simula and N.Tantalo, arXiv:1904.08731

- Definition of G_F at $O(\alpha_{\rm em})$. This must be consistent with the procedure being used.

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1-\frac{8m_e^2}{m_\mu^2}\right]\left[1+\frac{\alpha_{\rm em}}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right]$$

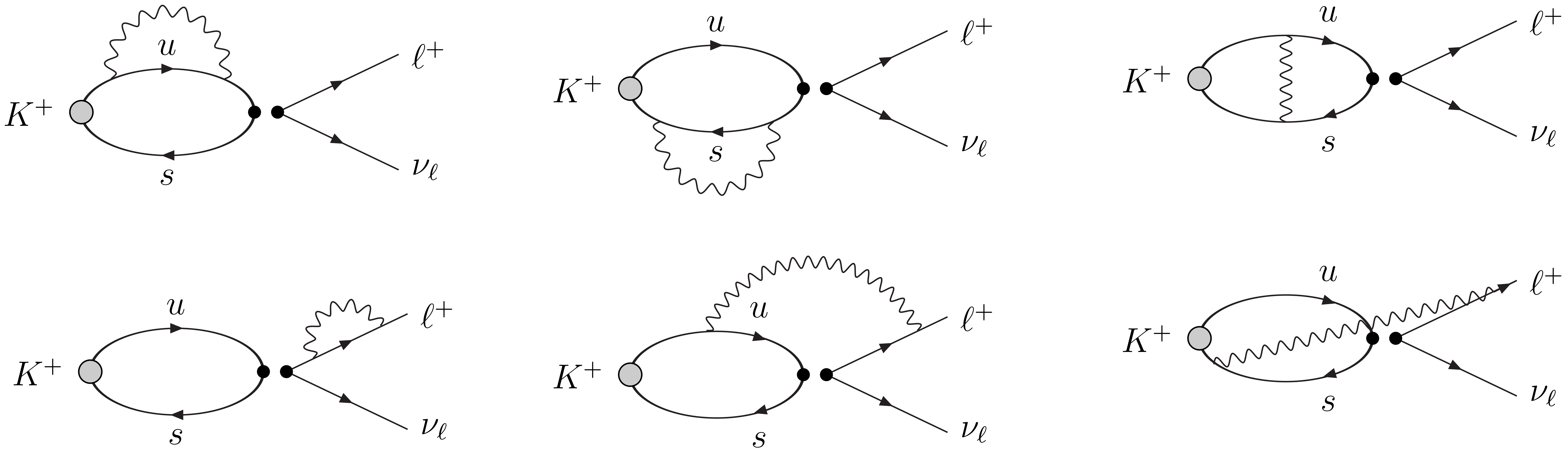
- Renormalization of the lattice operators including $O(\alpha_{\rm em})$ effects.

Non-perturbative renormalization in QCD+QED and its application to weak decays
M.Di Carlo, D Giusti, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo , S.Simula and N.Tantalo, arXiv:1911.00938

- Perturbative evaluation of $\Gamma_0^{\rm pt} + \Gamma_1^{\rm pt}(\Delta E_\gamma)$.

QED Corrections to Hadronic Processes in Lattice QCD
N.Carrasco, V.Lubicz, G.Martinelli, CTS, N.Tantalo, C.Tarantino and M.Testa, arXiv:1502.00257

- Evaluation of the diagrams.



+ disconnected diagrams + real photon emission

Finite-Volume Corrections

Finite-Volume QED corrections to decay amplitudes in lattice QCD
V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo, arXiv:1611.08497

- The photon is massless ⇒ difficulties in a finite volume.

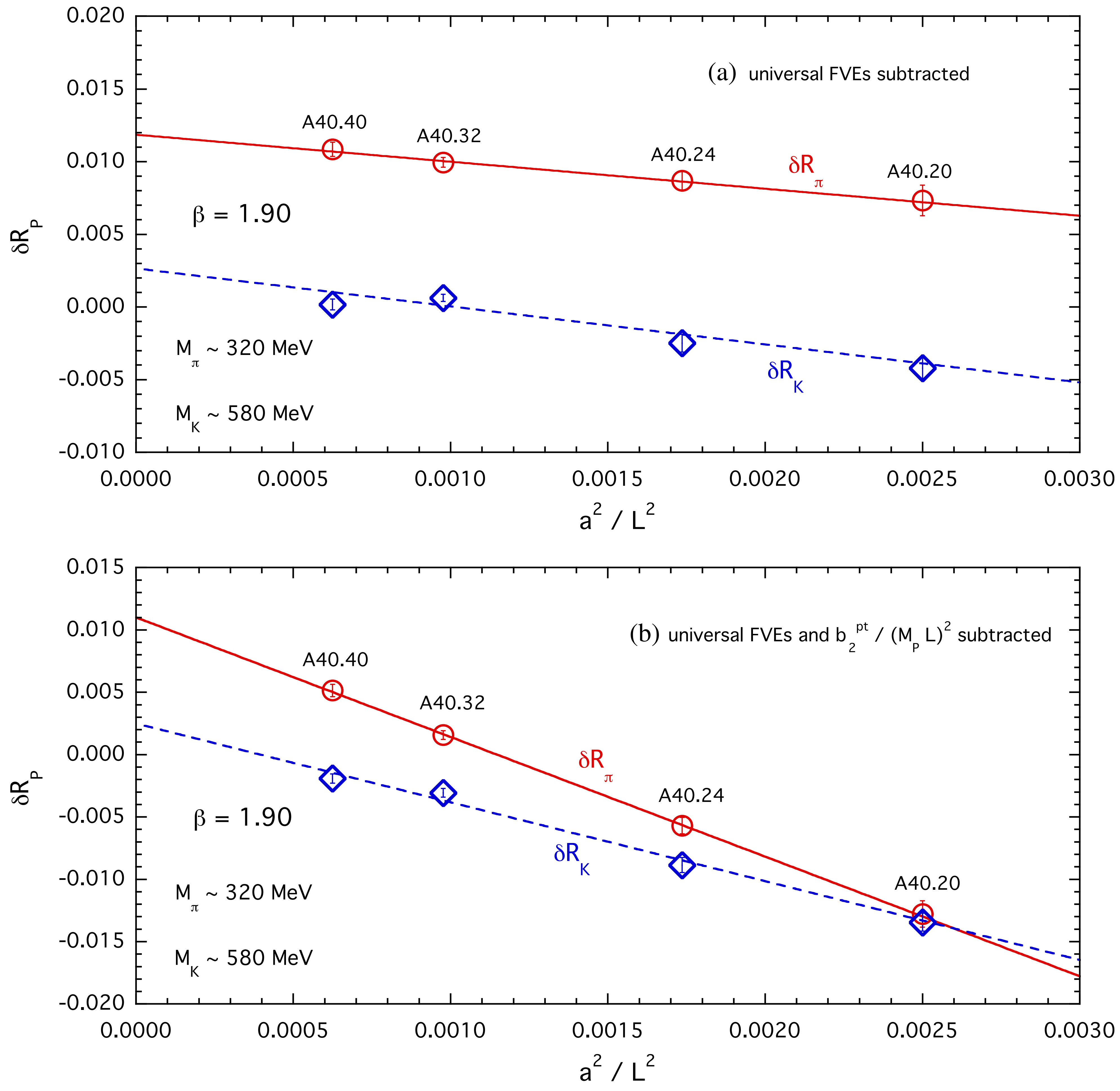
- We have implemented the framework in $\mathrm{QED_L}$ in which $A_\mu(\mathbf{k}=0,k_4)=0$ for all k_4. M.Hayakawa and S.Uno arXiv:0804.2044
  - Transfer matrix exists but locality is broken.
  - $L\to\infty$ limit should be taken first.

- Evaluation of FV effects is based on the Poisson Summation formula, e.g. in one dimension

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} f(p_n^2)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,f(p^2)+\sum_{n\neq 0}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,f(p^2)\,e^{inpL}\,.$$

- For decay constants, form factors, etc. the FV effects decrease exponentially $\propto \exp[-cm_\pi L]$.
- This is not the case when $f(p^2)$ has a singularity.

- For decay amplitudes, the FV corrections take the form: $\Gamma_0^{\rm pt}(L)=C_0(r_\ell)+\tilde C_0(r_\ell)\log\left(m_K L\right)+\dfrac{C_1(r_\ell)}{m_K L}+\ldots$ where $r_\ell=m_\ell/m_K$.
  - The exhibited L-dependent terms are *universal*, i.e. independent of the structure of the meson!
  - We have evaluated these coefficients in $\mathrm{QED_L}$.
  - The leading structure-dependent FV effects in $\Gamma_0-\Gamma_0^{\rm pt}$ are of $O(1/L^2)$.



Light-meson leptonic decay rates in lattice QCD+QED
M.Di Carlo, D Giusti, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo , S.Simula and N.Tantalo,
arXiv:1904.08731

- Finite-volume behaviour of 4-points, obtained at the same value of β and quark masses using ETMC twisted mass ensembles.
  - The universal $O(1/L)$ terms have been subtracted.
  - The leading SD finite-volume terms appear to be of $O(1/L^2)$ as expected.

- However, it has subsequently been shown that the point-like $O(1/L^3)$ terms are not negligible together with an argument that the SD $O(1/L^2)$ terms are very small.

  M.Di Carlo, M.T.Hansen, A.Portelli and N.Hermansson -Truedsson
  Phys. Rev. D105 (2022) 074509

- This has motivated the Edinburgh group to modify $\mathrm{QED_L}$ by redistributing the photon's zero mode over shells around $\mathbf{k} = 0$, which removes/reduces the $O(1/L^3)$ corrections pushing them to higher orders. *Numerical studies in progress.*

  M.Di Carlo, arXiv:2401.07666

QED Corrections to V_{us}

- Writing

$$\frac{\Gamma(K_{\mu 2})}{\Gamma(\pi_{\mu 2})} = \left| \frac{V_{us}}{V_{ud}} \frac{f_K^{(0)}}{f_\pi^{(0)}} \right|^2 \frac{m_\pi^3}{m_K^3} \left(\frac{m_K^2 - m_\mu^2}{m_\pi^2 - m_\mu^2} \right)^2 (1 + \delta R_{K\pi}),$$

where $m_{K,\pi}$ are the physical masses, using numerous twisted mass ensembles we find

$$\delta R_{K\pi} = -0.0126(14) \qquad \left[\delta R_\pi = +0.0153(19),\ \delta R_K = +0.0024(10)\right]$$

- $f_P^{(0)}$ are the decay constants obtained in iso-symmetric QCD with the renormalized $\overline{\text{MS}}$ masses and coupling equal to those in the full QCD+QED theory extrapolated to infinite volume and to the continuum limit.

  - Using ChPT, $\delta R_\pi = +0.0176(21),\ \delta R_K = +0.0064(24).$ PDG(2018)
  - Boyle et al. $\delta R_{K\pi} = -0.0086(3)_{\text{stat}} \left({}^{+11}_{-4}\right)_{\text{fit}} (5)_{\text{disc}} (5)_{\text{quench}} (39)_{\text{FV}}$ arXiv:2211.12865

QED Corrections to V_{us} (cont.)

- We obtained

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(46)\,.$$

- Taking $V_{ud} = 0.97420(21)$ (J.Hardy and I.S.Towner, CKM(2016) 028) $\Rightarrow$ $V_{us} = 0.22538(46)$ and with $|V_{ub}| = 0.00413(49)$,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99988(46)\,.$$

- However, taking $|V_{ud}| = 0.97370(14)$ (C.Y.Seng et al., arXiv:1807.10197), $|V_{us}| = 0.22526(46)$,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99885(34)\,.$$

- The latest PDG value is $V_{ud} = 0.97373(31)$, which is the average of the 15 most precise determinations and with a more conservative error. (Unitarity within 2σ: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9989(7)\,.$)

Infinite-volume reconstruction

- IVR is an idea by Xu Feng and Luchang Jin originally introduced to avoid non-exponential FV effects in calculations of QED corrections to the spectrum. X.Feng and L.Jin, arXiv:1812.09817

- We have been extending the technique to QED corrections to leptonic decay amplitudes.

N.H.Christ, X.Feng, L.Jin and CTS, PoS **LATTICE2019** (2020), 259

Radiative Corrections to Leptonic Decays using Infinite Volume Reconstruction

N.H.Christ, X.Feng, L.Jin, CTS and T.Wang, arXiv:2304.08026

- Two attractive features of IVR for leptonic decays:

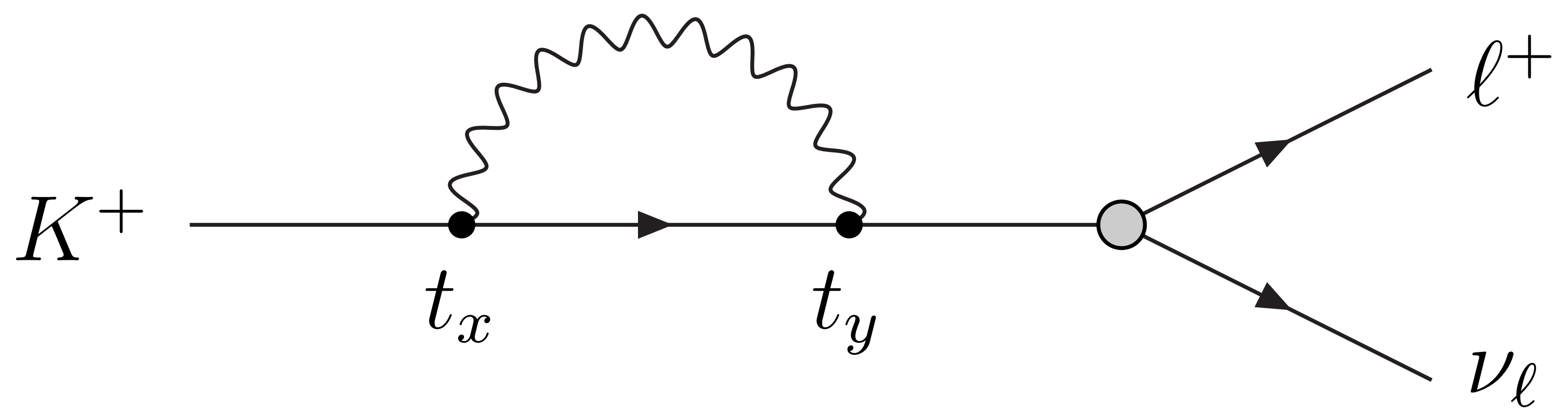
  1. Infrared divergences cancel analytically;
  2. Finite-volume corrections are exponentially small in the volume.

- The difficulties: large distance behaviour of correlation functions is generically of the form $e^{-m\sqrt{x^2+t^2}}$
- We use the large time behaviour to isolate the state we are interested in.
- For $t \gg x$, for example, $e^{-m\sqrt{x^2+t^2}} \simeq e^{-mt}\left(1+\dfrac{x^2}{2t^2}\right)$ and the coefficient of e^{-mt} has large FV corrections.

- Numerical computations of leptonic decay rates are under way. X.-Y.Tuo et al. (in preparation)

Illustration of Infinite-Volume Reconstruction

- For illustration consider the following diagram which contributes both to the electromagnetic mass-shift and to the wave function renormalisation of the kaon:



- For large $|t_y - t_x|$, $|t_y - t_x| > t_s$ say, the only state propagating between the two currents is $|K^+\gamma\rangle$.
- As will be demonstrated on the following slide, it is then sufficient to evaluate the correlation functions with $|t_y - t_x| \leq t_s$ and avoid non-exponential FV effects. For example consider:

$$H_2(z, t_z) = \langle K^+(\vec{0}\,)|T[J^\mu(\vec{z}, t_z)J^\nu(0)]|K^+(\vec{0}\,)\rangle$$

where J^μ and J^ν are electromagnetic currents.

Demonstration

$$H_2(z,t_z) = \langle K^+(\vec{0})|T[J^\mu(\vec{z},t_z)J^\nu(0)]|K^+(\vec{0})\rangle$$

$$H_2(\vec{z},t_s) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\langle K^+(\vec{0})|J^\mu(\vec{z},t_s)|K^+(\vec{p})\rangle\,\langle K^+(\vec{p})|J^\nu(0)|K^+(\vec{0})\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}e^{i\vec{p}\cdot\vec{z}}\,e^{-(E_p-m_K)t_s}\langle K^+(\vec{0})|J^\mu(0)|K^+(\vec{p})\rangle\,\langle K^+(\vec{p})|J^\nu(0)|K^+(\vec{0})\rangle$$

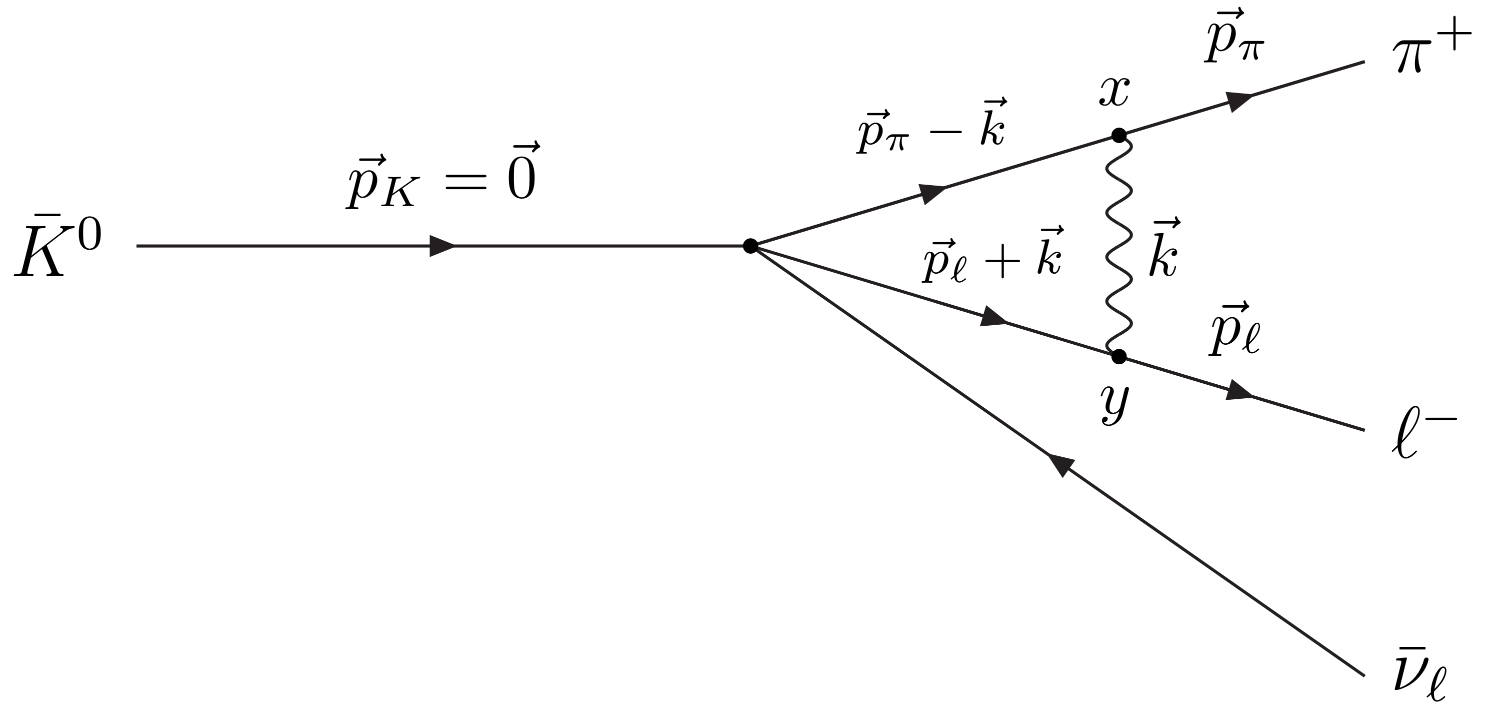
Performing the inverse Fourier transform:

$$\frac{1}{2E_p}\langle K^+(\vec{0})|J^\mu(0)|K^+(\vec{p})\rangle\langle K^+(\vec{p})|J^\nu(0)|K^+(\vec{0})\rangle = e^{(E_p-m_K)t_s}\int d^3z\, e^{-i\vec{p}\cdot\vec{z}}H_2(\vec{z},t_s)$$

So that finally:

$$H_2(z,t_z)|_{tz>ts} = \int \frac{d^3p}{(2\pi)^3}\int d^3z' H_2(\vec{z}',t_s)e^{-(E_p-m_K)(t_z-t_s)}\,e^{i\vec{p}\cdot(\vec{z}-\vec{z}')} \qquad \left(E_p = \sqrt{\vec{p}^2+m_K^2}\right)$$

IVR and $K_{\ell 3}$ Decays



- For $K_{\ell 3}$ decays, the amplitude contains an imaginary part which creates the following difficulties in a conventional lattice calculation:

- In a finite Euclidean space, the presence of internal $\pi\ell$ states with energies lower that those of the external $\pi\ell$ pair $\Rightarrow$ the corresponding contributions grow exponentially with x_4 relative to the pure QCD diagram leading to dominance by unphysical terms.

- In a finite volume Euclidean calculation the imaginary contribution to the amplitude is missed and the approximation of the principle part by a discrete sum introduced potentially large finite-volume corrections.

- In an appendix to our paper we sketch how these difficulties are overcome by using IVR.

3. K → $\pi\pi$ Decays

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_{ud}[MeV]	4.1.1	3.427(51)	[7–9]	3.387(39)	[10–16]		
m_s[MeV]	4.1.1	93.46(58)	[7–9, 17, 18]	92.4(1.0)	[11–15, 19]		
m_s/m_{ud}	4.1.2	27.227(81)	[7, 8, 20, 21]	27.42(12)	[12–14, 19, 22]		
m_u[MeV]	4.1.3	2.14(8)	[9, 23]	2.27(9)	[24]		
m_d[MeV]	4.1.3	4.70(5)	[9, 23]	4.67(9)	[24]		
m_u/m_d	4.1.3	0.465(24)	[23, 25]	0.485(19)	[24]		
$\overline{m}_c$(3 GeV)[GeV]	4.2.2	0.989(10)	[7–9, 18, 26, 27]	0.991(6)	[15, 28–32]		
m_c/m_s	4.2.3	11.766(30)	[7–9, 18]	11.82(16)	[29, 33]		
$\overline{m}_b(\overline{m}_b)$[GeV]	4.3	4.200(14)	[9, 34–37]	4.171(20)	[15]		
$f_+(0)$	5.3	0.9698(17)	[38, 39]	0.9677(27)	[40, 41]		
$f_{K^\pm}/f_{\pi^\pm}$	5.3	1.1934(19)	[20, 42–45]	1.1917(37)	[12, 46–50]		
$f_{\pi^\pm}$[MeV]	5.6			130.2(8)	[12, 46, 47]		
$f_{K^\pm}$[MeV]	5.6	155.7(3)	[21, 42, 43]	155.7(7)	[12, 46, 47]		
Re(A_2)[GeV]	6.2			$1.50(4)(14) \times 10^{-8}$	[51]		
Im(A_2)[GeV]	6.2			$-8.34(1.03) \times 10^{-13}$	[51]		
$\hat{B}_K$	6.3	0.717(18)(16)	[52]	0.7533(91)	[12, 53–56]	0.727(22)(12)	[57]
B_2	6.4	0.46(1)(3)	[52]	0.488(15)	[55, 56, 58]	0.47(2)(1)	[57]
B_3	6.4	0.79(6)	[52]	0.757(27)	[55, 56, 58]	0.78(4)(2)	[57]
B_4	6.4	0.78(2)(4)	[52]	0.903(14)	[55, 56, 58]	0.76(2)(2)	[57]
B_5	6.4	0.49(3)(3)	[52]	0.691(14)	[55, 56, 58]	0.58(2)(2)	[57]

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268

- A_2 - Thesis work of two Southampton students, Elaine Goode and Tadeusz Janowski.

"Lattice determination of the $K \to (\pi\pi)_{I=2}$ decay amplitude A_2"
T.Blum, E.Goode, A.Lytle, CTS et al.
Phys.Rev.D86 (2012) 074513

This paper won the 2012 Ken Wilson Lattice Award.

"$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit"
T.Blum, T.Janowski, CTS et al.,
Phys.Rev.D91 (2015) 074502

$\Delta I = 1/2$ Rule and ϵ'/ϵ

- For these decays $|\,f\,\rangle$ consists of two hadrons which interact in the finite volume.
- $K \to \pi\pi$ decays are a very important class of processes with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2 ,

$$ {}_{I=2}\langle \pi\pi\,|\,H_W\,|\,K^0\rangle = A_2\,e^{i\delta_2}\,, \qquad {}_{I=0}\langle \pi\pi\,|\,H_W\,|\,K^0\rangle = A_0\,e^{i\delta_0}\,. $$

- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule ($\mathrm{Re}A_0/\mathrm{Re}A_2 \simeq 22.5$) and an understanding of the experimental value of ϵ'/ϵ, the parameter which was the first experimental evidence for direct CP-violation.
- See the following two RBC-UKQCD papers, which however represent the culmination of many years of preparatory work:

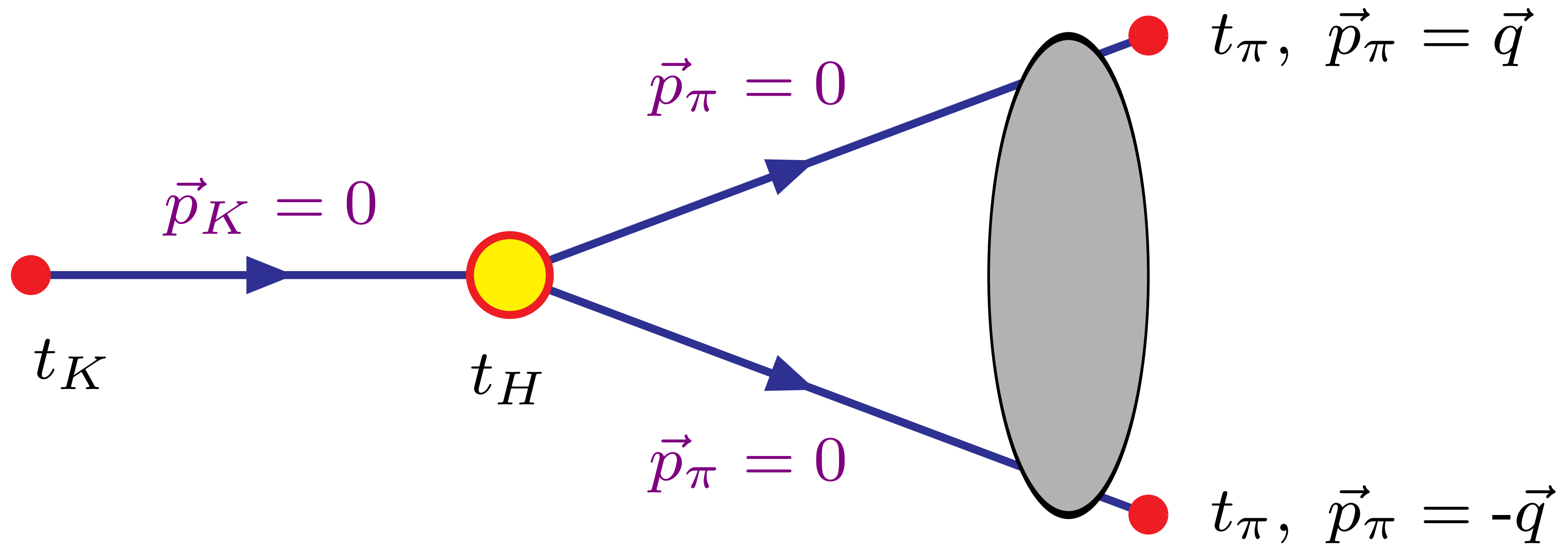
1. *"$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit"*
T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Janowski, C.Jung, C.Kelly, C.Lehner, A.Lytle, R.D.Mawhinney, CTS., A.Soni, H.Yin, and D.Zhang arXiv:1502.00263

2. *"Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay in the Standard Model"*
R.Abbott, T.Blum, P.A.Boyle, M.Bruno, N.H.Christ, D.Hoying, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, D.J.Murphy, CTS, A. Soni, M.Tomii and T.Wang, arXiv:2004.09440

(Building on RBC-UKQCD, Z.Bai et al. arXiv:1505.07863)

- Detailed references to earlier work can be found in these papers.

Why are the amplitudes difficult to compute?



- $K \to \pi\pi$ correlation function is dominated by the lightest intermediate state. L.Maiani and M.Testa, Phys.Lett. B245 (1990) 585
  - With periodic boundary conditions this is the $\pi\pi$ state with both pions at rest for A_2 and the vacuum state for A_0.
  - We have chosen to use anti periodic boundary conditions for the d-quark for A_2 and G-parity boundary conditions for A_0.
  - Work is in progress to compute the amplitudes with periodic boundary conditions with excited $\pi\pi$ states.
- Volume must be tuned to ensure $E_{\pi\pi} = m_K$. M.Tomii, arXiv:2501.18077
  - Moreover, the s-wave $I = 0$ and $I = 2$ channels are attractive and repulsive respectively and so the two cases must be treated separately.
- Finite-volume effects are not exponentially small and must be corrected. L.Lellouch and M.Lüscher, hep-lat/00030023, C.J.D.Lin, G.Martinelli, CTS and M.Testa, hep-lat/0104006 C-h.Kim, CTS and S.Sharpe, hep-lat/0507006

Summary of our Results

- $\mathrm{Re}\,A_0 = 2.99\,(0.32)\,(0.59) \times 10^{-7}\,\mathrm{GeV}$ (Experiment $3.3201(18) \times 10^{-7}\,\mathrm{GeV}$);

  $\mathrm{Im}\,A_0 = -\,6.98\,(0.62)\,(1.44) \times 10^{-11}\,\mathrm{GeV}$.

- $\mathrm{Re}\,A_2 = 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\,\mathrm{GeV}$, (Experiment $1.4787(31) \times 10^{-8}\,\mathrm{GeV}$) ;

  $\mathrm{Im}\,A_2 = -\,6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\,\mathrm{GeV}$.

- We find $\dfrac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 19.9 \pm 2.3 \pm 4.4$ in good agreement with the experimental result of $22.45(6)$.

- Combining the result for $\mathrm{Im}\,A_0$ and $\mathrm{Im}\,A_2$ and using the experimental results for the real parts we obtain

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = 0.00217\,(26)_{\mathrm{stat}}\,(62)_{\mathrm{syst}}\,(50)_{\mathrm{IB}}\,.$$

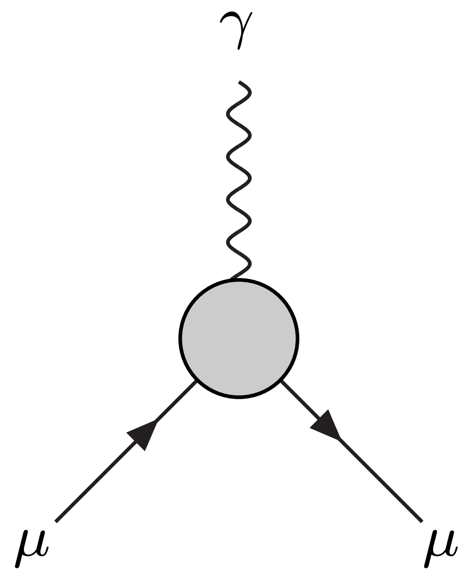
  The result is consistent with the experimental value of $0.00166\,(23)$.

  An IB effect of -50 was estimated using ChPT and model input and we take this as our uncertainty.
  V.Cirigliano et al., arXiv:1911.01359

- Compatible results with larger uncertainties (for now) have been obtained using periodic boundary conditions.
  M.Tomii et al., arXiv:2306.06781

- The RBC/UKQCD Collaboration continues work to reduce the uncertainties. Important priority is to control the IB effects.
  E.Lundstrum and N.Christ, Lattice 2025 (to appear in proceedings)

4. The anomalous magnetic moment of the muon



$$\langle \mu(p') \,|\, J^\mu(0) \,|\, \mu(p) \rangle = (-ie)\,\bar{u}(p') \left[F_1(q^2)\,\gamma^\mu + F_2(q^2)\,\frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu} \right] u(p)$$

$$\vec{\mu} = g\,\frac{e}{2m}\,\vec{S}$$

Anomalous magnetic moment: $a = \dfrac{g-2}{2} = F_2(0) = \dfrac{\alpha}{2\pi} + \cdots$

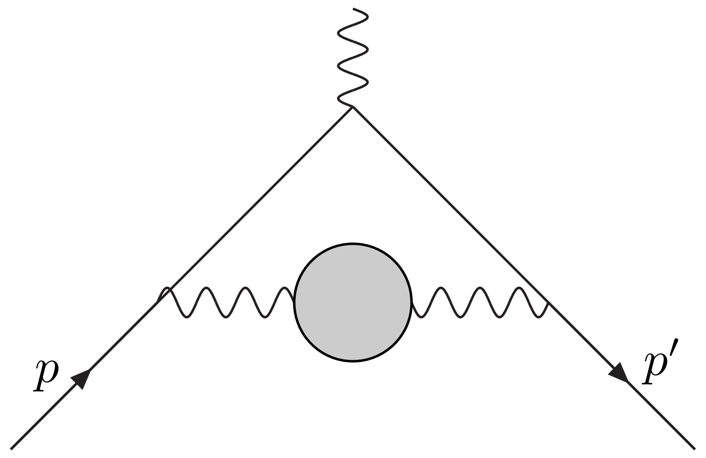
- Both the experimental measurement and the Standard Model theoretical prediction are very precise
  $\Rightarrow$ excellent test of the Standard Model.

- There have been two extensive reviews of the status of theoretical calculations of $(g-2)_\mu$ ("White Papers"), one in 2020 and the other in 2025, with diametrically opposite conclusions.
  i) In 2020 the main result came from the "data driven" approach and there was a claimed 3.7σ discrepancy between theory and experiment. T.Aoyama + 131 co-authors, arXiv:2006.04822
  ii) In 2025 the central result came from lattice computations and there was no discrepancy claimed
  $$a_\mu^{\rm exp} - a_\mu^{\rm th} = (38 \pm 63) \times 10^{-11}\,.$$
  R.Aliberti + 233 co-authors, arXiv:2505.21476

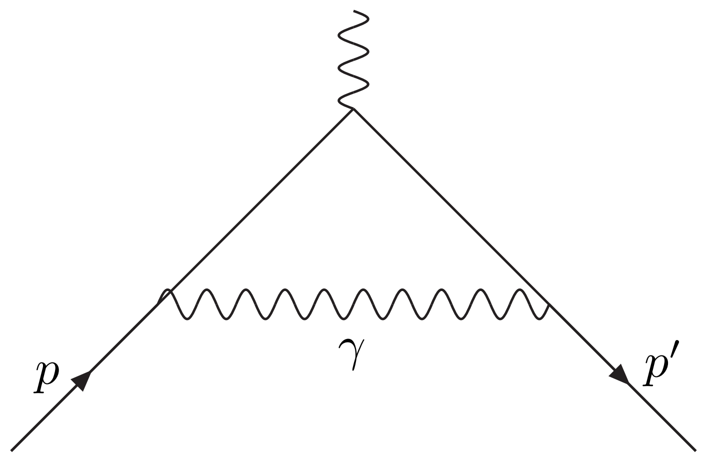
$2020 \Rightarrow 2025$

	2020	2025
a_μ^{exp}	$116\,592\,089 \pm 63$	$116\,592\,071.5 \pm 14.5$
Total SM Value	$116\,591\,810 \pm 43$	$116\,592\,033 \pm 62$
$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	279 ± 76	38 ± 63
QED	$116\,584\,718.931 \pm 0.104$	$116\,584\,718.8 \pm 0.2$
Electroweak	153.6 ± 1.0	154.4 ± 0.4
HVP	6845 ± 40	7045 ± 61
HLbL	92 ± 18	115.5 ± 9.9

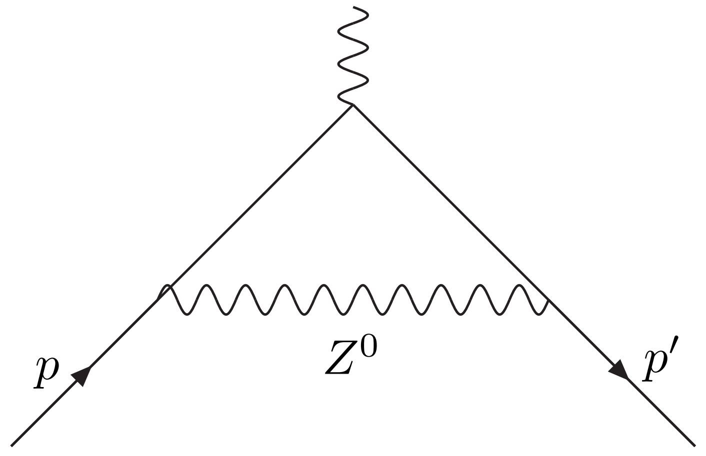
Units of 10^{-11}

- HVP 

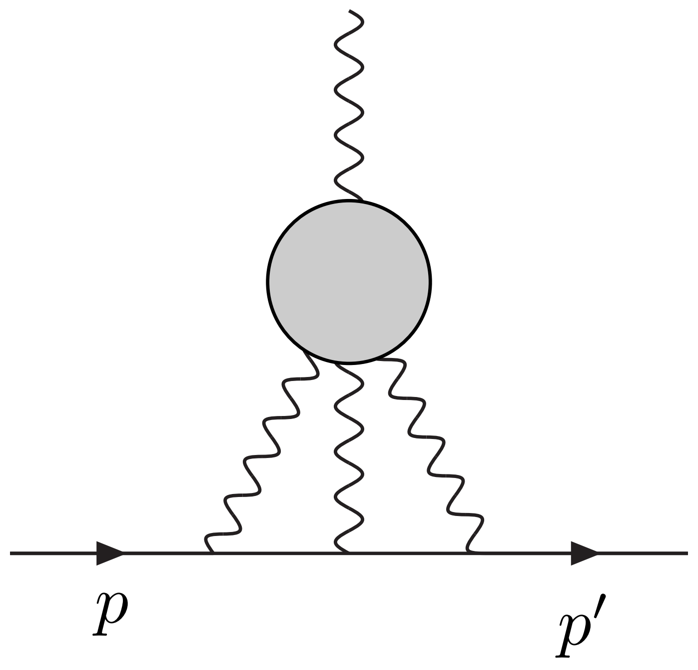
$+ \cdots$ (NNLO)

- QED 

$+ \cdots$ (5 loops)

- EW 

$+ \cdots$ (2 loops)

- HLbL 

$+ \cdots$ (NLO)

$2020 \Rightarrow 2025$

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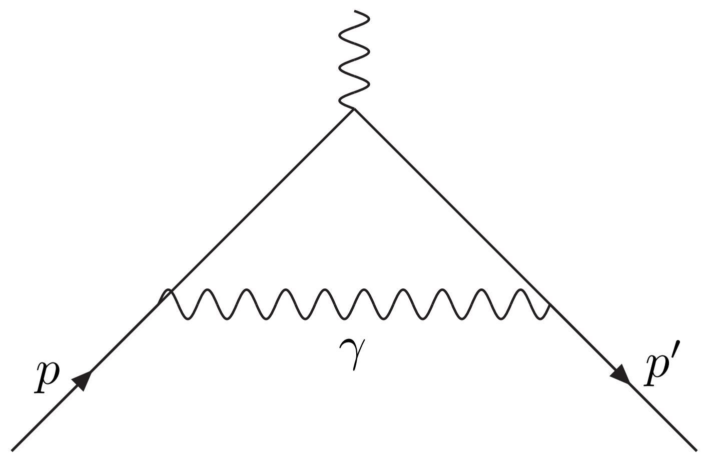
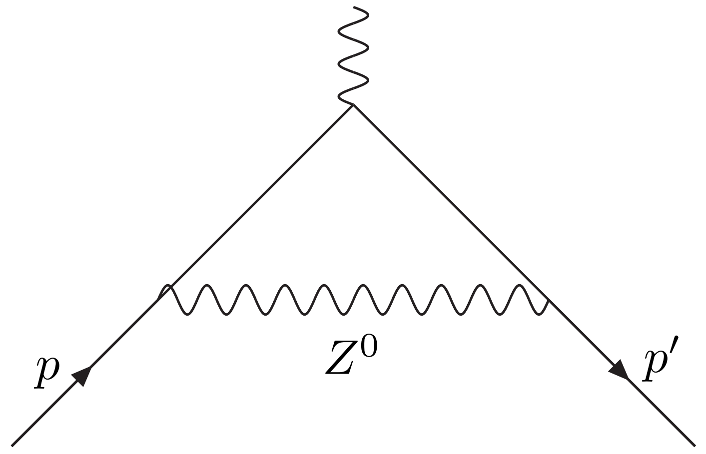
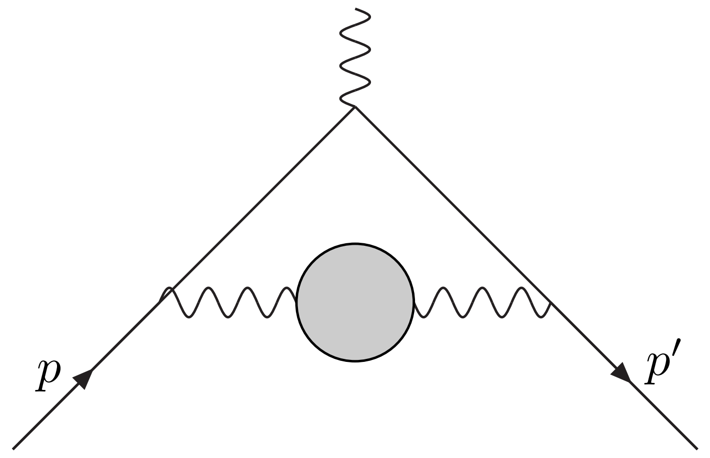
- QED 

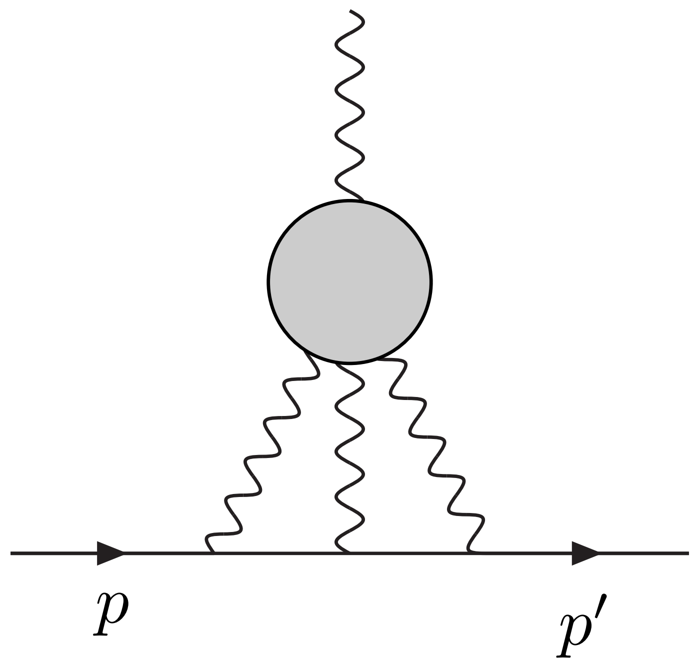
$+ \cdots$ (5 loops)

- EW 

$+ \cdots$ (2 loops)

- HVP 

$+ \cdots$ (NNLO)

- HLbL 

$+ \cdots$ (NLO)

The HVP contribution dominates the theoretical uncertainty and the comparison between the data-driven approach and lattice computations is a major area of investigation.

HVP: Data-driven approach

- In the data-driven approach the HVP contributions is related to the $e^+e^- \to$ hadrons cross section:

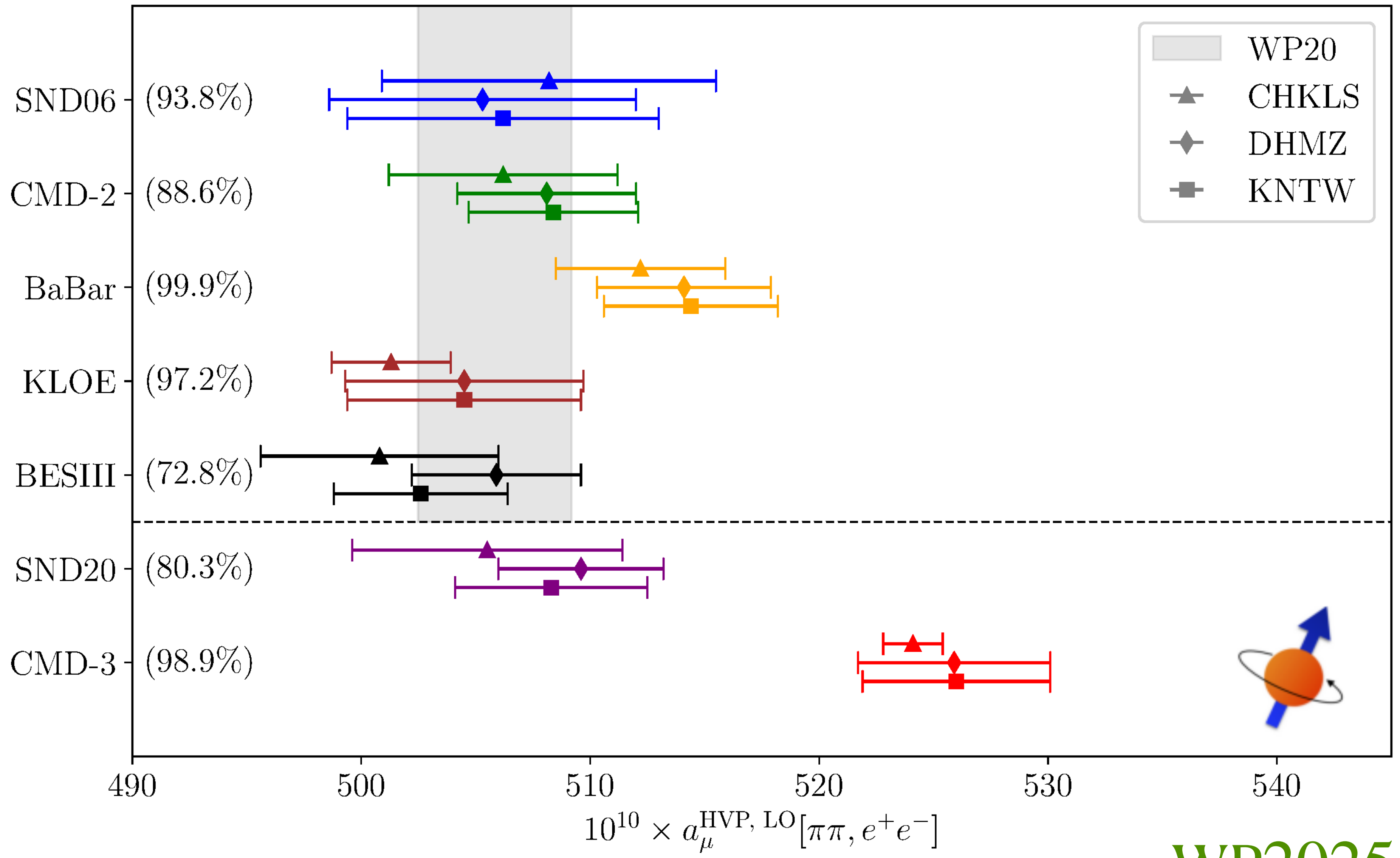
$$a_\mu^{\text{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{thr}}}^{\infty} ds\, \frac{\hat{K}(s)}{s^2}\, R_{\text{had}}(s)\,,$$

S.J.Brodsky and E. de Rafael, Phys.Rev. 168 (1968) 1620

B.E.Lautrup and E. de Rafael, Phys.Rev. 174 (1968) 1835

where $\hat{K}(s)$ is a known analytic kernel and $R_{\text{had}}(s) = \dfrac{3s}{4\pi\alpha^2}\,\sigma[\,e^+e^- \to \text{hadrons}\,(\gamma)\,]\,.$ $\left[\,s = (p_{e^+} + p_{e^-})^2\,\right]$

- "The challenge in evaluating $a_\mu^{\text{HVP,LO}}$ at sub-percent level lies in the extraordinary precision requirements for the measurement of $e^+e^- \to$ hadrons cross sections, especially for the crucial $e^+e^- \to \pi^+\pi^-$ channel."



WP2025

G.Colangelo, M.Hoferichter, B.Kubis, T.Leplumey, P.Stoffer
M.Davier, A.Hoecker, A.Malaescu, Z.Zhang
A.Keshavarzi, D.Nomura, T.Teubner, A.Wright

- "The unsatisfactory situation regarding the knowledge of the cross section for the process $e^+e^- \to \pi^+\pi^-$, which is known to contribute $> 70\,\%$ to the total HVP dispersion integral, presents a significant limitation to the data-driven evaluation of the HVP contribution $\cdots$ "

HVP : Lattice Approach

- In the lattice approach the principal ingredient is $\Pi_{\mu\nu}(Q) = \int d^4x\, \langle 0 \,|\, J_\mu(x)\, J_\nu(0) \,|\, 0 \rangle\, e^{-iQ\cdot x} \equiv (Q_\mu Q_\nu - \delta_{\mu\nu})\, \Pi(Q^2)$ and

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 \int_0^\infty dQ^2\, f(Q^2) \left[\Pi(Q^2) - \Pi(0)\right].$$

($f(Q^2)$ is a known function.)

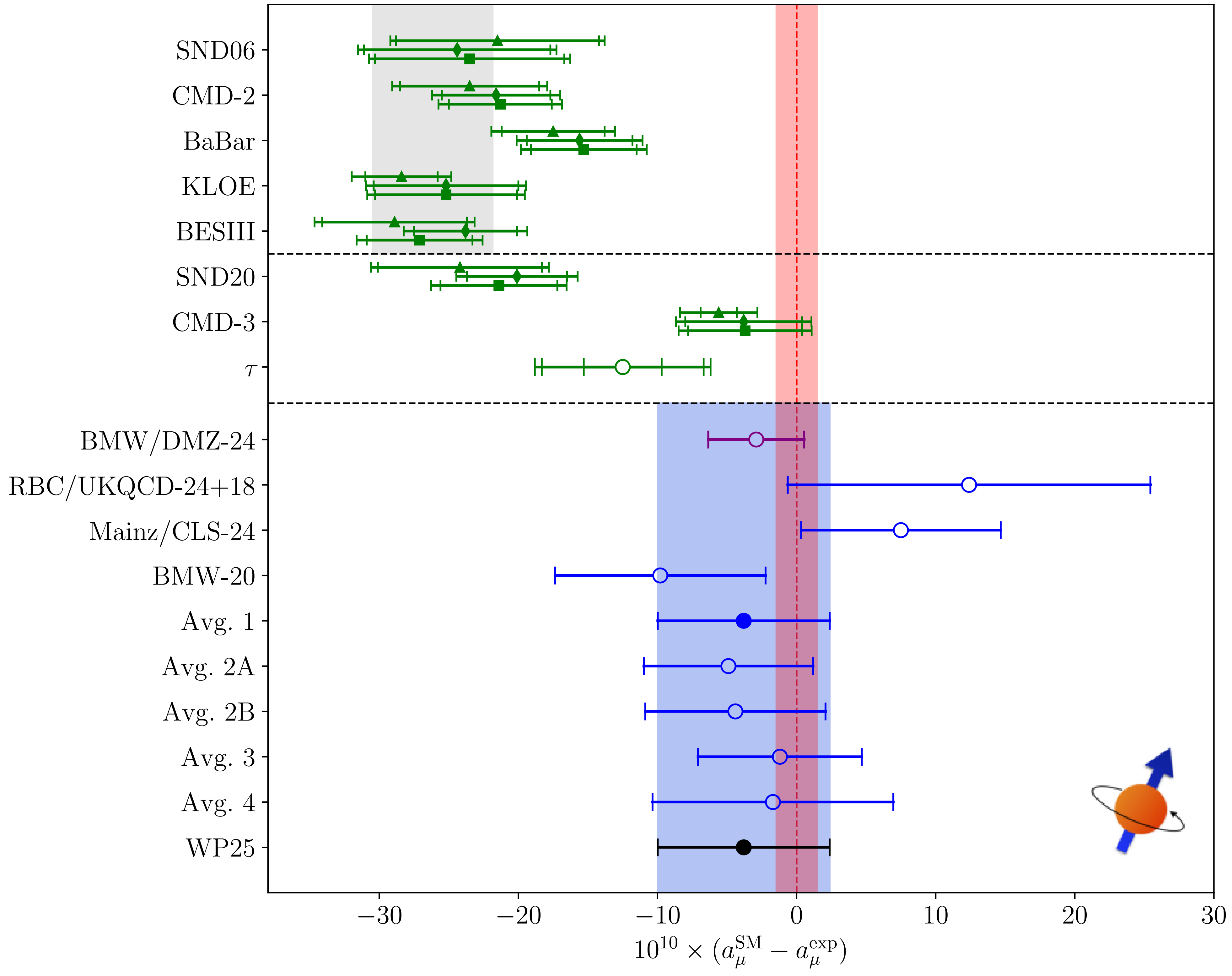
- In 2020 the lattice results were not sufficiently precise to be competitive in the determination of $a_\mu^{\text{HVP,LO}}$.
  - Between 2020 and 2025 there was a huge activity in the global lattice community to improve the precision including detailed studies of finite-volume corrections, determination of the lattice spacing, isospin-breaking effects and with simulations performed on lattices with different lattice actions.

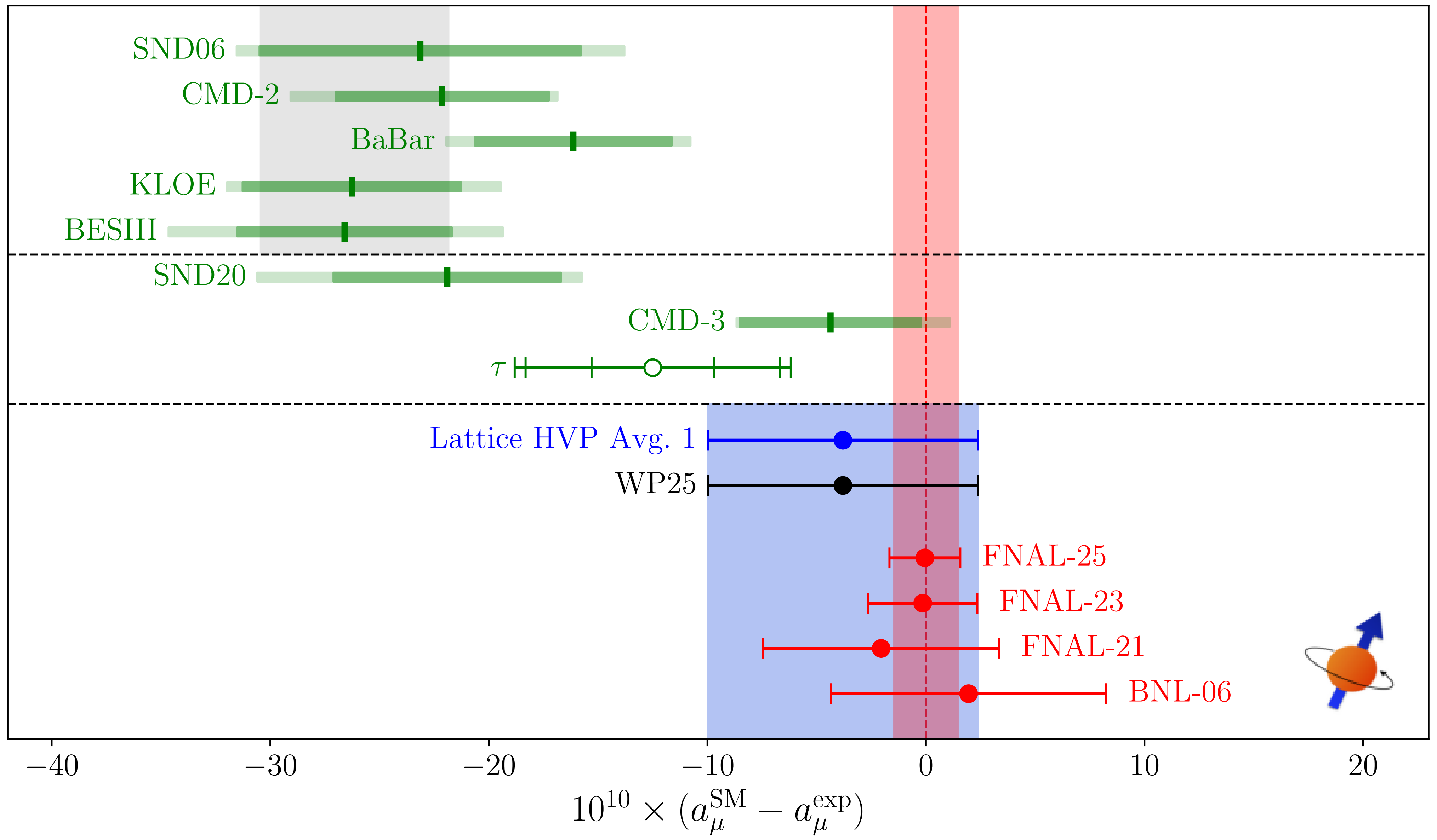
- The final result in WP2025, $a_\mu^{\text{HVP,LO}} = \left(7132 \pm 61\right) \times 10^{-11}$, was obtained using results using results from 17 papers from 8 independent collaborations, including 3 which computed the entire LO HVP contribution (BMW, RBC/UKQCD, Mainz).

- In order to better separate the different sources of uncertainty and facilitate comparisons between different calculations RBC-UKQCD suggested breaking up the time integration into three regions (with smeared boundaries):

  i) Short distance: $0 < t \lesssim 0.4\,\text{fm}$, ii) Intermediate $0.4\,\text{fm} \lesssim t \lesssim 1.0\,\text{fm}$ and iii) Long Distance $1.0\,\text{fm} \lesssim t \lesssim \infty$.

Summary plots from WP2025



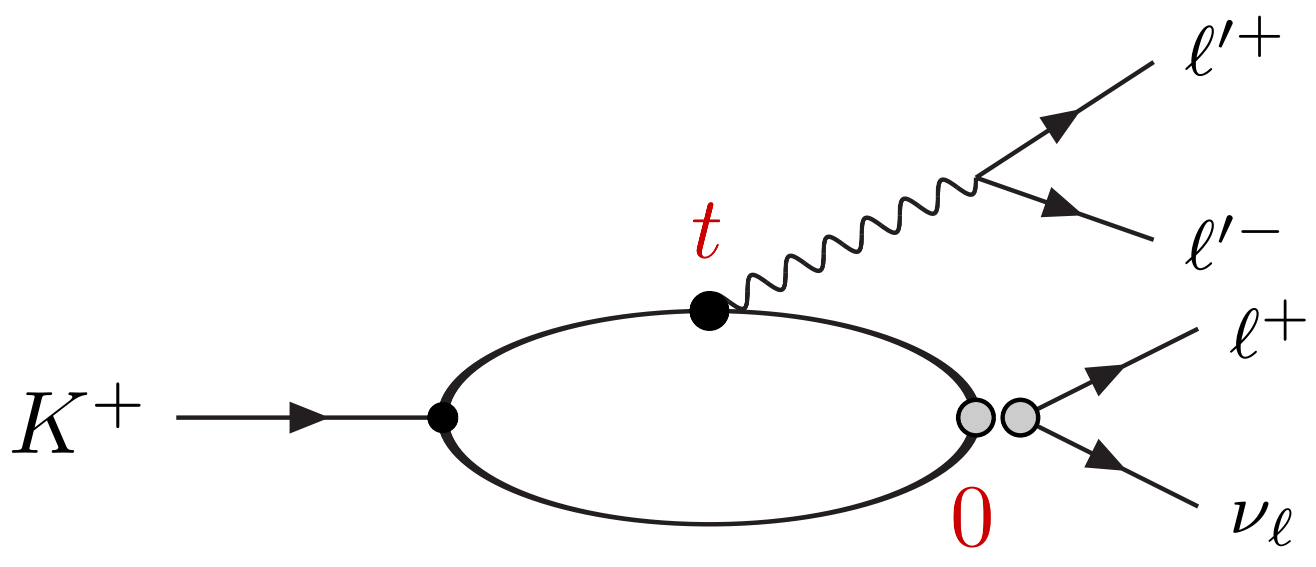


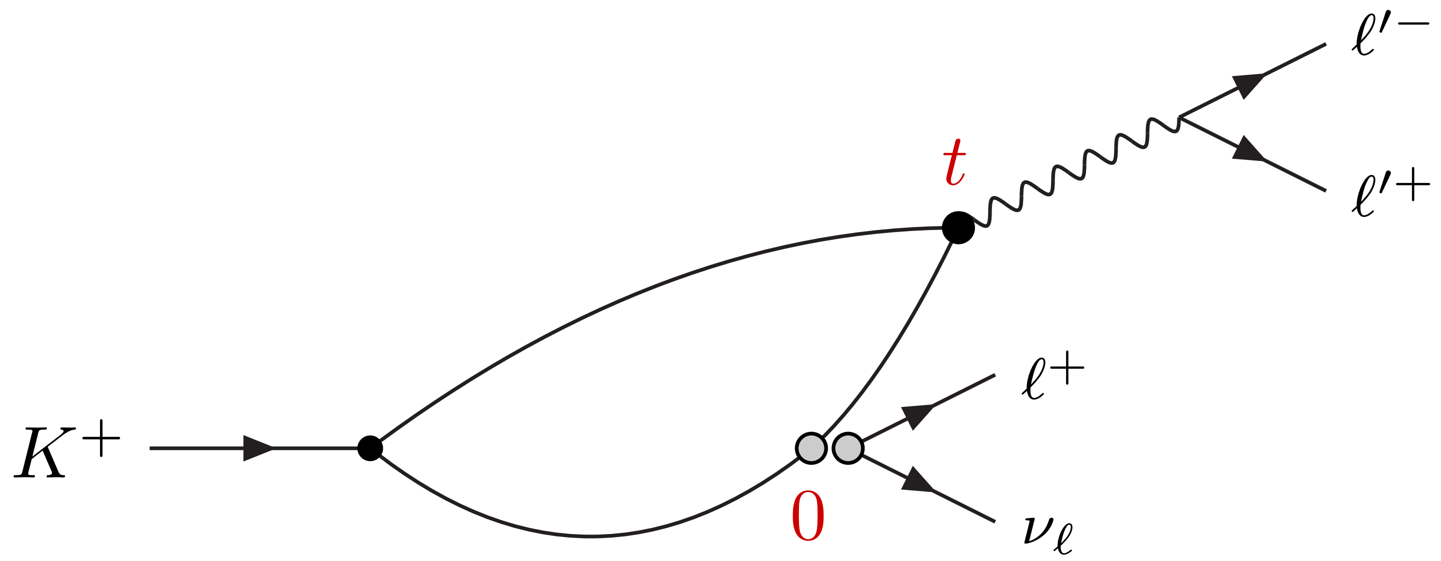
- The discrepancy between the SM calculation of a_μ and its experimental measurement appears to have been resolved.
- The puzzle now is to understand the reason for the experimental discrepancy between the CMD-3 measurement of the $e^+e^- \to \pi^+\pi^-$ cross section and the earlier results.
  - Much analysis being done in an attempt to resolve this puzzle.

5. Spectral Density Techniques

- Consider a process for which the hadronic factor in the amplitude is of the form: $\int d^4x\ e^{iq\cdot x}\ \langle F\,|\,T[O_1(x)\,O_2(0)]\,|\,I\,\rangle\,.$
  - There are very many such processes. A simple example $K^+ \to \ell^+\nu_\ell\,(\ell'^+\ell'^-)$ $\qquad \tilde{J}^\mu_{\rm em}(t,\mathbf{q}) = \int d^3x\ e^{-i\mathbf{q}\cdot\mathbf{x}}\ J^\mu_{\rm em}(t,\mathbf{x})$

$$\int_{-\infty}^{\infty} dt\ e^{iq_0t}\ \langle 0\,|\,T\,[\tilde{J}_{\rm em}(t,\mathbf{q})\,J_W(0)]\,|\,K^+(\mathbf{p_K})\,\rangle = \int_{-\infty}^{0} dt\ e^{iq_0t}\ \langle 0\,|\,J_W(0)\,\tilde{J}_{\rm em}(t,\mathbf{q})\,|\,K^+(\mathbf{p_K})\,\rangle + \int_{0}^{\infty} dt\ e^{iq_0t}\ \langle 0\,|\,\tilde{J}_{\rm em}(t,\mathbf{q})\,J_W(0)\,|\,K^+(\mathbf{p_K})\,\rangle$$





- For $t > 0$ and for $q^2 > 4m_\pi^2$ on-shell two-pion states can propagate between the two operators and contribute an imaginary part to the amplitude.
- Previous (exploratory) computations were performed at unphysical light-quark masses, so that there was no imaginary part in the amplitude ($m_K < 2m_\pi$). X.-Y.Tuo et al., arXiv:2103.11331; G.Gagliardi et al., arXiv:2202.03823

Spectral Density Techniques (Cont.)

- Consider the case $t > 0$. In the rest frame of the meson P,

$$H_W^{\mu\nu\,+}(E_\gamma, \mathbf{q}) = \int_0^\infty dt \int d^3x \; e^{iE_\gamma t} e^{-i\mathbf{q}\cdot\mathbf{x}} \; < 0 \,|\, J^\mu_{\rm em}(t, \mathbf{x}) \, J^\nu_W(0) \,|\, P(\mathbf{0}) >$$

$$= \sum_n \int_0^\infty dt \int d^3x \; e^{iE_\gamma t} e^{-i\mathbf{q}\cdot\mathbf{x}} \; < 0 \,|\, J^\mu_{\rm em}(t, \mathbf{x}) \,|\, n > < n \,|\, J^\nu_W(0) \,|\, P(\mathbf{0}) >$$

$$= \sum_n \int_0^\infty dt \int d^3x \; e^{i(E_\gamma - E_n)t} e^{-i(\mathbf{q}-\mathbf{p}_n)\cdot\mathbf{x}} \; < 0 \,|\, J^\mu_{\rm em}(0,\mathbf{0}) \,|\, n > < n \,|\, J^\nu_W(0) \,|\, P(\mathbf{0}) >$$

$$= i \int_{E^*}^\infty \frac{dE}{2\pi} \; \frac{\rho^{\mu\nu}(E, q)}{E - E_\gamma - i\epsilon},$$

- If $E_\gamma > E^*$ then the pole is in the range of integration.

where $q = (E_\gamma, \mathbf{q})$, E^* is below the threshold for the states $|n\rangle$ and the spectral density is defined as

$$\rho^{\mu\nu}(E, \mathbf{q}) = \langle 0 \,|\, J^\mu_{\rm em}(0) \, (2\pi)^3 \delta^{(3)}(\hat{\mathbf{P}} - \mathbf{q}) \, (2\pi)\delta(\hat{H} - E) \, J_W(0) \,|\, P(\mathbf{0})\rangle.$$

- The challenge is to determine $\rho^{\mu\nu}(E, q)$, which is independent of t, from Euclidean correlation functions.

The HLT/SFR Method

HLT - M.Hansen, A.Lupo and N.Tantalo, arXiv:1903.06476, Spectral Function Reconstruction (SFR) - R.Frezzotti et al., arXiv:2306.07228

(See references therein for earlier history and other approaches.)

$$H_W^{\mu\nu\,+}(E_\gamma, \mathbf{q}) = \int_0^\infty dt\ e^{iE_\gamma t} \int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}} < 0\,|\,J_{\rm em}^\mu(t,\mathbf{x})\, J_W^\nu(0)\,|\,P(\mathbf{0}) > = \int_0^\infty dt\ e^{iE_\gamma t}\ C_M^{\mu\nu}(t,\mathbf{q})$$

where the Minkowski correlation function $C_M^{\mu\nu}(t,\mathbf{k})$ is

$$C_M^{\mu\nu}(t,\mathbf{q}) = \int d^3x\ e^{-i\mathbf{q}\cdot\mathbf{x}} \langle 0\,|\,J_{\rm em}^\mu(t,\mathbf{x})\, J_W^\nu(0)\,|\,P(\mathbf{0})\rangle = \int \frac{dE}{2\pi}\ e^{-iEt}\ \rho^{\mu\nu}(E,\mathbf{q})\,.$$

- The formal continuation to Euclidean space is now straightforward:

$$C_M^{\mu\nu}(t,\mathbf{q}) \to C_E^{\mu\nu}(t,\mathbf{q}) = \int_{E^*}^\infty \frac{dE}{2\pi}\ e^{-Et}\ \rho^{\mu\nu}(E,\mathbf{q})\,,$$

  but still results in the inverse problem to determine the spectral density.

- Note that if there are states with $E^* < E < E_\gamma$, then it is **wrong** to write $H_W^{\mu\nu\,+}(E_\gamma,\mathbf{q}) = -\,i \int_0^\infty dt\ \ e^{E_\gamma t}\ C_E^{\mu\nu}(t,\mathbf{q})\,.$
  - In such a situation the t integration diverges.

The HLT/SFR Method (cont.)

- We have $H_W^{\mu\nu\,+}(E_\gamma\,,\mathbf{q}) = i\int_{E^*}^{\infty}\frac{dE}{2\pi}\,\frac{\rho^{\mu\nu}(E,q)}{E-E_\gamma-i\epsilon}$ and $C_E^{\mu\nu}(t,\mathbf{q}) = \int_{E^*}^{\infty}\frac{dE}{2\pi}\,e^{-Et}\,\rho^{\mu\nu}(E,\mathbf{q})\,.$

- The HLT approach is to expand $\frac{1}{E-E_\gamma-i\epsilon} \simeq \sum_{n=1}^{N_{\max}} g_n(E_\gamma,\epsilon)\;e^{-anE}$, at fixed finite ϵ and subsequently take the $\epsilon\to 0$ limit.

$$H_W^{\mu\nu\,+}(E_\gamma\,,\mathbf{q}) = i\sum_{n=1}^{N_{\max}} g_n(E_\gamma,\epsilon)\int_{E^*}^{\infty}\frac{dE}{2\pi}\;\rho^{\mu\nu}(E,q)\;e^{-anE} = i\sum_{n=1}^{N_{\max}} g_n(E_\gamma,\epsilon)\,C_E^{\mu\nu}(an,\mathbf{q})\,.$$

  - Thus at fixed ϵ the *smeared* amplitude is obtained in terms of Euclidean correlation functions.

- Significant technical issues remain.

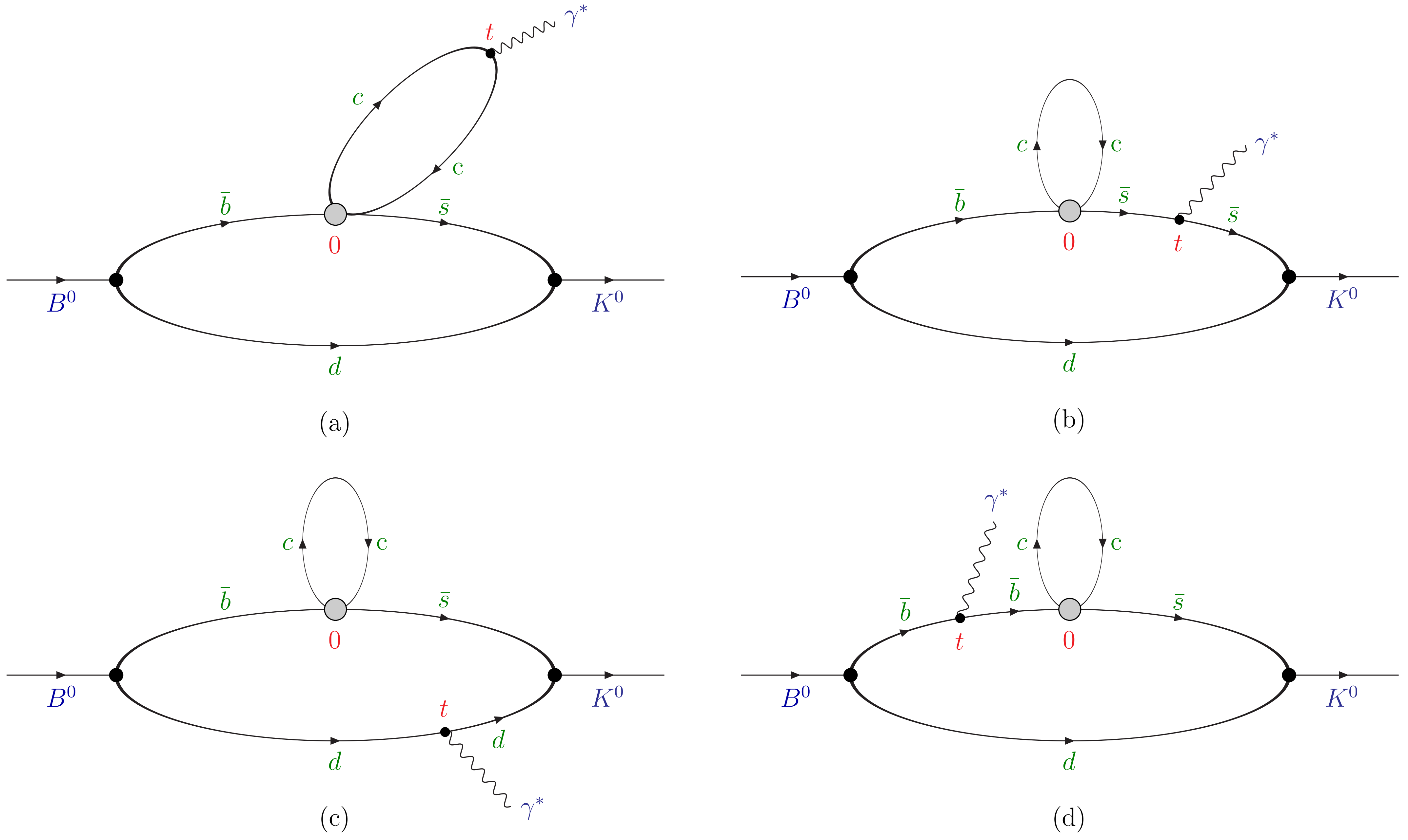
A computation of the form factors for $K\to\ell\nu_\ell\,(\ell'^+\ell'^-)$ decays is underway.

R. Di Palma, G.Gagliardi et al.

- Other applications of spectral-density methods include the energy-smeared R-ratio of $e^+e^-\to$ hadrons (ETMC, arXiv:2212.08467), inclusive τ-decays (ETMC, arXiv:2308.03125), inclusive semileptonic $D_s$decays (A.De Santis et al., arXiv:2504.06063/4), $B\to K^{(*)}\ell^+\ell^-$ (R.Frezzotti et al., arXiv:2508.03655) and $B\to\gamma\ell^+\ell^-$ decays (R.Frezzotti et al., arXiv:2402.03262).

Further Applications of Spectral Density Methods

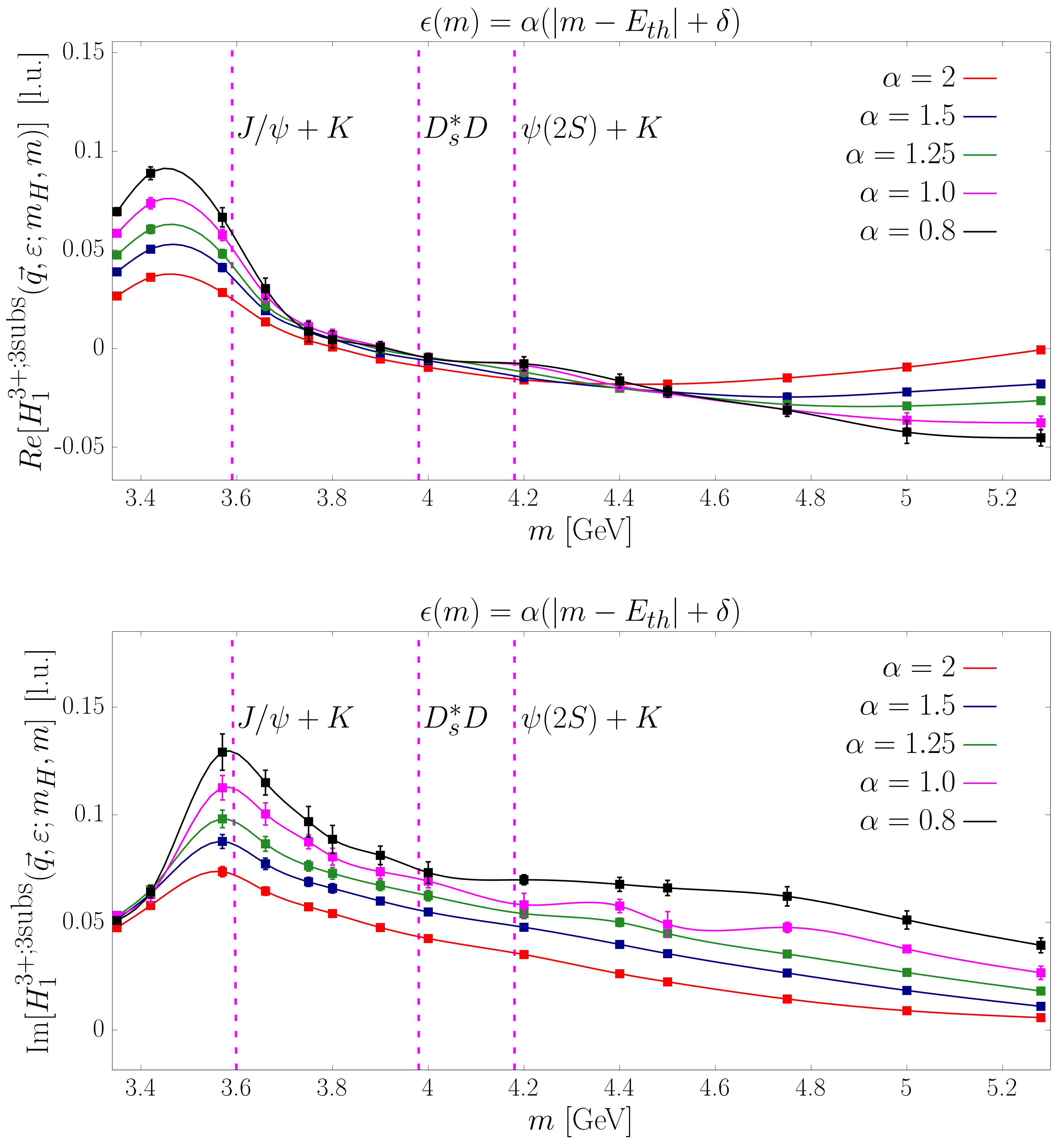
"Theoretical framework for lattice QCD computations of $B \to K\ell^+\ell^-$ decays and $\bar{B}_s \to \ell^+\ell^-\gamma$ decay rates, including contributions from charming penguins" R.Frezzotti et al., arXiv:2508.03655



$$O_1^{(c)} = (\bar{s}^i \gamma_L^\mu c^j)(\bar{c}^j \gamma_{\mu L} b^i) \qquad O_2^{(c)} = (\bar{s}^i \gamma_L^\mu c^i)(\bar{c}^j \gamma_{\mu L} b^j)$$

- The major difficulty in the phenomenology of this $b \to s$ decay is the evaluation of such contributions, particularly in the region of charmonium resonances.

    - Much exciting physics to be explored !



Results of an exploratory computation of diagram (a) with unphysical parameters.

$m_b = 2m_c$, $|\mathbf{q}| \simeq 250$ MeV .

Conclusions

- I have presented a selection of examples to illustrate the impact which lattice computations are having on particle physics phenomenology.
  - The precision of the computations of "standard" quantities is now such that isospin-breaking corrections, including QED, must be included to make further progress.
  - The $\Delta I = 1/2$ rule is confirmed as a QCD effect and the first computations of ϵ'/ϵ have been performed.
  - Lattice computations of the HVP (and HLbL) contribution to $(g-2)_\mu$, and the recent CMD-3 experimental measurement of the $e^+e^- \to \pi^+\pi^-$ cross-section, suggest that there is no discrepancy in the experimental and Standard Model values of a_μ. (Much work still remains to be done.)
    - An important focus now is to understand the discrepancy between the CMD-3 result and those from earlier experiments.
  - New spectral density techniques allow for an extension of lattice computations to new classes of phenomenologically important quantities, including the energy-smeared R-ratio of $e^+e^- \to$ hadrons, inclusive τ-decays, inclusive semileptonic $D_s$decays, $B \to K^{(*)}\ell^+\ell^-$ and $B \to \gamma\ell^+\ell^-$ decays.

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Congratulations to the GGI on this special occasion and best wishes for a long and fruitful continuation of its important role both in advanced education and in stimulating progress through the exchange of scientific ideas.