STRING THEORY AND MODERN FIELD THEORY II

ou l'histoire d'une surface

Novels use various means to describe a long and complex historical period.

One is to use a multitude of characters, all roughly equally important, seen through the viewpoint of an omniscient narrator *(The Dream of the Red Chamber, War and Peace, Les Miserables, The Tale of Genji…)*

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One is to use a multitude of characters, all roughly equally important, seen through the viewpoint of an omniscent narrator *(The Dream of the Red Chamber, War and Peace, Les Miserables, The Tale of Genji…)*

The other is to see history through a single character, or family, (*Buddenbrocks, The Sea of Fertility…)*

Here we'll adopt the second technique by tracking the development of string and field theory through the evolution and refinement of a minimal surface, which geometrizes entanglement entropy and appeared in 2006, at the beginning of the GGI adventure

$$AREA \propto ENTROPY$$

has a long history starting from Bekenstein and Hawking (1973, 1974) who established that a Black Hole has entropy

$$S = A/4G$$

A wealth of subsequent work, starting with Strominger and Vafa (1996) that convincingly shows

$$S = \log N_{micorstates}$$

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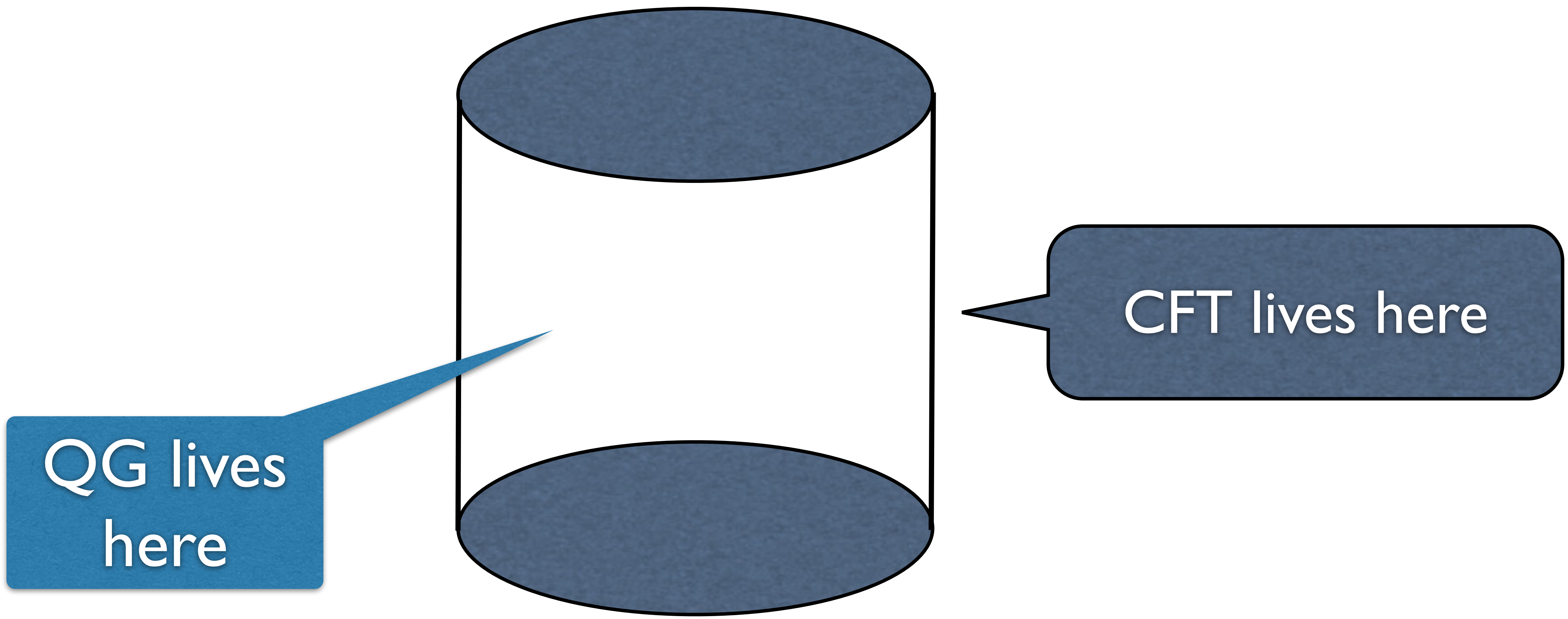
The original result used Supergravity in a fundamental way:

- In SUGRA there exist extremal BH solutions that preserve some of the spacetime supersymmetries

- the INDEX of supersymmetric states, $I \sim N_B - N_F$ is coupling-constant independent ($g \ll 1$ = perturbative counting, $g \gg 1$ = geometric GR limit)

AdS/CFT Maldacena (1997) :
Quantum Gravity in AdS D+1 = CFT in D “on” boundary of AdS



Gold-plated example: 4D supersymmetric Yang-Mills theory with 16 supercharges / Type IIB superstring on $AdS_5 \times S_5$

Many checks possible because of supersymmetric non-renormalization theorems and other SUGRA properties

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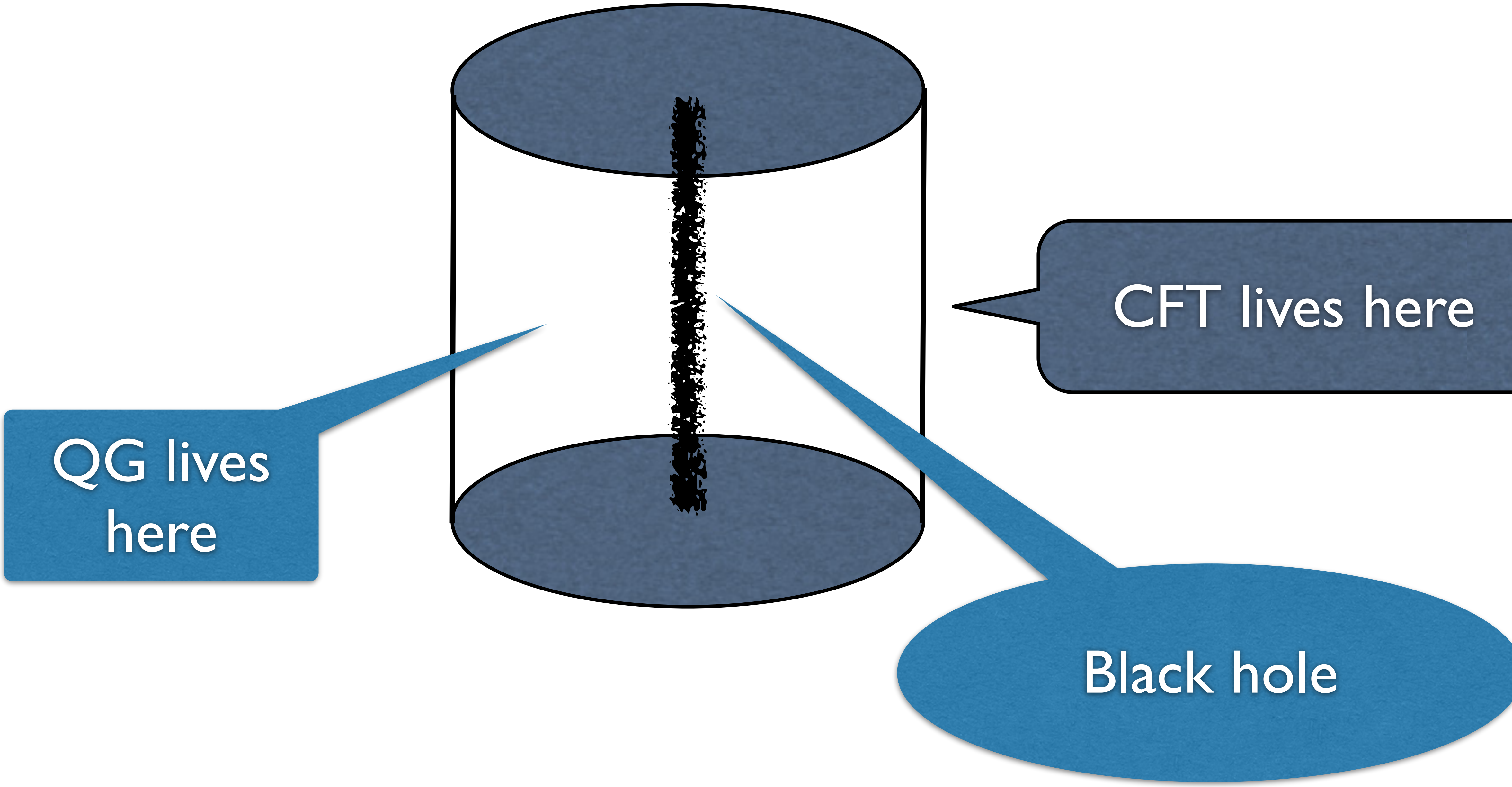
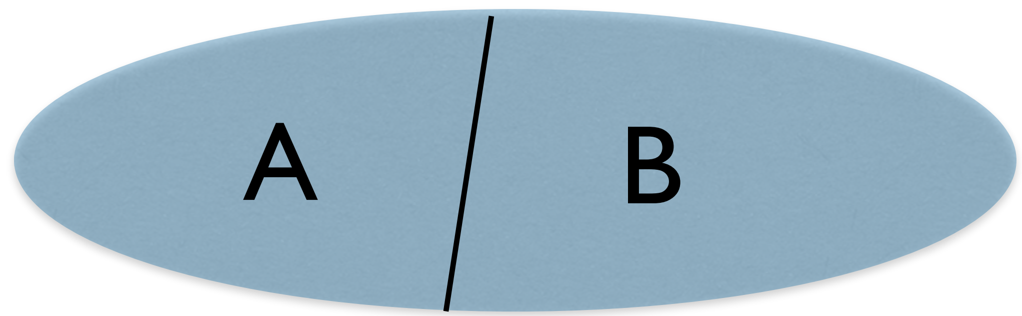


Holography: BH = Typical Thermal State with $\frac{1}{T} = \frac{dS}{dE}$

Quantum Information has left the realm of theoretical speculation to become a a tool relevant to physics, cryptography, computer science … even finance and banking!

One of the most basic quantities in QI is entanglement entropy



$$H = H_A \times H_B \qquad \mathrm{Tr}_A \rho = \rho_B \qquad \mathrm{Tr}_B \rho = \rho_A$$

$$S(\rho_A) = -\,\mathrm{Tr}_A \rho_A \log \rho_A$$

Can be nonzero even when $\rho^2 = \rho$ (pure state)

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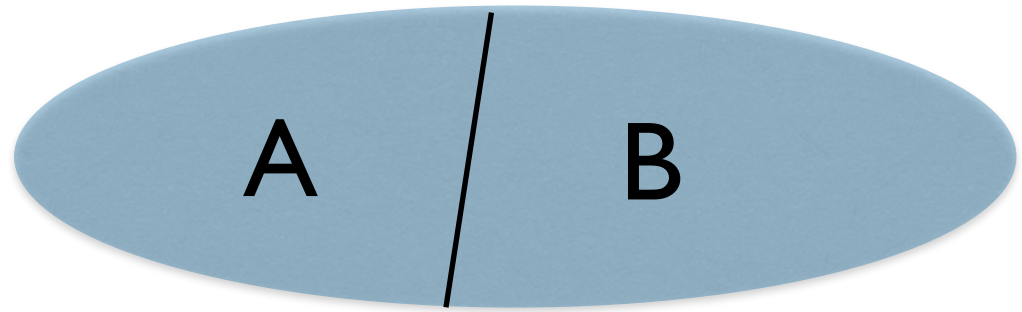

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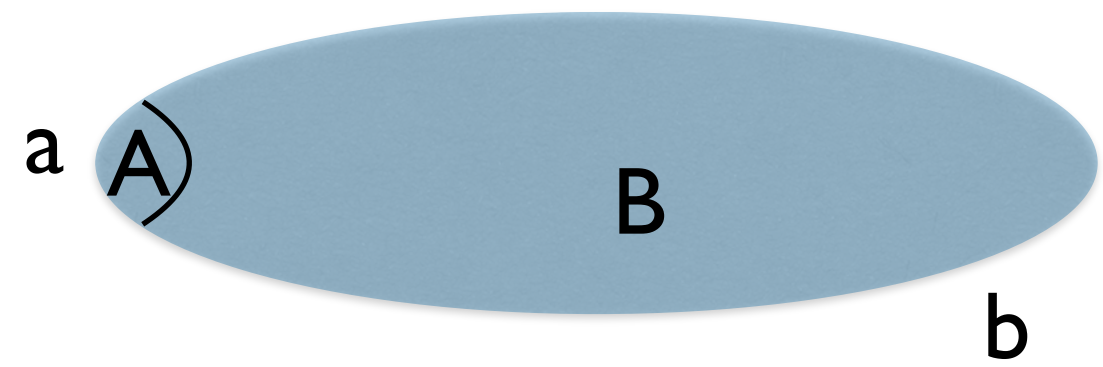
Generalizes to Rényi entropy

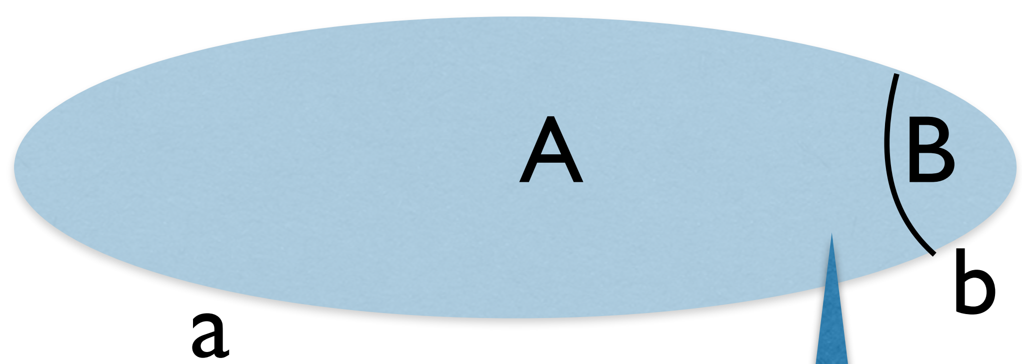
$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \mathrm{Tr}\, \rho^\alpha$$

$$S(\rho_A) = \lim_{\alpha \to 1} S_\alpha(\rho_A)$$

Holography: can we geometrize entanglement entropy?



If only a small part of the boundary (a) is known it is reasonable to think that only a small portion of the bulk (A) can be reconstructed



a larger boundary (a) reconstructs a bigger bulk (A) what's the dividing surface?

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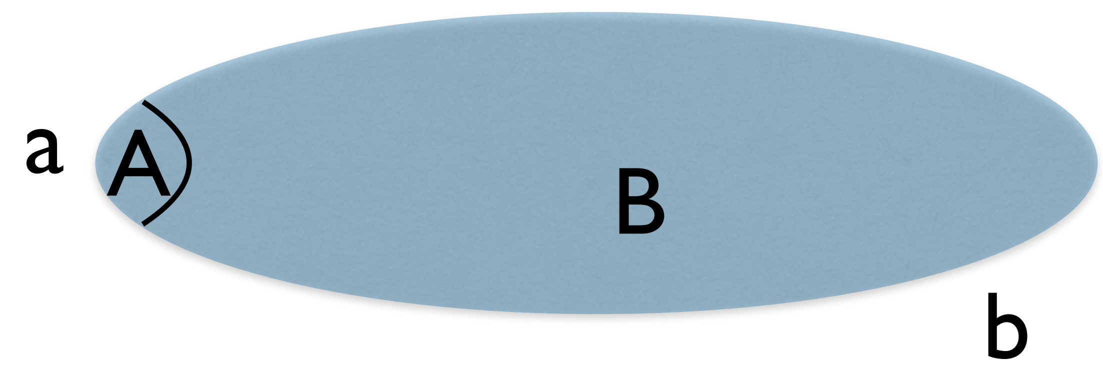

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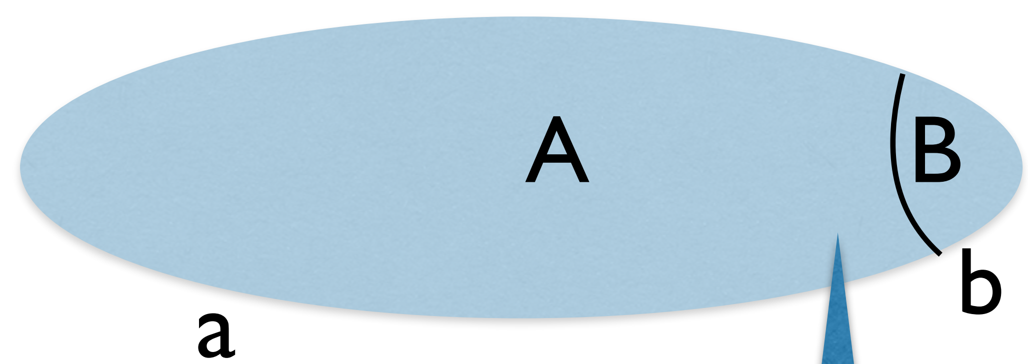
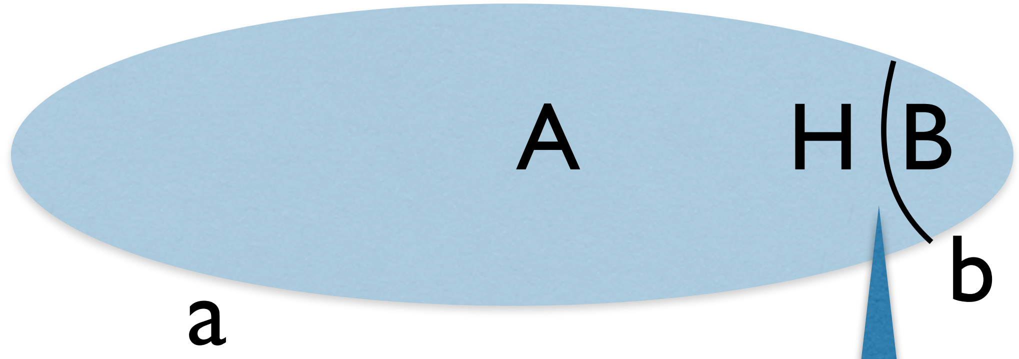


a larger boundary (a) reconstructs a bigger bulk (A) what's the dividing surface?

Ryu & Takayanagi (2006): it is the minimal area surface homologous to (a) and with the same boundary

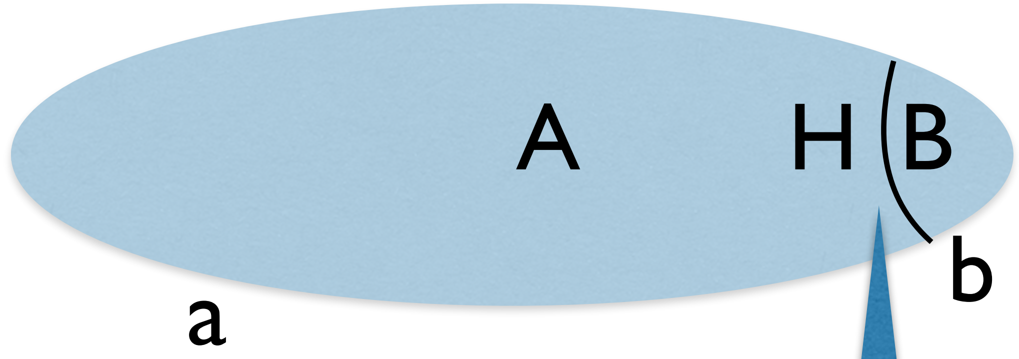


What's the dividing surface H?

Ryu & Takayanagi (2006): (H) is the minimal area surface homologous to (a) and with the same boundary

Covariant generalization: Hubeny, Rangamani, Takayanagi (2007)

Quantum extremal surface
[Faulkner et al. (2013) Engelhardt & Wall (2014)]:



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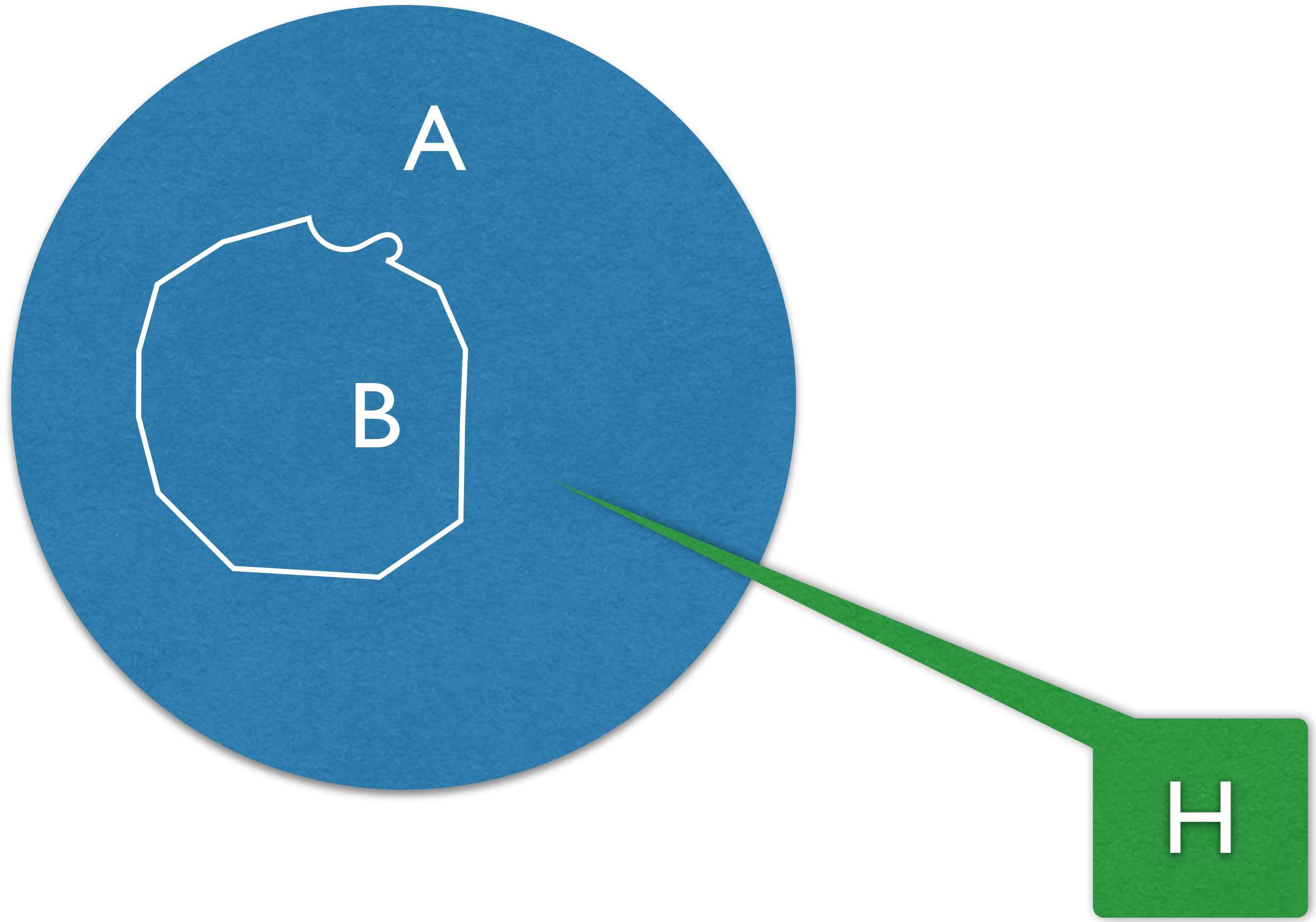
The QES is the largest minimum of $S_{gen}(H) = \frac{A(H)}{4G} + S_{bulk}(A)$
on a spacelike surface anchored at $a \cap b$

Trivial case: $S_{bulk}(H) \ll \frac{A(H)}{4G}$ (e.g. pure state in bulk)



Minimum area: $H = \emptyset$ - all the boundary reconstructs all the bulk

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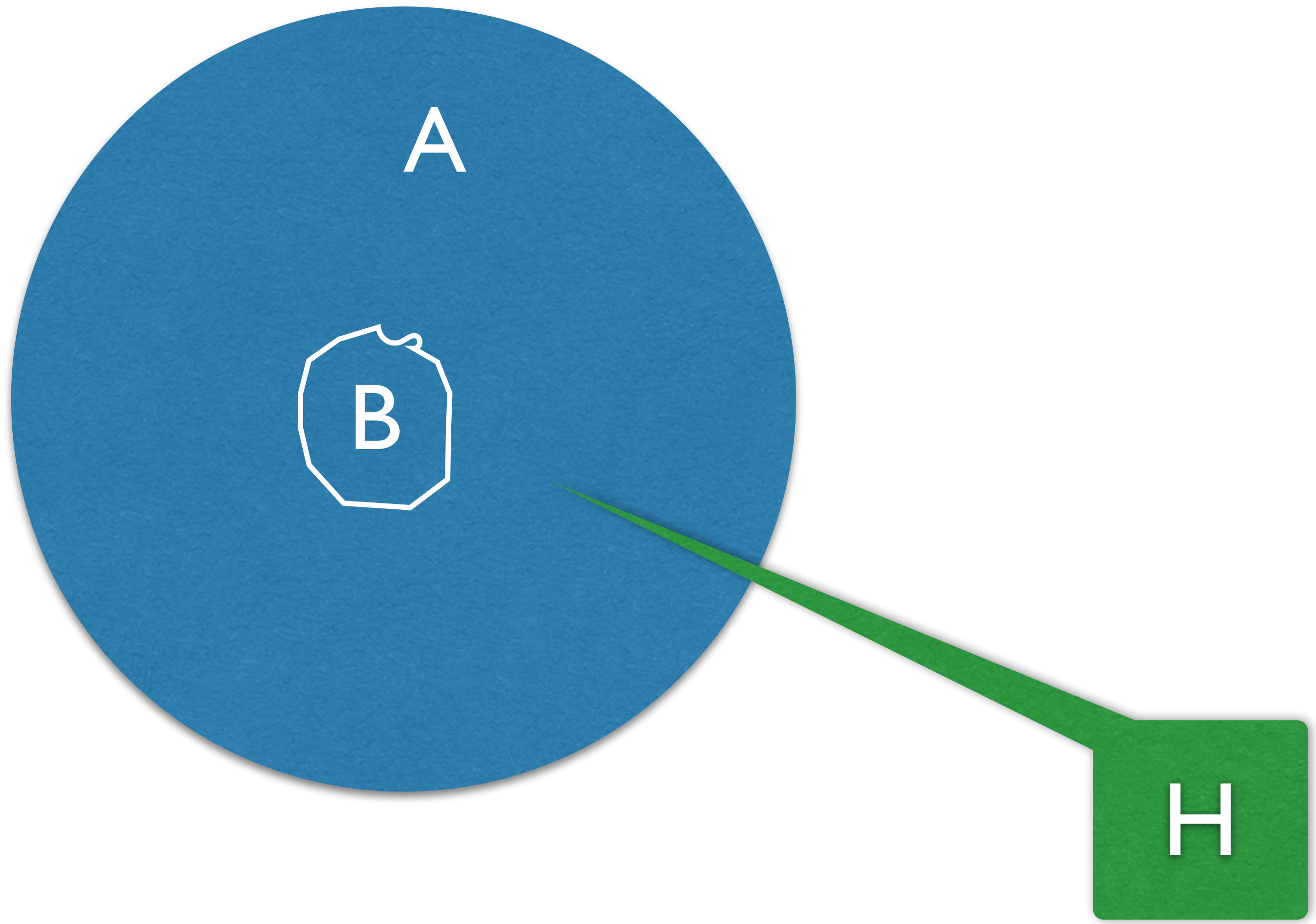

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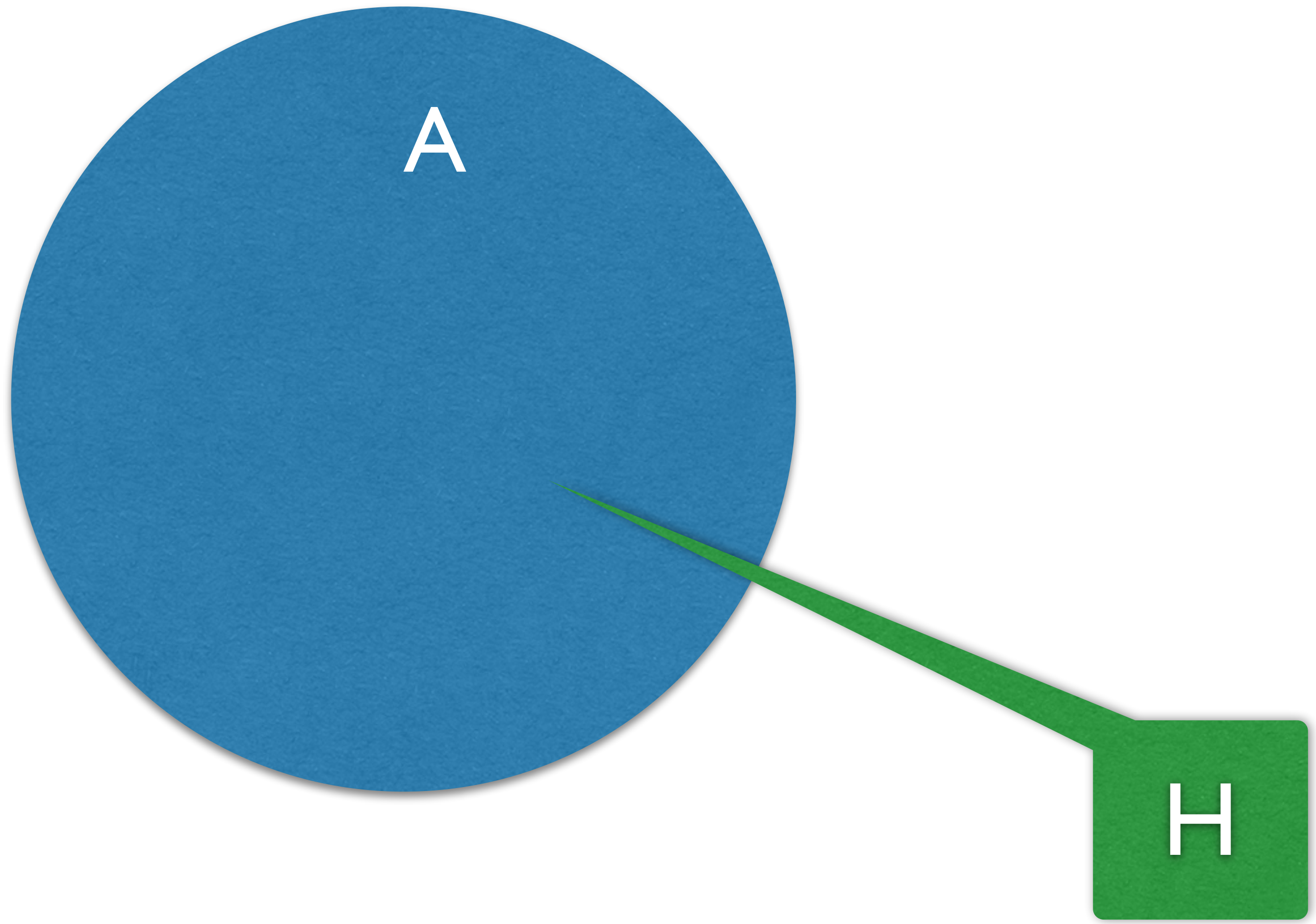

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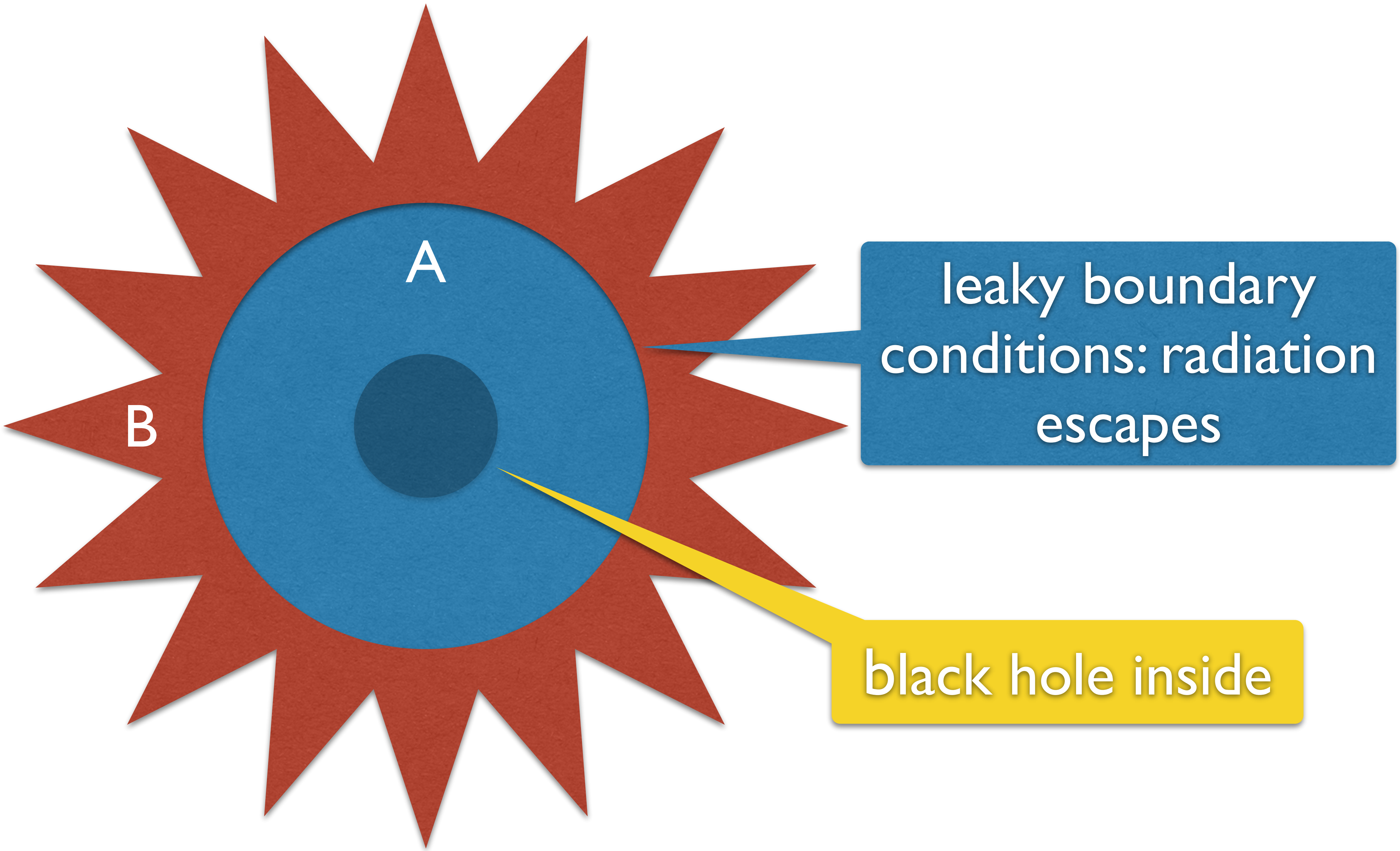
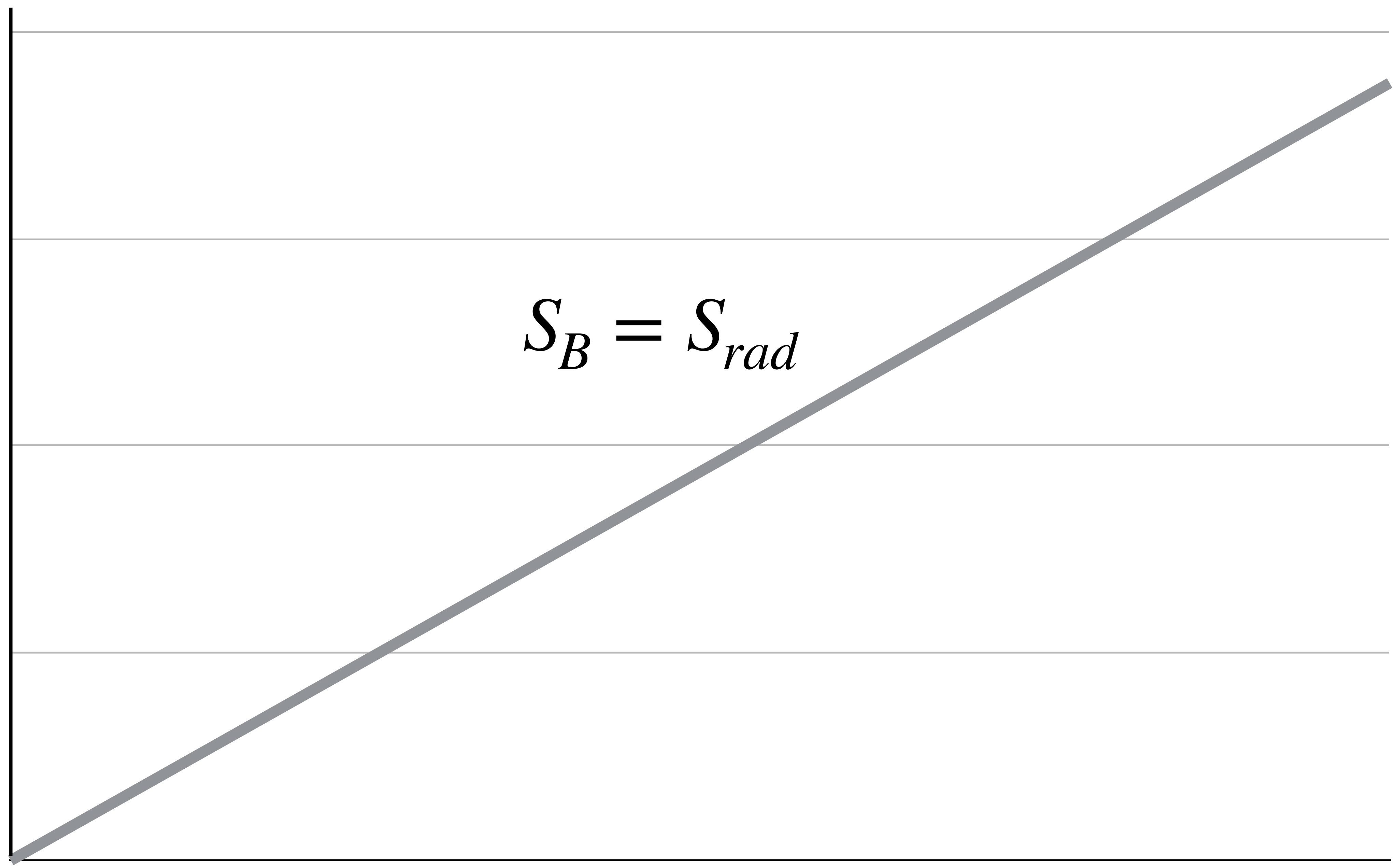
Less trivial case: BH + leaky boundary condition



THE PAGE CURVE PROBLEM

Hawking: entropy of radiation = entanglement entropy



$$\frac{dS_{Rad}}{dt} \propto \frac{1}{R_{BH}}$$

Time

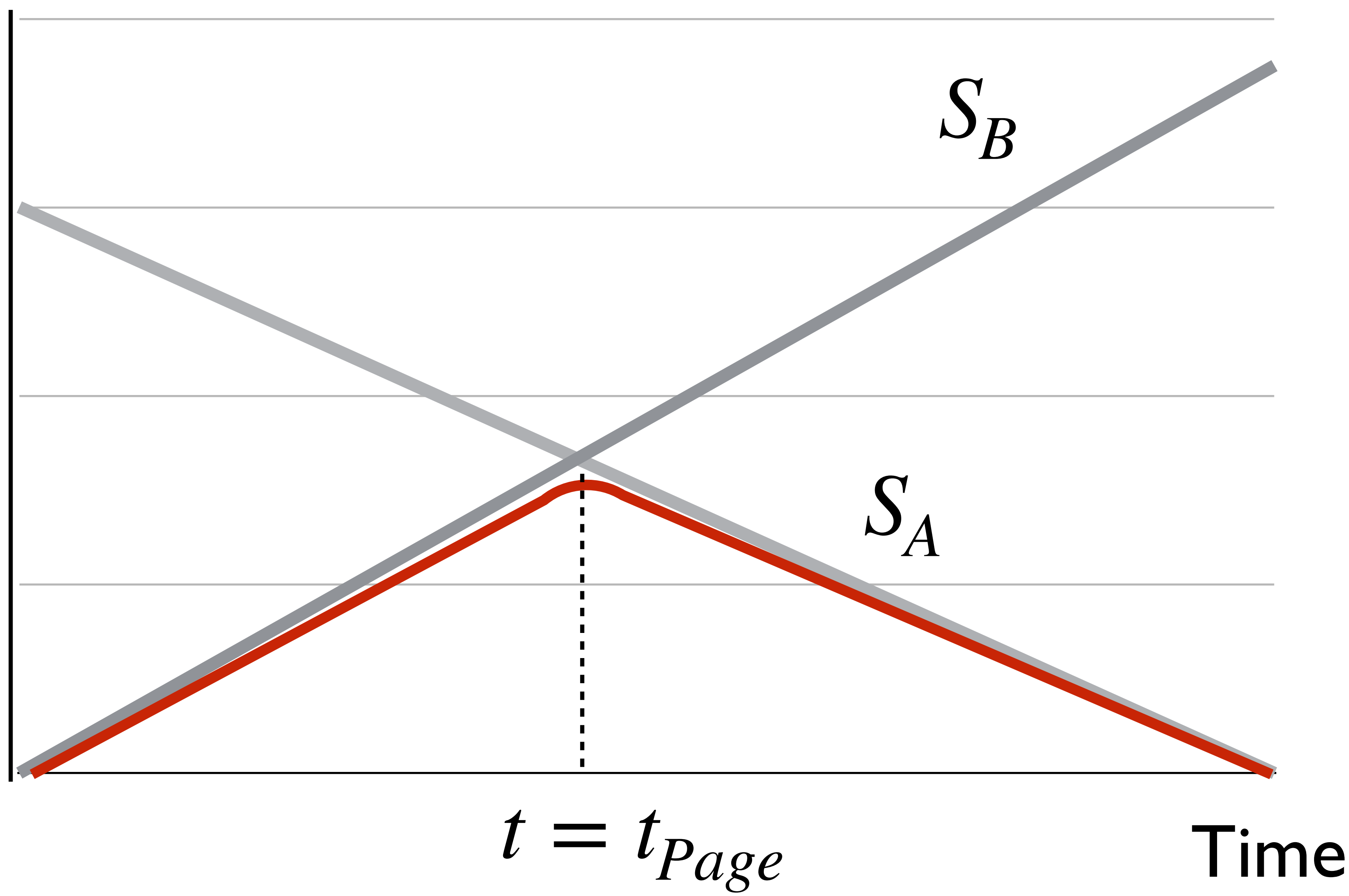
THE PAGE CURVE PROBLEM

Hawking: entropy of radiation = entanglement entropy

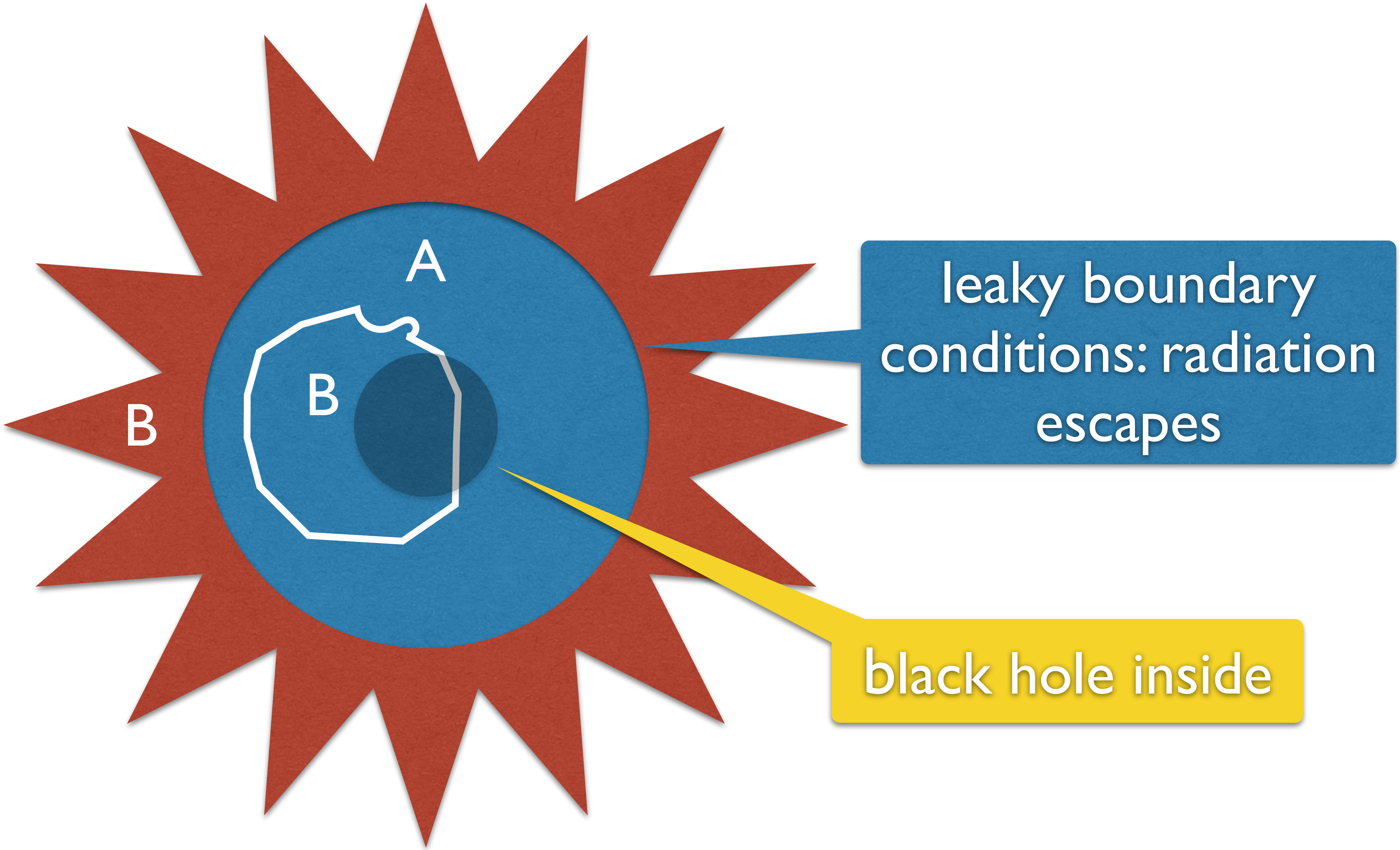
Entanglement entropy obeys $|S_A - S_B| \leq S_{A\cup B}$

Evolution from a pure state: $S_{A\cup B} = 0$

Evaporating BH: $S_{BH}(t) \sim S_A \to 0$



QES to the rescue: $S_{bulk}(H) = \mathrm{Tr}\,\rho_A \log \rho_A \gtrsim \frac{A(H)}{4G}$



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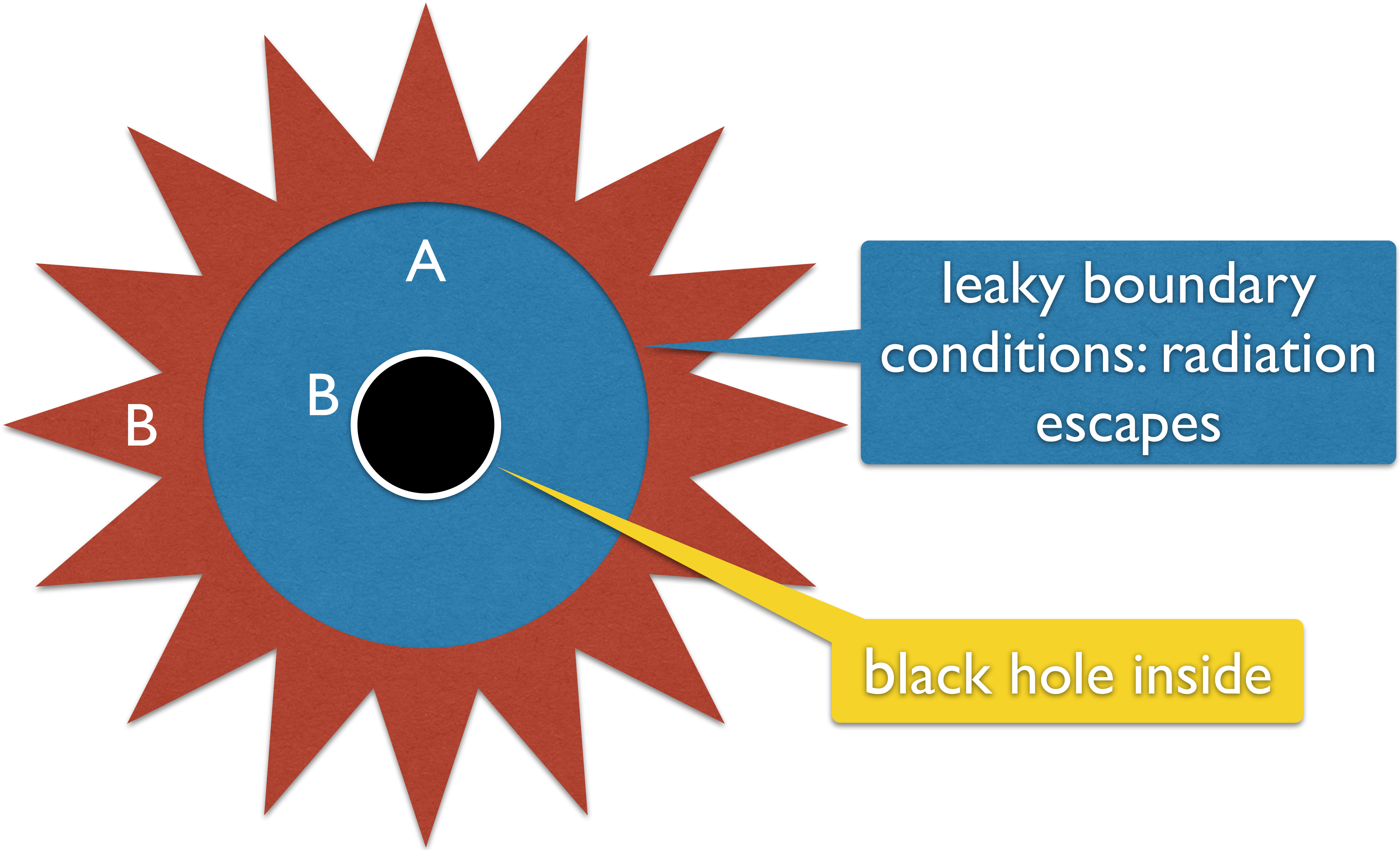
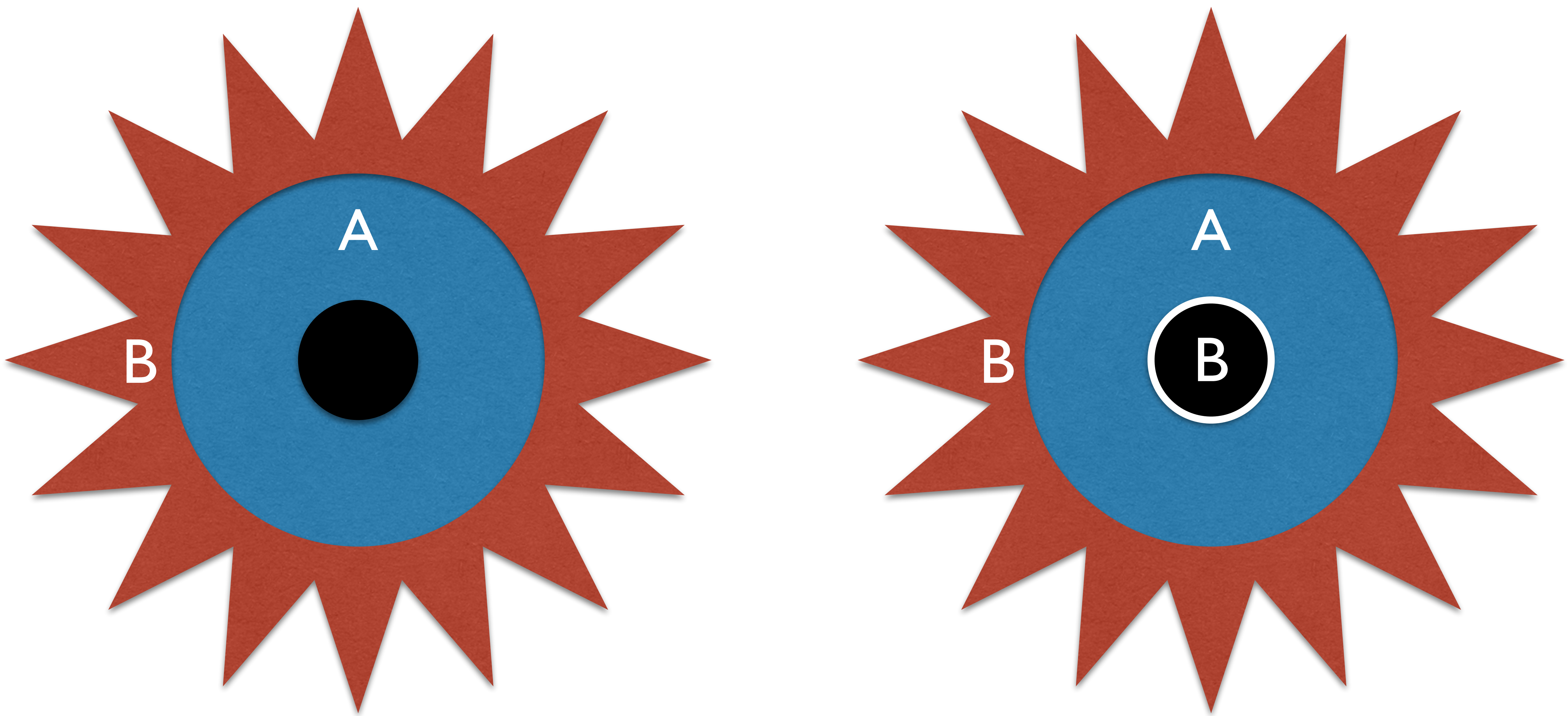


Minimum generalized entropy: $H = 4\pi R_{BH}^2$ - the holographic theory does not reconstruct the BH interior. Information is encoded in radiation that escaped from the leaky boundary

QES to the rescue, Penington, Penington & al., Almheiri et al. (2019)

$$S_{gen} = S_{AdS} \approx S_{BH}$$



$t \ll t_{Page}$ few Hawking quanta

$$S_{gen} = S_A$$

Radiation entropy ($S_{A \cup B} = 0$)

$$S_{rad} = S_B = S_A$$

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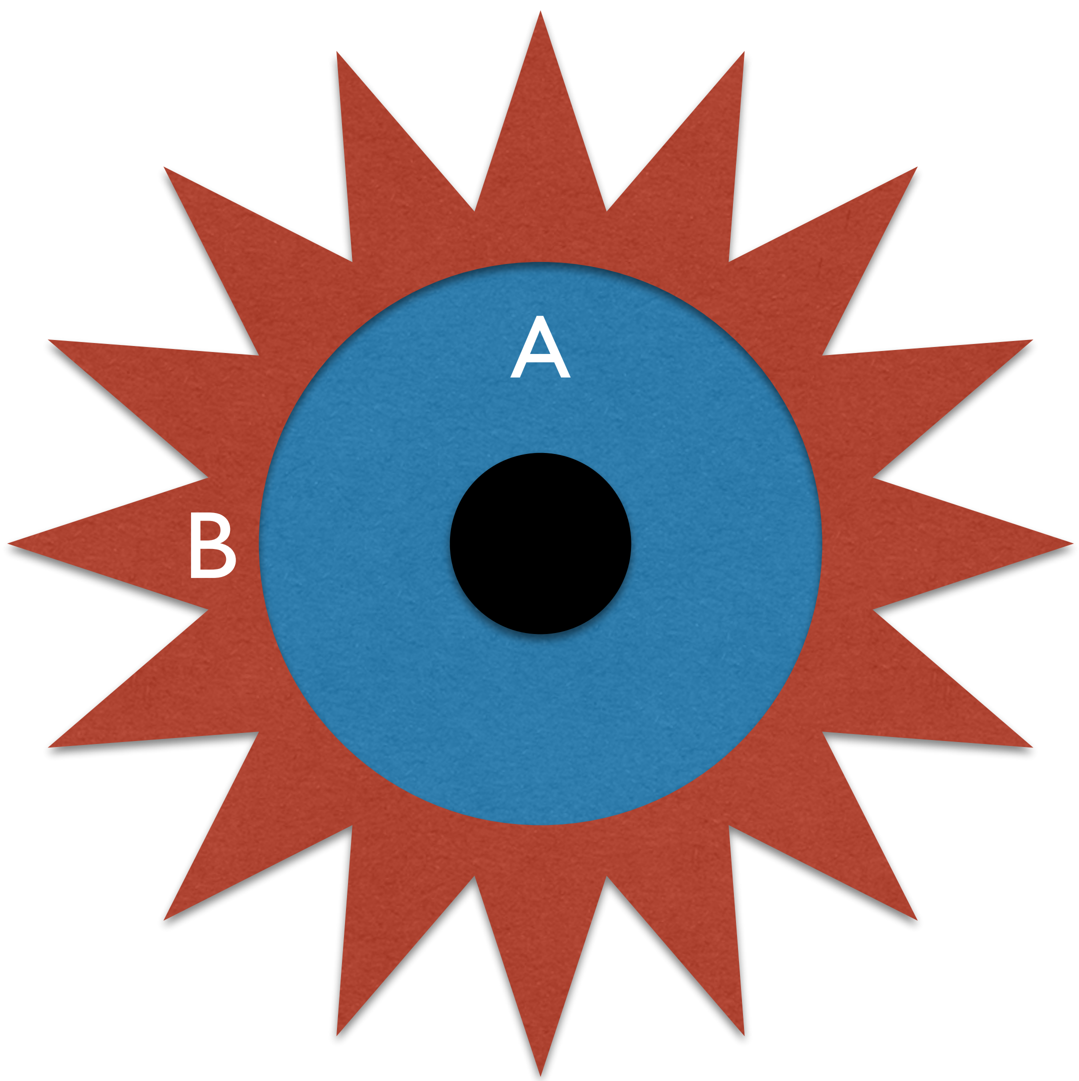
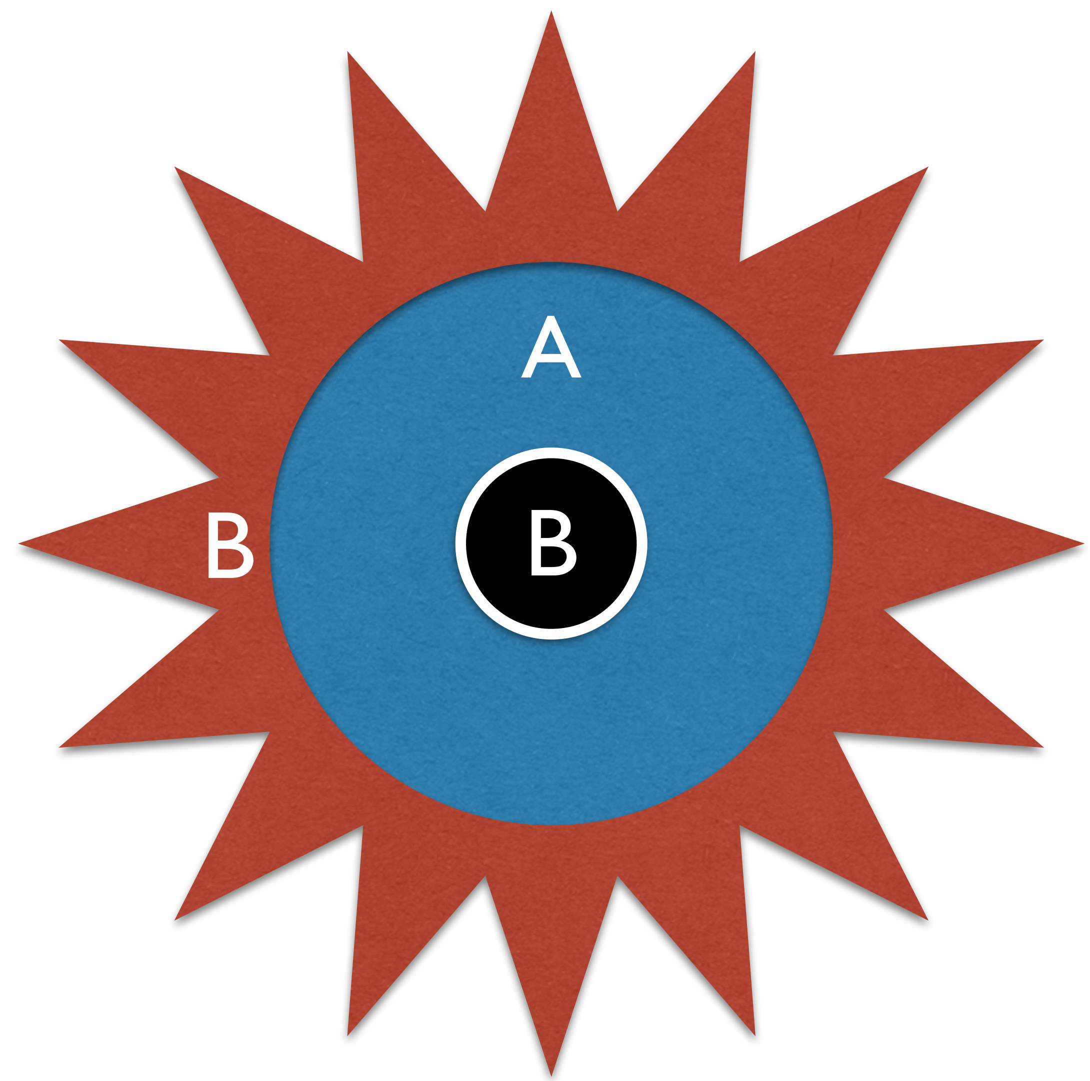




$t \ll t_{Page}$ few Hawking quanta $S_{gen} = S_A$

Radiation entropy ($S_{A \cup B} = 0$) $S_{rad} = S_B = S_A$

$t \gg t_{Page}$ many Hawking quanta $S_{gen} = A/4G$

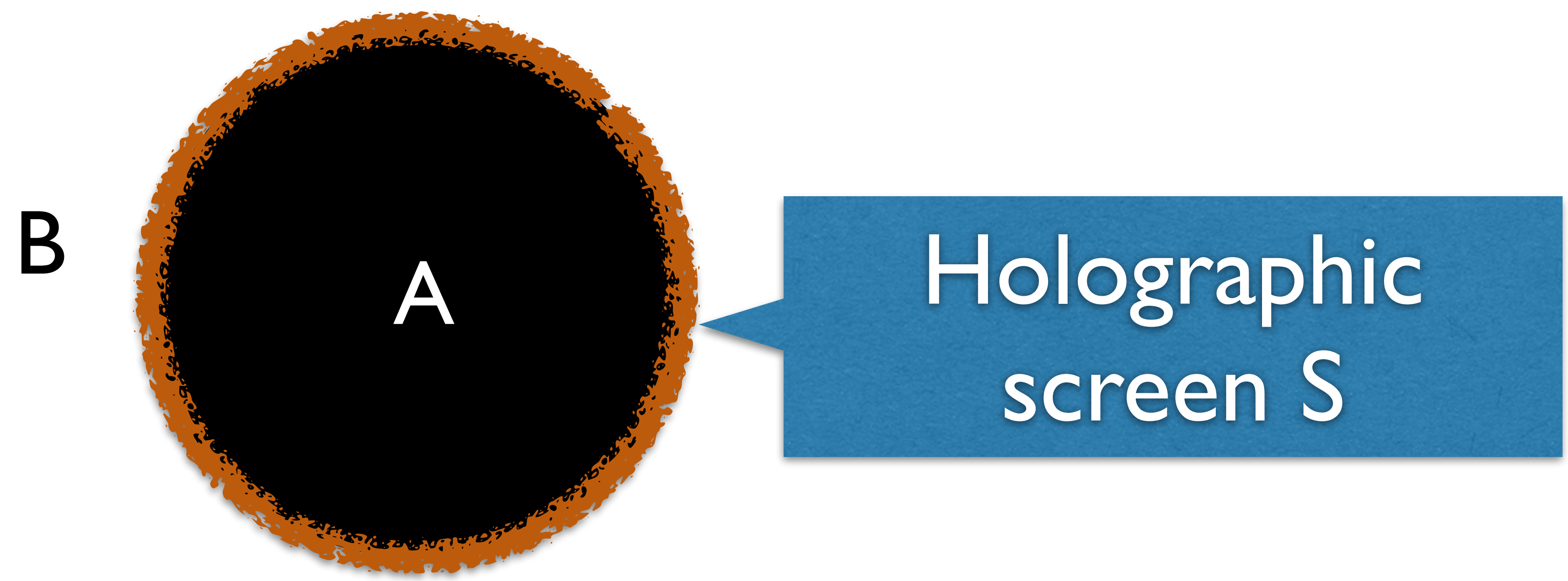
Radiation entropy $S_{rad} = S_{gen} \sim A/4G \underset{t \to +\infty}{\rightarrow} 0$

As a formula for entropy of region (A) bound by the QES

$$S_{gen}(H) = \frac{A(H)}{4G} + S_{bulk}(A)$$

applies also to asymptotically flat spacetime

To understand better how the Page curve problem is resolved notice that the "AdS leaky boundary" story can be generalized to flat space



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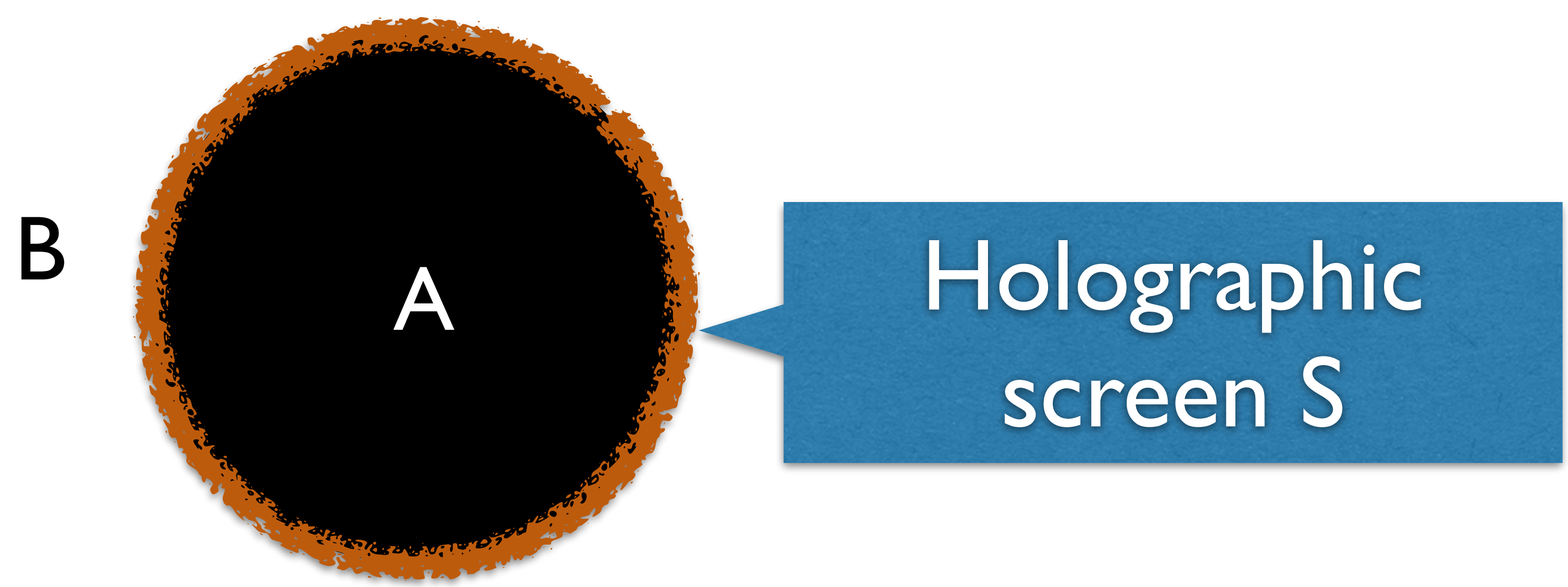
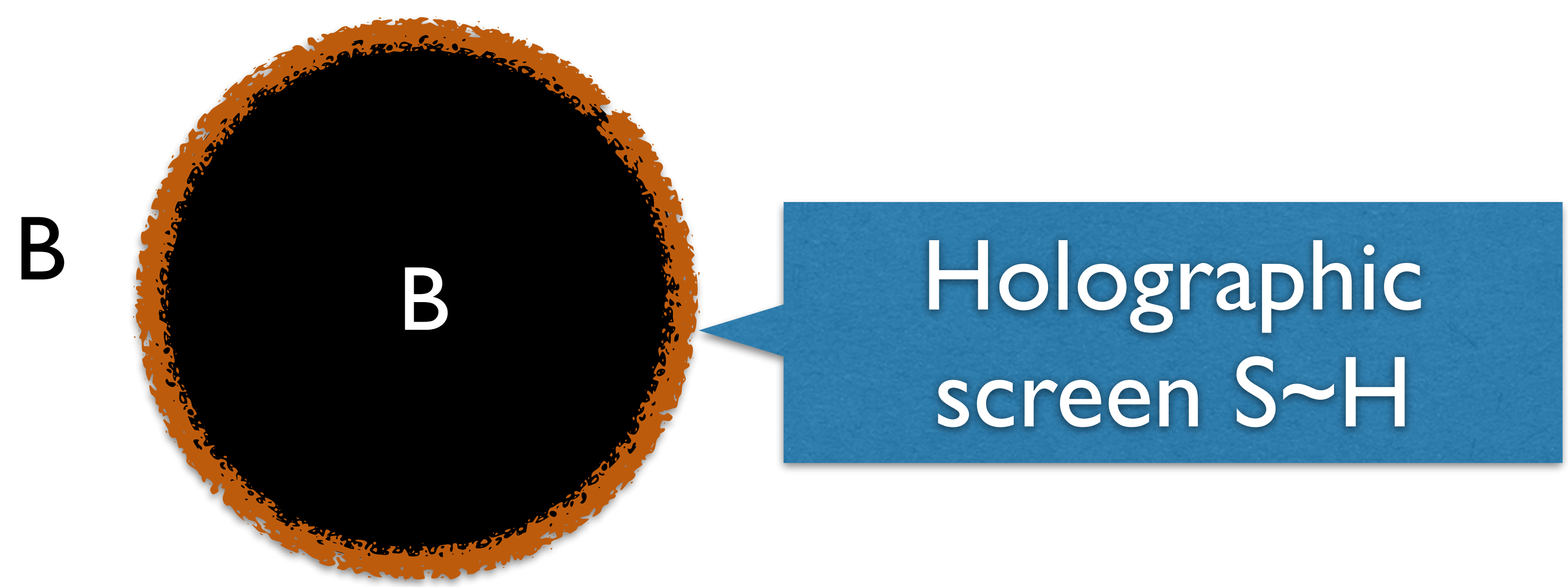


At $t \ll t_{Page}$ all the region (A) inside the BH is encoded on the holographic screen (S) and $H = \emptyset$.

All entropy is due to Hawking quanta that entangle (A) with (B) $S_{gen}(H) = S_{bulk}(B) = S_{bulk}(A)$

At $t \sim t_{Page}$ $S_{bulk}(A) > A/4G$ so the value for $S_{gen}(H)$ can be made smaller by putting the surface (H) at the horizon —price to pay: $A/4G$

Gain: $S_{bulk}(B) = 0$ because on the new (B) the radiation state is pure $\Psi = \sum |n_{in}\rangle \otimes |n_{out}\rangle$



At $t \sim t_{Page}$ $S_{bulk}(A) > A/4G$ so the value for $S_{gen}(H)$ can be made smaller by putting the surface (H) at the horizon —price to pay: $A/4G$

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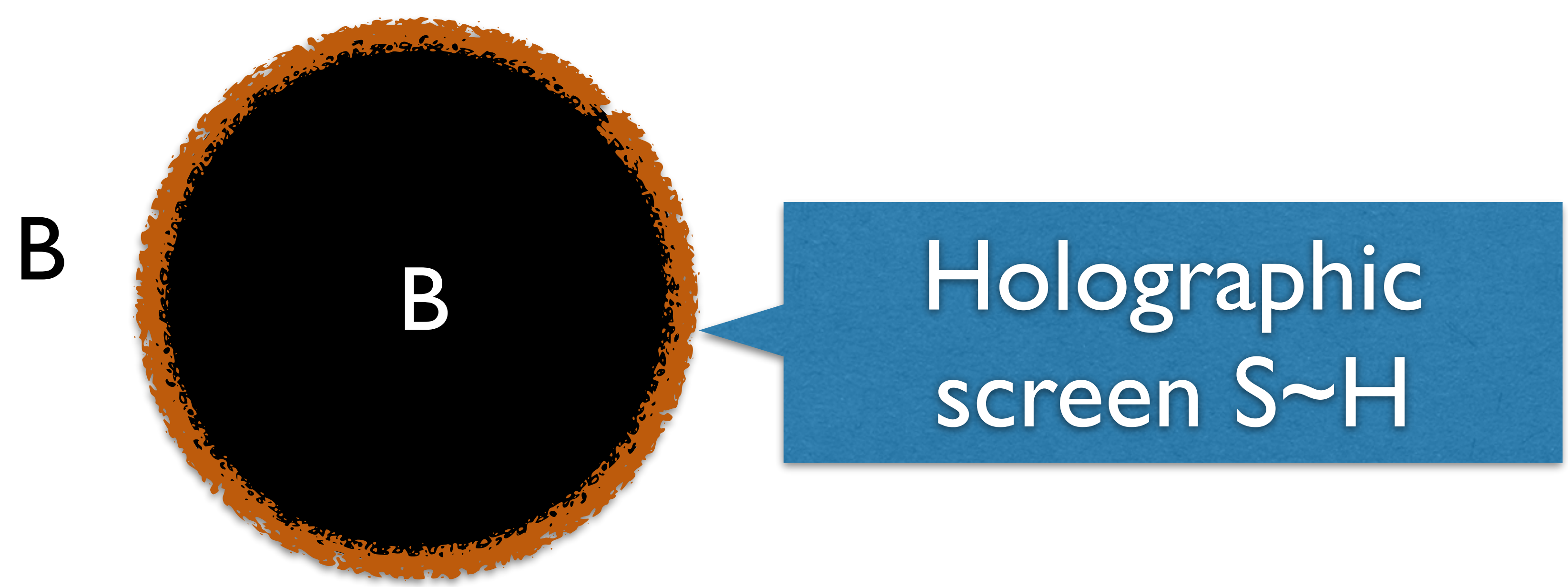
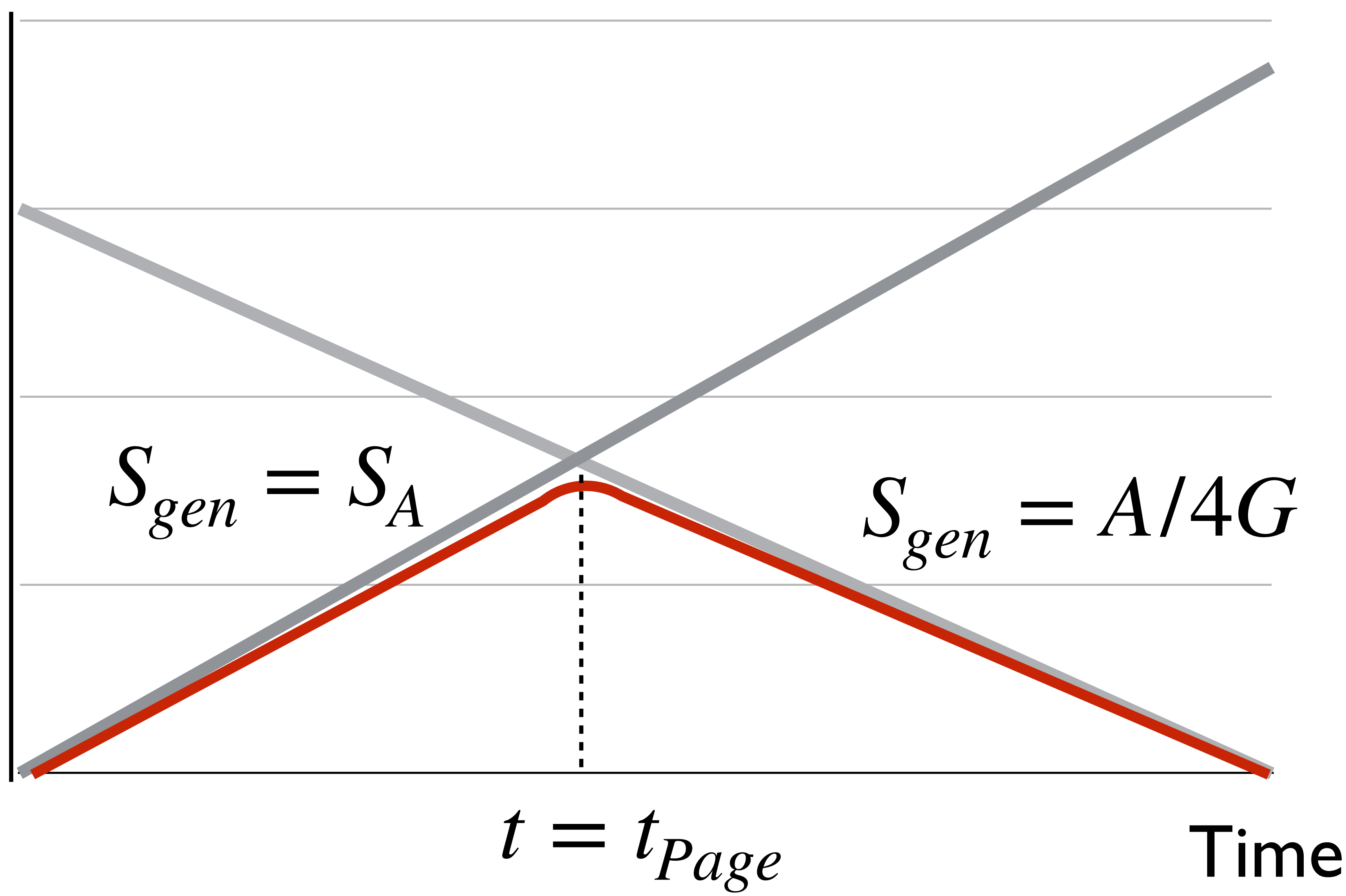


At $t \gtrsim t_{Page}$ the holographic screen (S) does not encode ANY information on the BH which is instead encoded in the Hawking radiation

PAGE CURVE

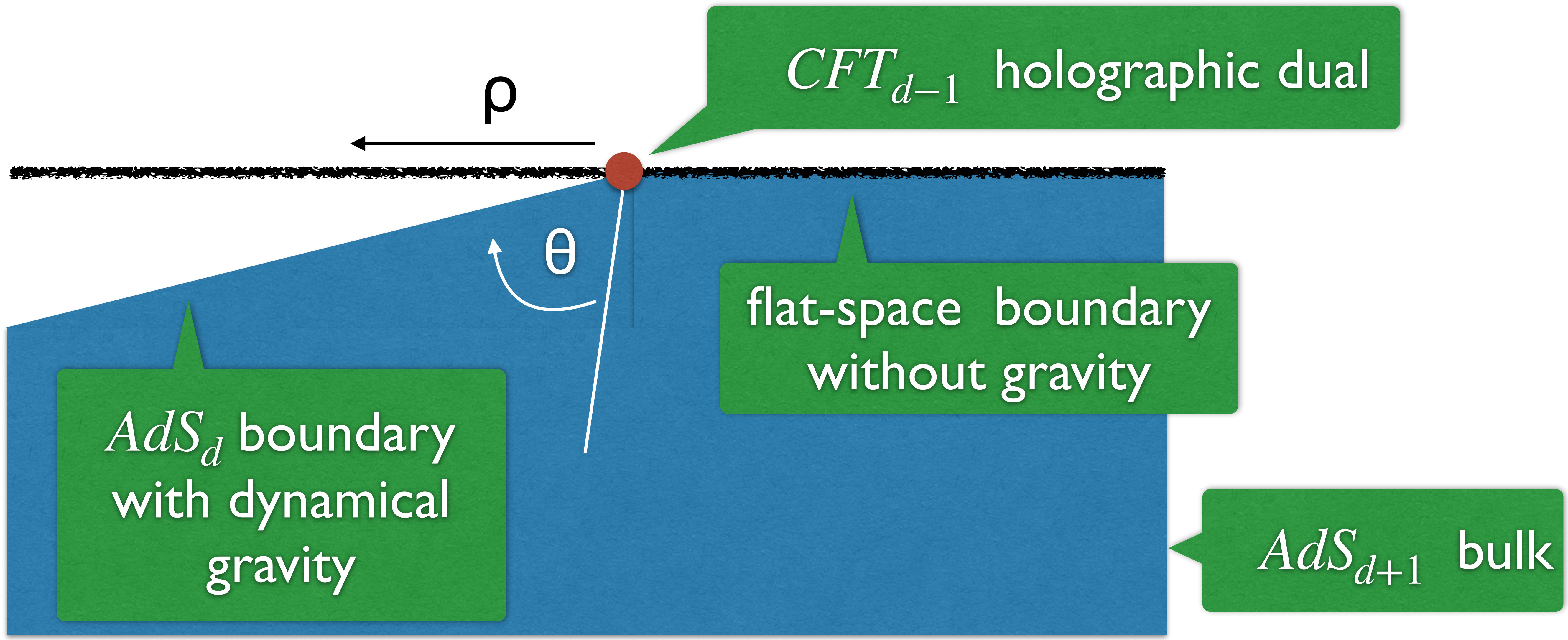
At $t \sim t_{Page}$ S_{gen} stops being dominated by the entanglement entropy of Hawking quanta across the horizon and becomes dominated by the Bekenstein-Hawking term $A/4G$



AdS boundary at $\theta = \pi/2 - \epsilon$ in coordinates

$$ds^2 = \frac{d\rho^2 + \rho^2 d\theta^2 - dt^2}{\rho^2 \cos^2 \theta}$$

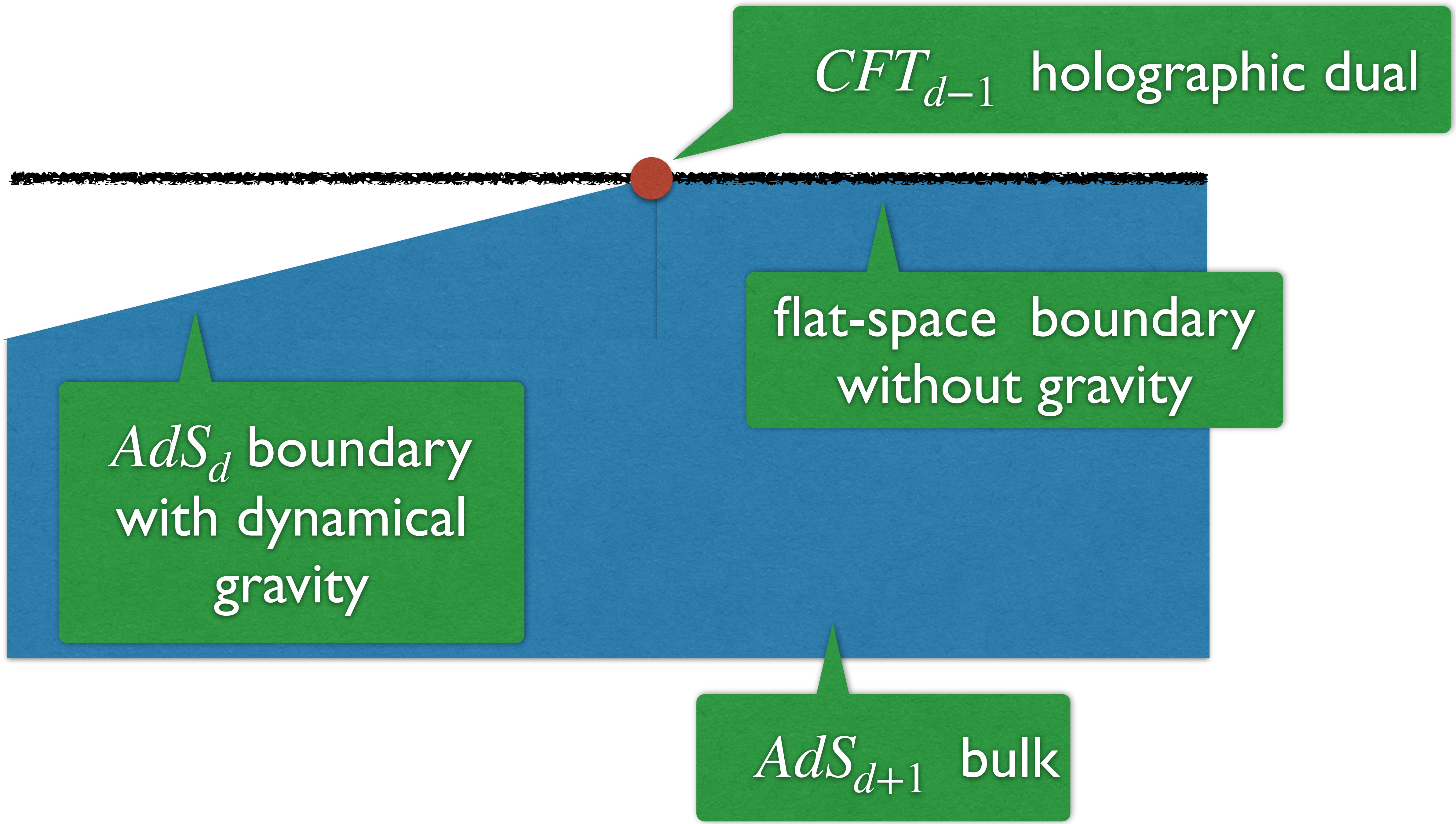
$t =$ constant section:



This is the Karch-Randall model (2000)

QES from HRT

QES is HRT in a double-holography model



In the bulk $S_{bulk}(A)$ is already encoded in the purely geometric HRT surface

QES from HRT

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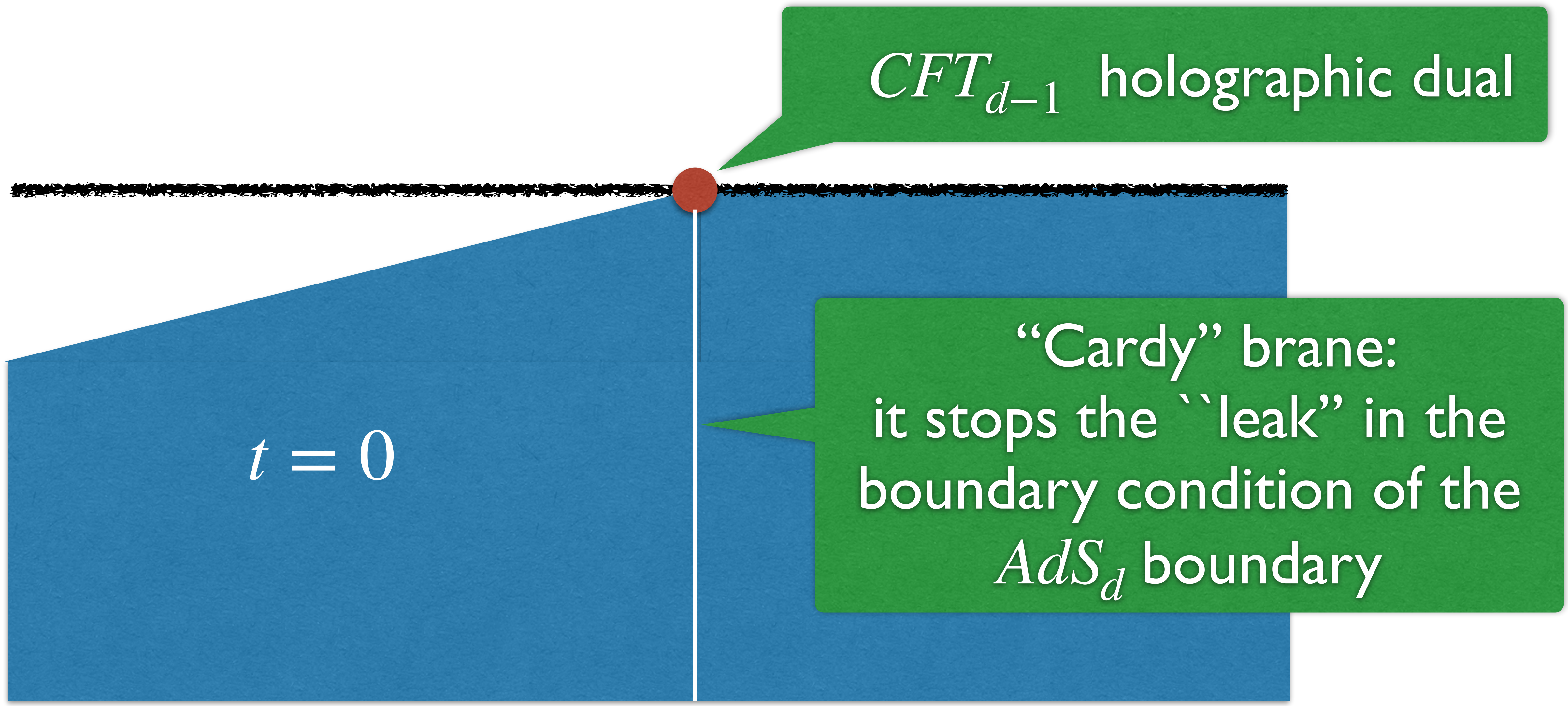


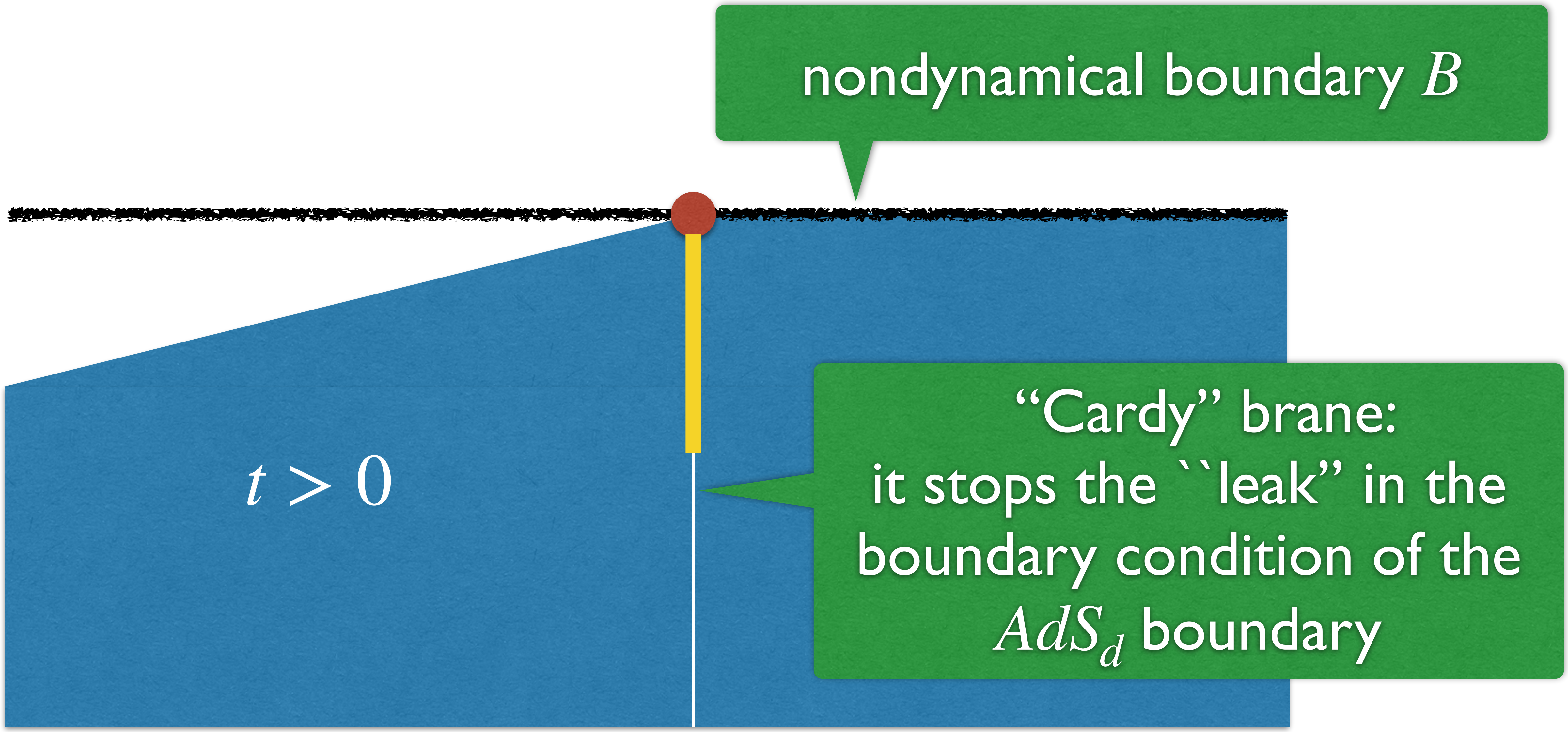
At t=0 the state of the dynamical AdS boundary is pure

$$S(AdS_d) = 0$$

The HRT is simply ● and has zero area

QES from HRT

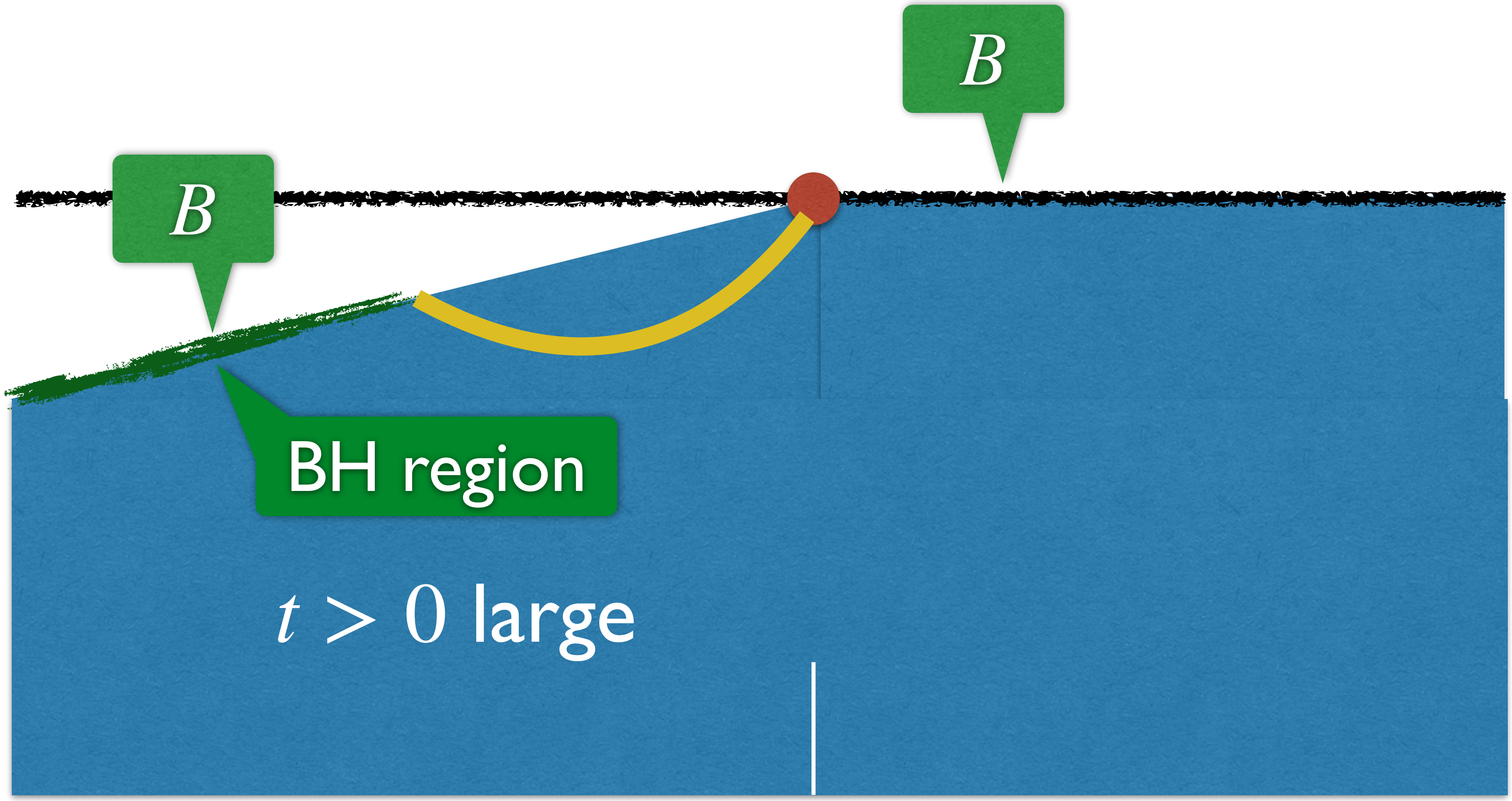
QES is HRT in a double-holography model



At t>0 the state of the dynamical AdS boundary is entangled with radiation into the nondynamical boundary so $S(AdS_d) > 0$
The minimal HRT is the yellow line and equals the full
$S_{gen}(\partial AdS_d) = S_{bulk}(B)$

QES from HRT

QES is HRT in a double-holography model



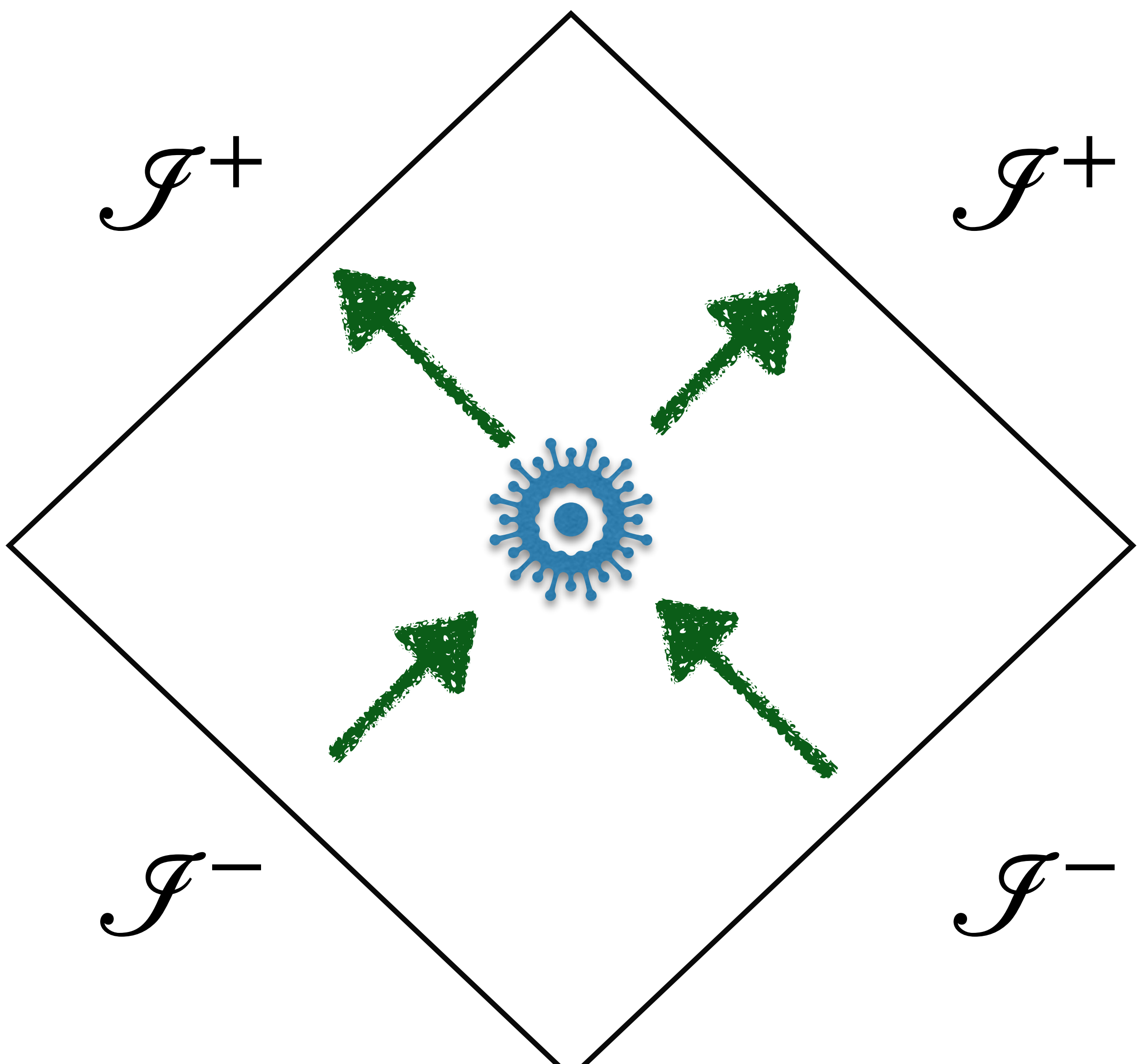
At large t>0 the state of the dynamical AdS boundary is entangled with radiation into the nondynamical boundary.
The minimal HRT changes to the orange line and now equals
$$S_{gen}(\partial AdS_d) \approx A/4G$$

INTERLUDE

Another surface that has a story to tell is the CELESTIAL SPHERE

Asymptotically flat space in Bondi coordinates

$$ds^2 = -du^2 - 2dudr + r^2\left(h_{AB} + \frac{C_{AB}}{r}\right)d\theta^A d\theta^B + \frac{2m}{r}du^2 +$$
$$+\frac{1}{r}\left(\frac{4}{3}(N_A + u\partial_A m) - \frac{1}{8}\partial_A(C_{BD}C^{BD})\right)dud\theta^A + \ldots.$$



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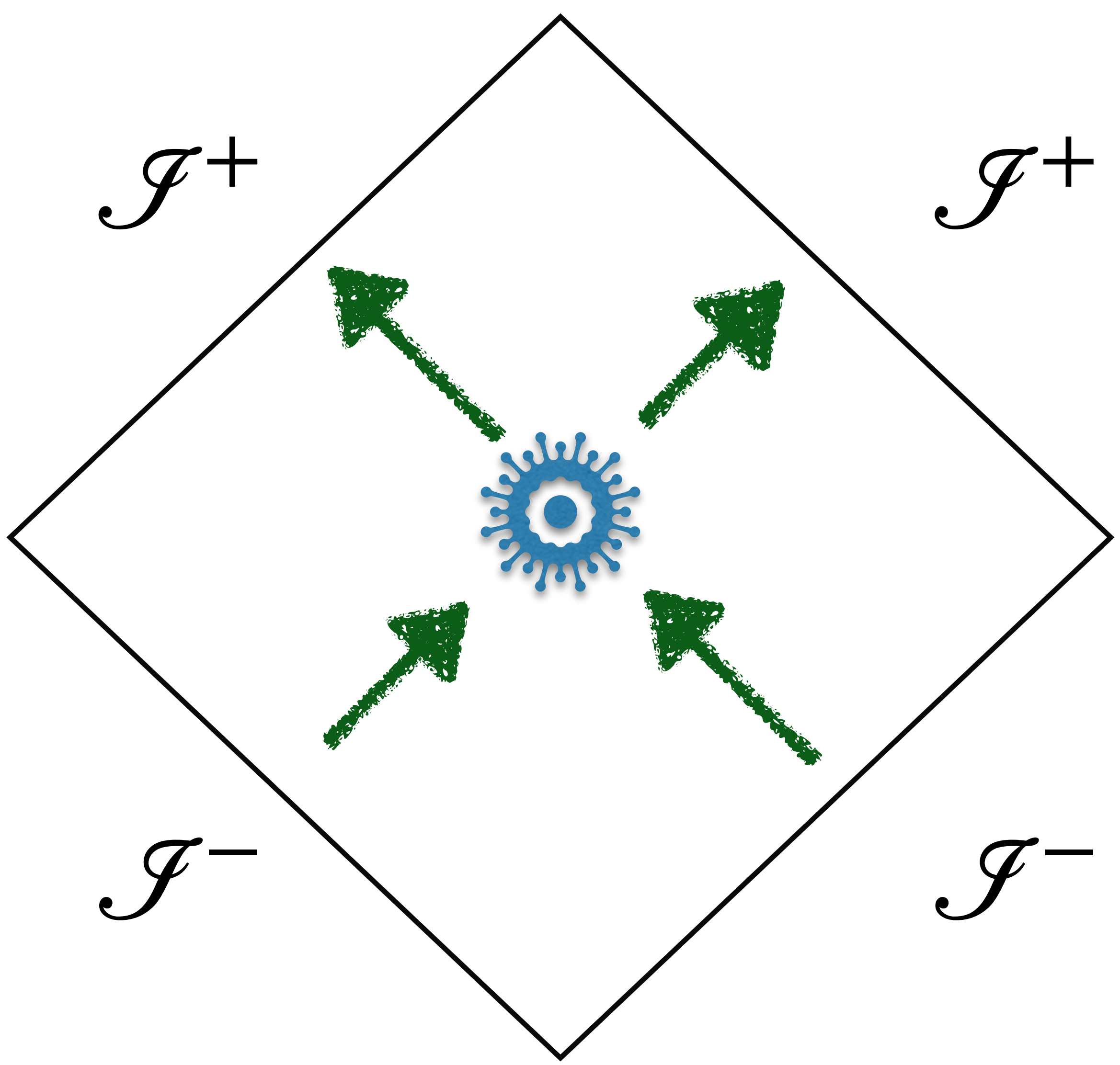

Boundary $\mathscr{I}^\pm$ is $\mathbb{R} \times S^2$

S^2=Celestial Sphere - coord. θ^A

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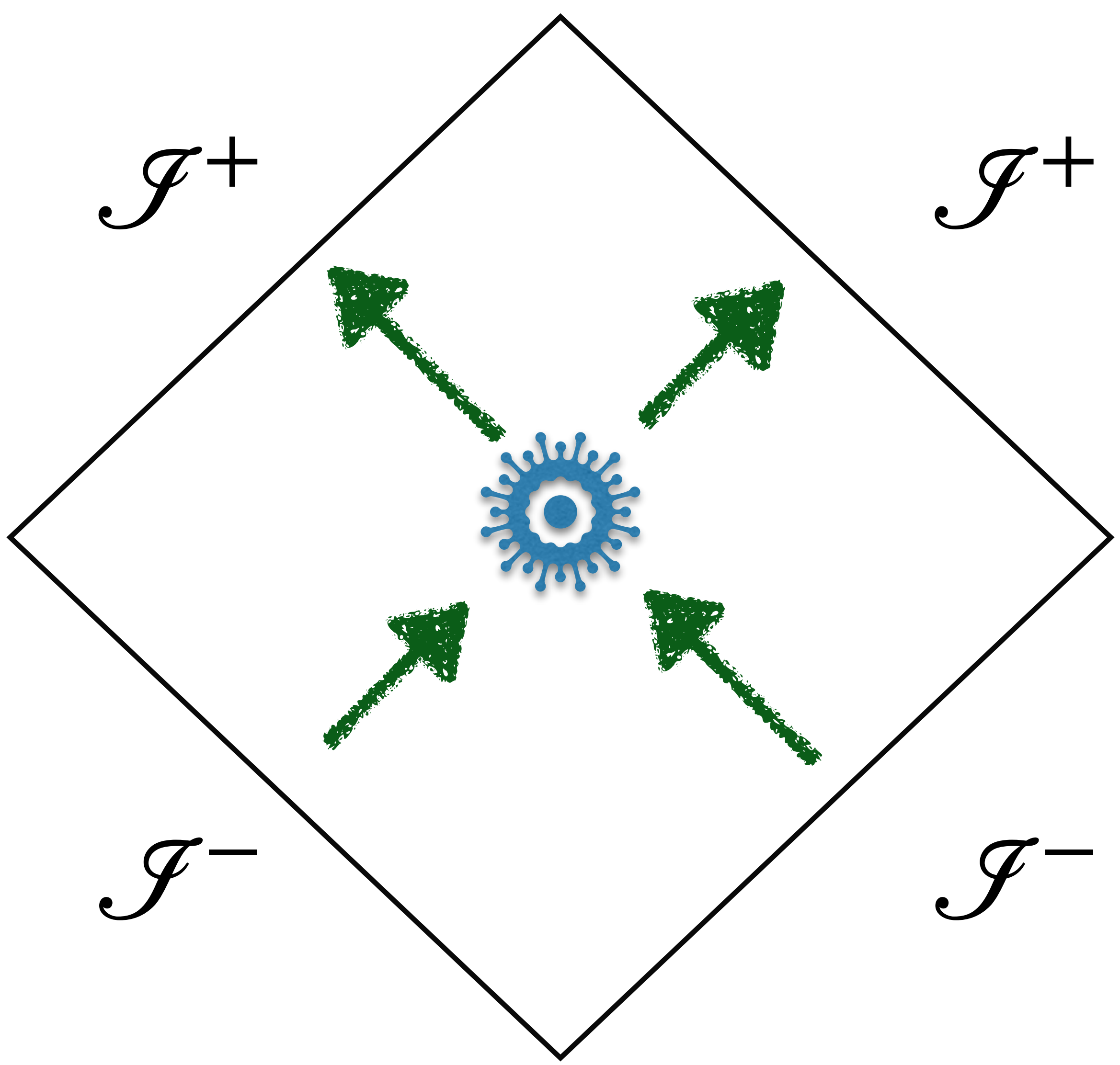

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S-matrix in momentum basis is transformed into a formal correlator of primaries of a generically non-unitary CFT by a change of basis

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For scalars

$$\langle 0 | \prod_I O_{\Delta_I}(\theta_I) | 0 \rangle = \left(\prod_I \int \frac{d^3 p_I}{2 p_I^0} G_{\Delta_I}(\theta, p_I) \right) A(p_I)$$

Correlation function of primaries in (unknown) CFT

Bulk-to-boundary propagator for $(-\Box + m^2)\Psi = 0$

S-matrix amplitude in momentum basis

OPEs of CFT operators reproduced at coincident points on S^2 =collinear singularities in momentum space

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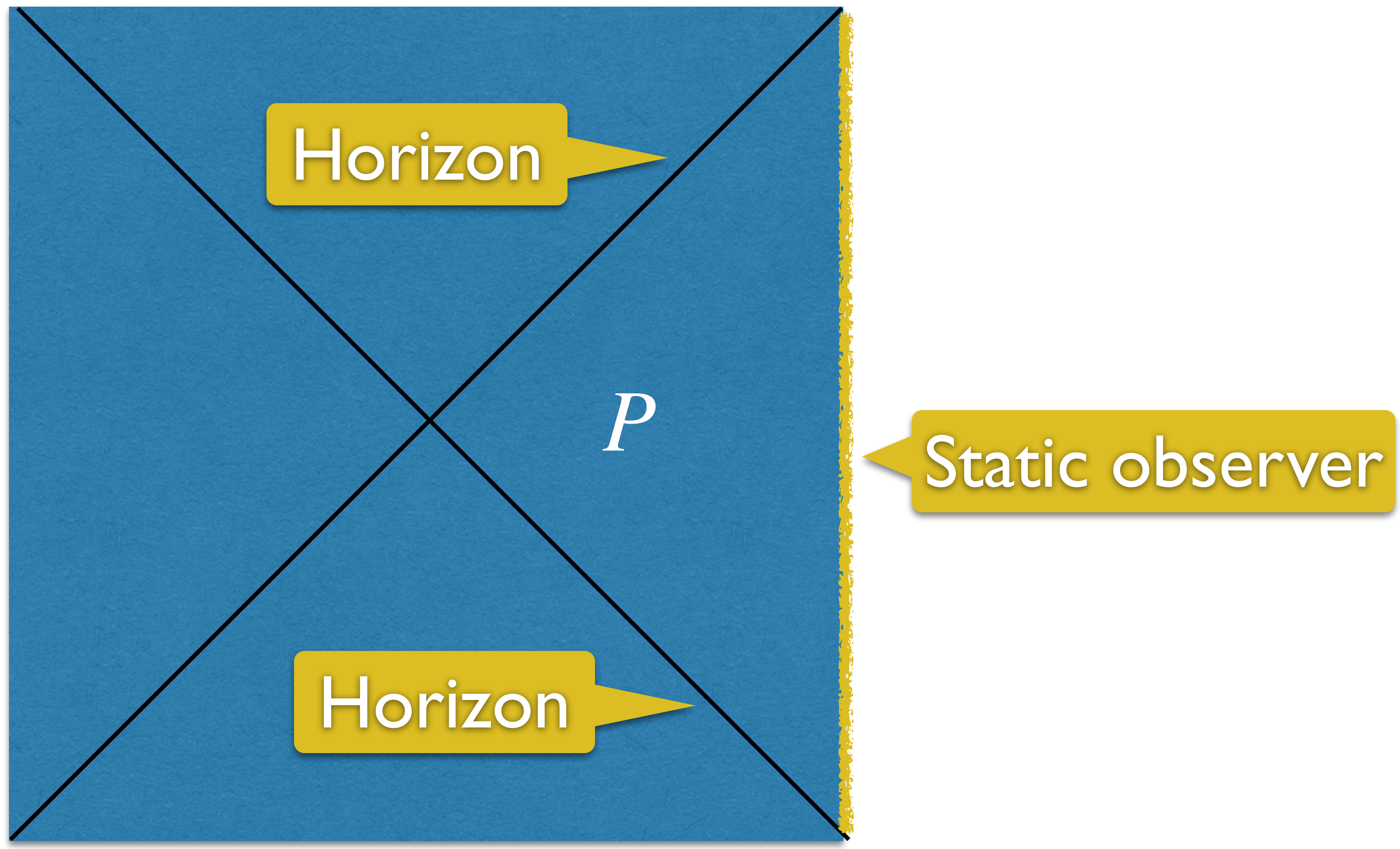
Many developments of this initial idea as in GGI workshop "From Asymptotic Symmetries to Flat Holography" (2025)

The QES formula can be extended to de Sitter space
Where is the boundary?



Instead of the boundary the surface term in S_{gen} is the horizon of a static observer in static de Sitter $r = H$

$$ds^2 = -\left(1 - \frac{r^2}{H^2}\right) dt^2 + \left(1 - \frac{r^2}{H^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

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The generalized entropy of the static patch P is

$$S_{gen}(P) = \frac{A(P)}{4G} + S_{bulk}(P) \leq \frac{4\pi H^2}{4G}$$

It is less than the entropy $\dfrac{4\pi H^2}{4G}$ of the pure de Sitter state Ψ_{HH}

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$$S_{gen}(P) = \frac{A(P)}{4G} + S_{bulk}(P) \leq \frac{4\pi H^2}{4G}$$

It is less than the entropy $\frac{4\pi H^2}{4G}$ of the pure de Sitter state Ψ_{HH}

Ψ_{HH} = Hartle-Hawking state

It entangles the two static coordinate patches making up a maximal Cauchy surface

The relative entropy of the static patch P is

$$S_{rel}(P) = \frac{4\pi H^2}{4G} - S_{gen}(P)$$

It can be computed rigorously in perturbation theory in G_{Newton} even though the local operator algebra of the bulk QFT is a Type III von Neumann factor —an algebra that admits no trace

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Key to the computation is the introduction of an "observer" with Hamiltonian q (a canonical coordinate) so that the Hamiltonian constraint becomes [Chandrasekaran & al. (2022)]

$$[H + q, O_{phys}] = 0 \Rightarrow O_{phys} = e^{ipH} O e^{-ipH}$$

$\mathscr{A} = \{e^{ipH} O e^{-ipH}, q\}$ is Type II when $\mathscr{B} = \{O\}$ is Type III

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For instance

separate $t \to t' = t + \Delta(t, \vec{x})$ into the global generator $H + q$ and generators of time reparametrizations such that $\int d^3x \hat{\Delta} = 0$

Call Q^B the BRST operator associated to $\hat{\Delta}$

The operator O used above is in reality a $\hat{O}$ which must obey

$$[Q^B, \hat{O}] = 0$$

PAID ADVERTISEMENT [Grassi & Porrati (2024)]

How do we get $\hat{O}$ from a local operator O?

Rewrite the physical state condition as the BRST cohomology

$$[Q^B, \hat{O}]_\pm = 0 \qquad \hat{O} \sim \hat{O} + [Q^B, X]_\mp$$

and next write the BRST operator as the sum

$$Q^B = Q_0 + Q_I$$

The charge Q_0 is quadratic in the ghost, gauge and matter fields all interaction terms are in Q_I

Suppose that an operator Ω exists with the property $Q_0\Omega = \Omega Q^B$ (i.e. Ω is a conjugation or intertwiner)

and that the Q_0 cohomology is known

PAID ADVERTISEMENT [Grassi & Porrati (2024)]

Ω defines an isomorphism
between the cohomologies of Q_0 and Q^B because

$$[Q^B, \Omega^\dagger O \Omega]_\pm = \Omega^\dagger [Q_0, O]_\pm \Omega$$

implies

$$[Q_0, O]_\pm = 0 \Leftrightarrow [Q^B, \Omega^\dagger O \Omega]_\pm = 0$$

$$X = [Q_0, O]_\pm \Leftrightarrow \Omega^\dagger X \Omega = [Q^B, \Omega^\dagger O \Omega]_\pm$$

PAID ADVERTISEMENT [Grassi & Porrati (2024)]

Ω can be constructed using methods described in [Grassi & Porrati (2024)]

We do not need to know the explicit form of Q^B
We need to know only its quadratic part Q_0 and that its interacting terms have positive S-charge

The construction of Ω requires $(Q^B)^2 = 0$ so it does not apply to anomalous gauge theories

SUMMARY+CONCLUSIONS

- We used the QES to illustrate one of several pathways along which theoretical physics developed since the founding of GGI
- It can be described as "geometrization of quantum information": holography suggests that subtle QI quantitites become non-so-subtle geometrical objects in a dual theory
- Other quantities include complexity, which is conjectured to be dual to a certain volume (or classical action) of a bulk geometry.
- Of course other surfaces are still waiting to tell their *histoire* in full. Among them is the surface our story started with: the black hole horizon. Is it singular [Firewalls, Almheiri & al. (2013)]? Regular [Maldacena & Susskind (2013)]?