Sum rules for conformal defect data

Philine van Vliet

May 18, 2026

Based on:
Arxiv: 2509.26561 with B. Girault and M. Paulos
Work in progress with G. Bliard, J. Julius, M. Paulos, N. Suchel and B. Gabai, V. Gorbenko, B. Offertaler, J. Qiao

LABORATOIRE DE PHYSIQUE DE L'ÉCOLE NORMALE SUPÉRIEURE

Motivation

- Topic of today: conformal defects which break

$$SO(d+1,1) \to SO(p+1,1) \times SO(d-p)\,.$$

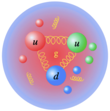
- Correspond to physical setups, e.g. boundaries, magnetic impurity (pinning line defect).
- In AdS/CFT: dual to branes, flux tubes.
- Wilson lines in gauge theories.
- Can be studied perturbatively and nonperturbatively (bootstrap).

Outline

1. Introduction
2. Broken Ward identities and defect sum rules
   Example I: Flux tube in AdS_3
3. Injecting the bulk
   Example II: The $1/2$ BPS Wilson line in $\mathcal{N} = 4$ SYM
4. Conclusion

Symmetry-breaking constraints

- Spontaneous symmetry breaking in QCD leads to sum rules on scattering amplitudes: Adler zero [Adler '62], soft pion theorems [Weinberg '66].



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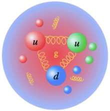

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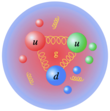
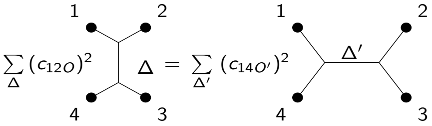


- Similar constraints for explicit breaking of symmetry by considering conformal defects.
- Some constraints already known [Kutasov '89, Friedan, Konechny '12, Billò, Goncalves, Lauria, Meineri '16, Behan '17, Drukker, Kong, Sakkas '22, Gabai, Sever, Zhong '25, Ferrando, Sever, Urisman '25 . . .]
- Here we present unified framework to derive general sum rules + new ones.
  See also [Belton, Kong '25, Drukker, Kong, Kravchuk '25]

The conformal bootstrap

- Way to obtain nonperturbative and rigorous results for CFT data.
- Solve crossing equations either analytically or numerically.



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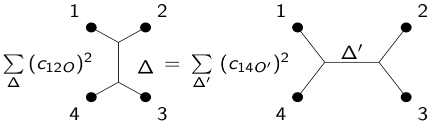
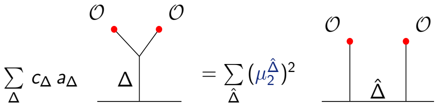


- For defects there are technical difficulties for numerics:
  - No information from the bulk [Gimenez-Grau, Lauria, Liendo, PvV '22].
  - No positivity for bulk operators.



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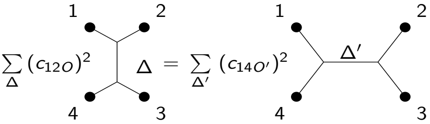

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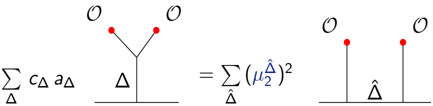

⇒ Combine bootstrap with defect soft sum rules and include bulk operators in a positive semi-definite way.

- See also [Lanzetta, Liu, Metlitski '25, Meineri, Radhakrishnan '25]

Broken Ward identities

&

defect sum rules

Symmetry breaking

- For a symmetry G we have conserved currents and charges

$$\partial_\mu \mathcal{J}^{S,\mu} = 0 \Leftrightarrow Q^S = \int_\Sigma \star \mathcal{J}^S \,, \quad [Q^S, Q^T] = f^{ST}{}_U Q^U \,.$$

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- Broken symmetry $G \to H$ lead to $Q^S \to Q^s, Q^{\tilde{s}}$:

$$[Q^s, Q^t] = f^{st}{}_u \, Q^u \,, \quad [Q^{\tilde{s}}, Q^{\tilde{t}}] = f^{\tilde{s}\tilde{t}}{}_{\tilde{u}} \, Q^{\tilde{u}} + f^{\tilde{s}\tilde{t}}{}_u \, Q^u \,,$$
$$[Q^s, Q^{\tilde{t}}] = \underbrace{f^{s\tilde{t}}{}_{\tilde{u}} \, Q^{\tilde{u}}}_{\text{choice of basis}}$$

Ward identities

- Defect breaks symmetry:
  $G \times SO(d+1,1) \to H \times SO(p+1,1) \times SO(q)$.

Ward identities

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  $G \times SO(d+1,1) \to H \times SO(p+1,1) \times SO(q)$.
- Lead to broken Ward identities

$$\text{flavour}: \quad Q^{\tilde{s}}_{\Sigma} |0\rangle_{\mathcal{D}} = \int_{\Sigma} d^p x\, d^q y\, \partial_\mu \mathcal{J}^{\tilde{s},\mu} |0\rangle_{\mathcal{D}} = \int_{\Sigma} d^p x\, t^{\tilde{s}} |0\rangle_{\mathcal{D}}$$

$$\text{conformal}: \quad Q^{a}_{\Sigma} |0\rangle_{\mathcal{D}} = \int_{\Sigma} d^p x\, d^q y\, \partial_\mu T^{\mu,a} |0\rangle_{\mathcal{D}} = \int_{\Sigma} d^p x\, \xi^a_m D^m |0\rangle_{\mathcal{D}}$$

$$\text{SUSY}: \quad Q_{\alpha,\Sigma} |0\rangle_{\mathcal{D}} = \int_{\Sigma} d^p x\, d^q y\, P_{\alpha\beta} \partial_\mu J^{\mu\beta} |0\rangle_{\mathcal{D}} = \int_{\Sigma} d^p x\, \psi_\alpha |0\rangle_{\mathcal{D}}$$

- with three protected operators:

$$\text{tilt } t: \quad \Delta_t = p\,,$$

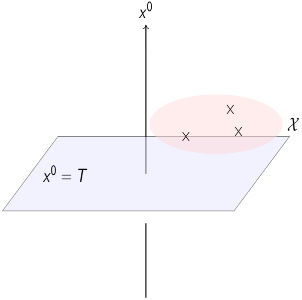
$$\text{displacement } D: \quad \Delta_D = p+1\,,$$

$$\text{displacino } \psi: \quad \Delta_\psi = p + \frac{1}{2}\,,$$

Defect soft identity

- Take correlators with bulk insertions $\mathcal{X} = \mathcal{O}_1(x_1, y_1) \ldots \mathcal{O}_i(x_i, y_i)$

$$\langle \mathcal{Q}^{\tilde{s}} \mathcal{X} \rangle = \langle \delta_{\tilde{a}} \mathcal{X} \rangle \, .$$



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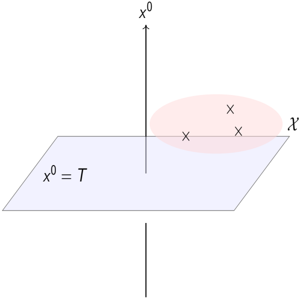

- Take correlators with bulk insertions $\mathcal{X} = \mathcal{O}_1(x_1, y_1) \dots \mathcal{O}_i(x_i, y_i)$

$$\langle \mathcal{Q}^{\tilde{s}} \mathcal{X} \rangle = \langle \delta_{\tilde{a}} \mathcal{X} \rangle \, .$$

- For insertion of $Q^{\tilde{s}}$ at time T, close contour to past or future:

$$-\int_{x^0 > T} d^p x \, \langle t^{\tilde{s}}(x) \mathcal{X} \rangle = \int_{x^0 < T} d^p x \, \langle t^{\tilde{s}}(x) \mathcal{X} \rangle + \langle \delta_{\tilde{s}} \mathcal{X} \rangle \, ,$$

- resulting in

$$\int d^p x \, \langle t^{\tilde{s}}(x) \mathcal{X} \rangle = -\langle \delta_{\tilde{s}} \mathcal{X} \rangle \, .$$

Double-soft identity

- What if we insert two charges?

$$\langle \mathcal{X}_L \, [Q^{\tilde{s}}, Q^{\tilde{t}}] \, \mathcal{X}_R \rangle = f^{\tilde{s}\tilde{t}}{}_{u} \langle \mathcal{X}_L Q^u \mathcal{X}_R \rangle + f^{\tilde{s}\tilde{t}}{}_{\tilde{u}} \langle \mathcal{X}_L Q^{\tilde{u}} \mathcal{X}_R \rangle \, .$$

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- Integrate to the past and to the future gives

$$\begin{aligned} & -\int_{x^0>T}\int_{x'^0<T} d^p x\, d^p x'\, \langle \mathcal{X}_L\, t^{[\tilde{s}}(x) t^{\tilde{t}]}(x') \mathcal{X}_R \rangle - \langle \delta^{[\tilde{s}} \mathcal{X}_L\, \delta^{\tilde{t}]} \mathcal{X}_R \rangle \\ & - \int_{x^0>T} d^p x \langle \mathcal{X}_L\, t^{[\tilde{s}}(x) \delta^{\tilde{t}]} \mathcal{X}_R \rangle - \int_{x'^0<T} d^p x'\, \langle \delta^{[\tilde{s}} \mathcal{X}_L\, t^{\tilde{t}]}(x') \mathcal{X}_R \rangle \\ & = f^{\tilde{s}\tilde{t}}{}_u \langle \mathcal{X}_L \delta_u \mathcal{X}_R \rangle + f^{\tilde{s}\tilde{t}}{}_{\tilde{u}} \left[\langle \mathcal{X}_L \delta_{\tilde{u}} \mathcal{X}_R \rangle + \int_{x^0<T} d^p x\, \langle \mathcal{X}_L\, t^{\tilde{u}}(x)\, \mathcal{X}_R \rangle \right]\ . \end{aligned}$$

Apply the identities

- Let's simplify: $p = 1$ and $O(N) \to O(N-1)$ s.t. $f^{\tilde{s}\tilde{t}}{}_{\tilde{u}} = 0$.

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$$\langle t^{\tilde{s}}(x) \rangle = 0 \,, \quad \langle t^{\tilde{s}}(x) t^{\tilde{t}}(0) \rangle = \delta^{\tilde{s}\tilde{t}} \frac{C_t}{x^2} \,,$$

$$\langle t^{\tilde{s}}(x_1) t^{\tilde{t}}(x_2) \hat{\mathcal{O}}(x_3) \rangle = \frac{\hat{c}_{tt\hat{\mathcal{O}}} \, \delta^{\tilde{s}\tilde{t}}}{x_{12}^{2-\hat{\Delta}} (x_{13} x_{23})^{\hat{\Delta}}} \,, \quad x_{ij} = x_i - x_j \,.$$

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- Want to constrain 4-point functions on the defect
  $\to$ infinite amount of defect CFT data ($\hat{\Delta}_i, \hat{c}_{ijk}, C_t, C_D$).
- 4-point correlator is function of single cross-ratio:

$$\langle t^{\tilde{s}}(x_1) t^{\tilde{t}}(x_2) t^{\tilde{u}}(x_3) t^{\tilde{v}}(x_4) \rangle = \frac{\mathcal{G}^{\tilde{s}\tilde{t}\tilde{u}\tilde{v}}(z)}{x_{12}^2 x_{34}^2}\,, \quad z = \frac{x_{12} x_{34}}{x_{13} x_{24}}\,.$$

Apply the identities - soft

- Start from $\langle Q^{\tilde{s}} \mathcal{J}^{\tilde{t}} \mathcal{J}^{\tilde{u}} \mathcal{J}^{\tilde{v}} \rangle$, and

$$\mathcal{J}_{\mu}^{\tilde{s}}(x, y) \sim \mu_J^t \, t^{\tilde{s}}(x) |y|^{\Delta_{\mathcal{J}} - \Delta_t} + \ldots \quad \text{defect OPE}$$

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- Soft identity gives: [Behan '17, Gabai, Sever, Zhong '25]

$$\int_{-\infty}^{\infty} dx \, \langle t^{\tilde{s}}(x) \mathcal{J}^{\tilde{t}}(x_1,y_1) \mathcal{J}^{\tilde{u}}(x_2,y_2) \mathcal{J}^{\tilde{v}}(x_3,y_3) \rangle$$

$$= \langle \delta_{\tilde{s}} \mathcal{J}^{\tilde{t}}(x_1,y_1) \mathcal{J}^{\tilde{u}}(x_2,y_2) \mathcal{J}^{\tilde{v}}(x_3,y_3) \rangle \, .$$

- Push bulk operators to defect: $y \to 0$

$$\boxed{\int_{-\infty}^{\infty} dx \, \langle t^{\tilde{s}}(x) t^{\tilde{t}}(x_1) t^{\tilde{u}}(x_2) t^{\tilde{v}}(x_3) \rangle = 0 \, .}$$

Apply the identities - double soft

- Start from $\langle \mathcal{J}^{\tilde{s}}\ [Q^{\tilde{t}}, Q^{\tilde{u}}]\ \mathcal{J}^{\tilde{v}} \rangle$, then double-soft identity and $y \to 0$

$$-\int_0^\infty dx \int_{-\infty}^0 dx' \, \langle t^{\tilde{s}}(x_1) t^{[\tilde{t}}(x) t^{\tilde{u}]}(x') t^{\tilde{v}}(x_2) \rangle = f^{\tilde{t}\tilde{u}}{}_m \langle t^{\tilde{s}} \delta^m t^{\tilde{v}} \rangle \, .$$

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- Rewrite as integral over single cross-ratio: [Friedan, Konechny '12, Drukker, Kong, Sakkas '22]

$$\boxed{\int_0^1 dz \Big[\mathcal{G}_1(z) \log z + \mathcal{G}_2(z) \log(1-z) \Big] = C_t \,.}$$

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- Displacement: more generators $P, K, M \to 5 = 2 + 3$ independent constraints. [Gabai, Sever, Zhong '25]

Functional sum rules

- Decompose 4-point correlator in conformal blocks:

$$\langle t(x_1)t(x_2)t(x_3)t(x_4)\rangle = \frac{\mathcal{G}(z)}{x_{12}^2 x_{34}^2} = \frac{1}{x_{12}^2 x_{34}^2} \sum_{\hat{\Delta}} \hat{c}_{tt\hat{\Delta}}^2 \, G_{\hat{\Delta}}(z) \, .$$

with $G_{\hat{\Delta}}(z) = z^{\hat{\Delta}} \, {}_2F_1(\hat{\Delta}, \hat{\Delta}, 2\hat{\Delta}; z)$.

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- Improve: replace $\mathcal{G}(z)$ by dispersion relation [Paulos '21]

$$\mathcal{G}(z) = -\int_0^1 d\omega\, h(z,\omega)\ \mathrm{dDisc}\ \mathcal{G}(\omega)$$

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leading to

$$\int_0^1 dz \int_0^1 d\omega\, \mathcal{K}(z,\omega)\ \text{dDisc}\ \mathcal{G}(\omega) = \text{cst} \implies \sum_{\hat{\Delta}} \hat{c}^2_{tt\hat{\Delta}}\, f^t[\hat{\Delta}] = \text{cst}\,.$$

Functional sum rules

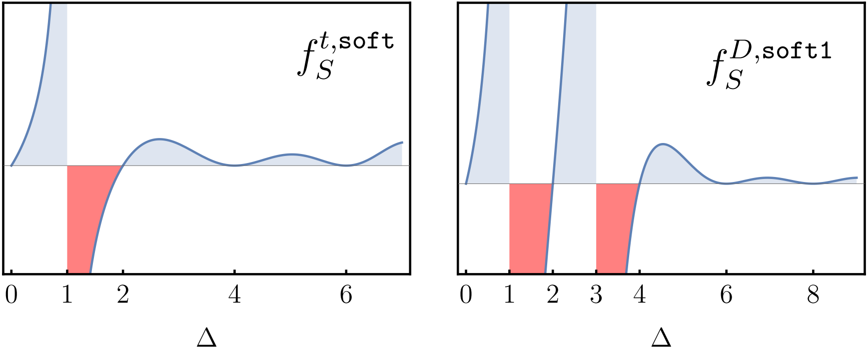

Figure: Soft sum rules and bounds for displacement and tilt.

Rigorous bounds just by looking!

Example I: Flux tube in pure YM in AdS_3

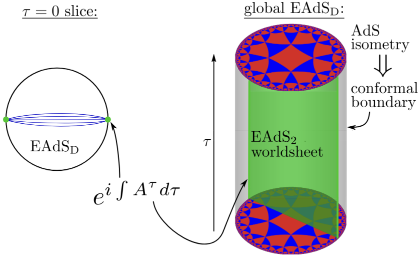

Figure: Setup: flux tube in AdS generated by a pair of Wilson lines on the boundary [Gabai, Gorbenko, Qiao '25]

- On the boundary: 1d dCFT with displacement operator $D \sim igF_{\tau i}$.
- Sum rules still hold despite nonlocal bulk CFT. [Gabai, Gorbenko, Qiao '25]
- See also: [Bianchi, de Sabbata, Meineri '26] [Qiao '26]

Example I: Flux tube in pure YM in AdS_3

- Two perturbative descriptions:
  - R_{AdS} small: expansion in weak coupling $\lambda \equiv \Lambda_{QCD} R_{AdS}$.
  - R_{AdS} large: Effective String Theory.

Operator	Dimension	R	CT
$\mathbb{D} = F_{\tau 1}$	2	-	+
$\partial_1 F_{\tau 1}$	3	+	+
$\partial_1^2 F_{\tau 1}$	4	-	+
$F_{\tau 1} F_{\tau 1}$	4	+	+
$\partial_1^3 F_{\tau 1}$	5	+	+
$[\partial_1 F_{\tau 1} F_{\tau 1}]^- = \partial_1 F_{\tau 1} F_{\tau 1} - F_{\tau 1} \partial_1 F_{\tau 1}$	5	-	-
$[\partial_1 F_{\tau 1} F_{\tau 1}]^+ = \partial_1 F_{\tau 1} F_{\tau 1} + F_{\tau 1} \partial_1 F_{\tau 1}$	5	-	+
$[F_{\tau 1} F_{\tau 1}]_1 = \partial_\tau F_{\tau 1} F_{\tau 1} - F_{\tau 1} \partial_\tau F_{\tau 1}$	5	+	+

Operator	Dimension	R	CT
$\mathbb{D} = X$	2	-	+
X^2	4	+	+
X^3	6	-	+
$[X^2]_2 \propto 4\,\partial_\tau^2 X X - 5\,(\partial_\tau X)^2$	6	+	+
$[X^3]_2 \propto 5X^2 \partial_\tau^2 X - 4\,X(\partial_\tau X)^2$	8	-	+
$[X^2]_4 \propto 10\,\partial_\tau^4 X X - 70\,\partial_\tau X \partial_\tau^3 X + 63\,(\partial_\tau^2 X)^2$	8	+	+
X^4	8	+	+
$[X^3]_3 \propto 4\,X^2 \partial_\tau^3 X - 18 X \partial_\tau X \partial_\tau^2 X + 15\,(\partial_\tau X)^3$	9	-	-

Figure: Dimensions of low-lying operators for weak (left) and strong (right) coupling from [Gabai, Gorbenko, Qiao '25].

Example I: Flux tube in pure YM in AdS_3

- Perturbative computation of dimensions, OPE coefficients, C_D on both sides and interpolate through Padé approximation [Gabai, Gorbenko, Offertaler '26].

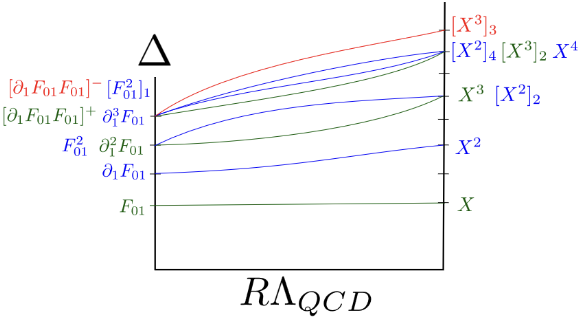

Figure: Dimensions of low-lying operators for weak (left) and strong (right) coupling from [Gabai, Gorbenko, Qiao '25].

Single correlator bootstrap

- Combined with numerical conformal bootstrap for $\langle DDDD \rangle$
  - bulk is 2d $\to$ no $SO(d-p)$ symmetry.
  - In $D \times D$ OPE first operator has $3 \leq \hat{\Delta}_{\text{gap}} \leq 4$.
  - Second operator has $\hat{\Delta}' = 0.5 + \hat{\Delta}_{\text{gap}}$.

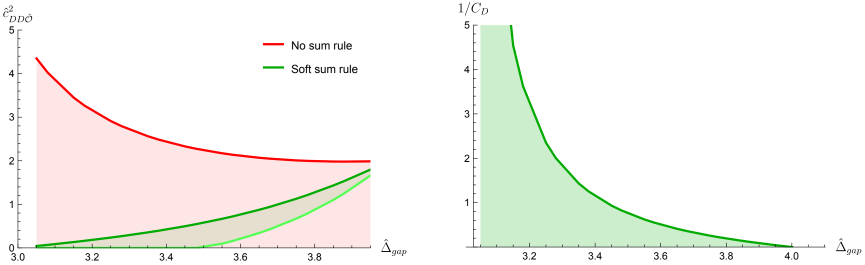

Figure: Bounds on $\hat{c}_{DD\hat{\mathcal{O}}}$ and C_D for displacement.

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  - Second operator has $\hat{\Delta}' = 0.5 + \hat{\Delta}_{\text{gap}}$.
  - Compared to perturbative computation [Gabai, Gorbenko, Offertaler '26]

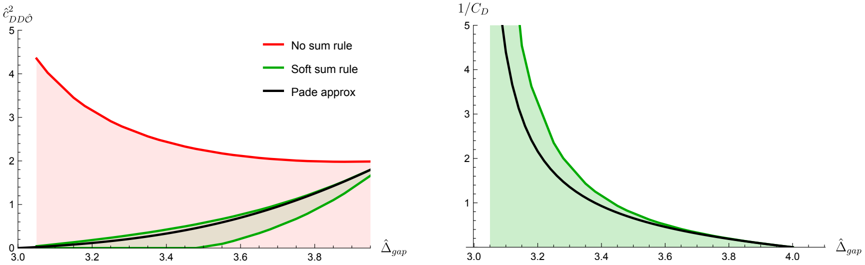
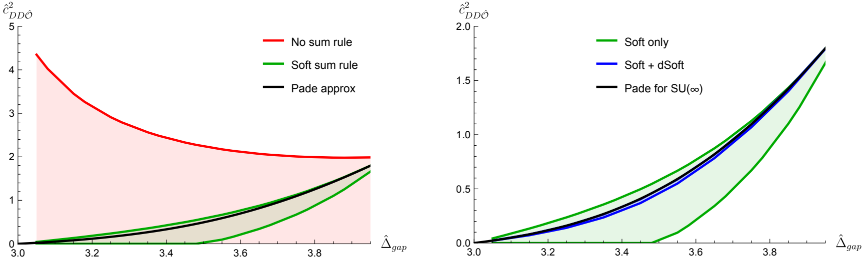


Figure: Bounds on $\hat{c}_{DD\hat{\mathcal{O}}}$ and C_D for displacement 4pt function.

Single correlator bootstrap

- Input: C_D from Padé approximation [Gabai, Gorbenko, Offertaler '26]



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Multi correlator bootstrap

- Now include $\hat{\mathcal{O}} =$"D^2" as second external operator:

$$\langle DDDD \rangle\,, \quad \langle D^2 D^2 D^2 D^2 \rangle\,, \quad \langle DDD^2 D^2 \rangle + \text{ perms}\,.$$

- Impose that $D \times D \sim D^2\,, \quad D^2 \times D^2 \sim D^2$ and $D \times D^2 \sim D$
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Multi correlator bootstrap

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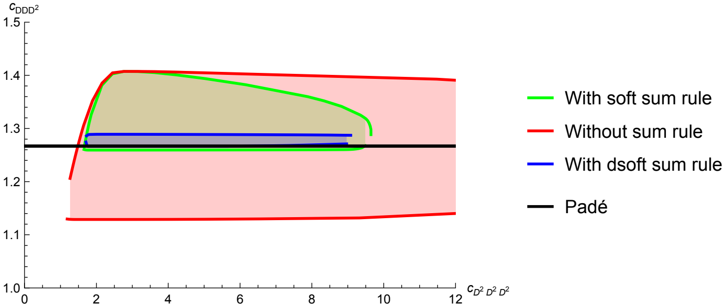

Access to mixed sum rules $\rightarrow$ even more constraints to be added.

Injecting the bulk

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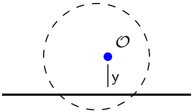
- We constrained some defect CFT data: $\hat{\Delta}_i, \hat{c}_{ijk}, C_t$. But there is more!
- For bulk operators in the presence of the defect: defect OPE



$$\mathcal{O}_\Delta(x,y) \sim \sum_{\hat{\Delta}} \mu_\Delta^{\hat{\Delta}} C(y,\partial_x) \hat{\mathcal{O}}_{\hat{\Delta}}(x)\,.$$

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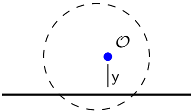

$$\mathcal{O}_\Delta(x,y) \sim \sum_{\hat{\Delta}} \mu^{\hat{\Delta}}_{\Delta} C(y,\partial_x)\hat{\mathcal{O}}_{\hat{\Delta}}(x)\,.$$

$$\langle \mathcal{O}_\Delta(x,y)\rangle = \frac{\mu^{\mathbf{1}}_{\mathcal{O}}}{|y|^\Delta}\,, \quad \langle \mathcal{O}_\Delta(x,y)\hat{\mathcal{O}}_{\hat{\Delta}}(0)\rangle = \frac{\mu^{\hat{\mathcal{O}}}_{\mathcal{O}}}{(y^2+x^2)^{\hat{\Delta}}|y|^{\Delta-\hat{\Delta}}}$$

- 3-point function depends on cross-ratio:

$$\langle \hat{\mathcal{O}}_{\hat{\Delta}}(x_1)\hat{\mathcal{O}}_{\hat{\Delta}}(x_2)\mathcal{O}_\Delta(x_3,y)\rangle = \frac{\mathcal{H}(\omega)}{x_{12}^{2\hat{\Delta}}|y|^\Delta}\,, \quad \omega = \frac{y^2 x_{12}^2}{(y^2+x_{13}^2)(y^2+x_{23}^2)}\,.$$

More sum rules

- Use soft identity [Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, '21]:

$$\int d^p x \, \langle t(x) \Phi(x_1, y_1) \rangle \sim \langle \Phi(x_1, y_1) \rangle \;\Rightarrow \mu^t_\Phi = -\frac{\mu^1_\Phi}{\pi \sqrt{C_t}} \, .$$

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- Repeat for 3-point function:

$$\int d^p x \, \langle t(x) J_\mu(x_1, y_1) \Phi(x_2, y_2) \rangle \sim -\langle J_\mu(x_1, y_1) \Phi(x_2, y_2) \rangle$$

$$\Rightarrow \int_0^1 \frac{d\omega}{\omega^{1+\frac{p}{2}} (1-\omega)^{1-\frac{p}{2}}} \mathcal{H}^\Phi_{tt}(\omega) = -\mu^t_\Phi \, .$$

- with

$$\mathcal{H}^\Phi_{tt}(\omega) = \sum_\Delta \mu^\Phi_\Delta \hat{c}_{tt\Delta} H_\Delta(\omega) \, .$$

Example II: The 1/2 BPS Wilson line in $\mathcal{N} = 4$ SYM

- $\mathcal{N} = 4$ SYM in $4d$, $SU(N)$ gauge group, integrable for $N \to \infty$.

- Introduce a 1/2-BPS Maldacena-Wilson line [Maldacena '98]

$$\mathcal{W}_\ell = \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \int_{-\infty}^{+\infty} d\tau \left(i\dot{x}^\mu A_\mu + |\dot{x}| \theta \cdot \Phi \right),$$

with expectation value $\langle \mathcal{W}_\ell \rangle = 1$ [Erickson, Semenoff, Zarembo '00].

- Wilson line breaks symmetries:

$$\mathfrak{psu}(2,2|4) \to \mathfrak{osp}(4^*|4)$$
$$\mathfrak{so}(5,1) \times \mathfrak{so}_R(6) \to \mathfrak{so}(2,1) \times \mathfrak{so}(3) \times \mathfrak{so}_R(5)$$

Broken Ward identities and supermultiplets

- Stress-tensor multiplet $\mathcal{O}_{20'} = \frac{1}{\sqrt{n_2}}\mathrm{Tr}[u^I u^J \Phi^I(x,y)\Phi^J(x,y)]$, $u^2 = 0$.

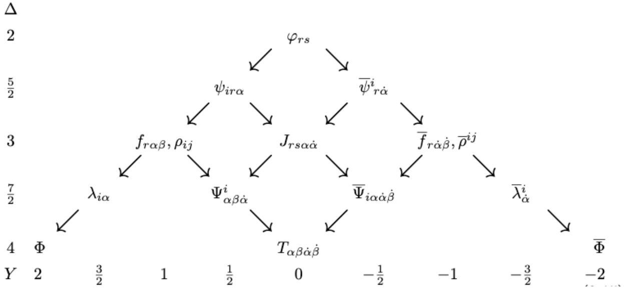

Figure: $\mathcal{O}_{20'}$ multiplet [Dolan, Osborn '01]

- Protected defect operators combined into supermultiplet $\mathcal{D}_1 = \frac{1}{\sqrt{\hat{n}_1}}\mathcal{W}_\ell[v\cdot\hat{\phi}]$:

$$\mathcal{D}_1(x): \quad t^{\tilde{a}}(x) \to \psi^a_\alpha(x) \to D^i(x)$$

Conformal bootstrap + Integrability = Bootstrability

- At $N \to \infty$ use Quantum Spectral Curve to obtain spectrum [Gromov, Kazakov, Leurent, Volin, '13 '14].

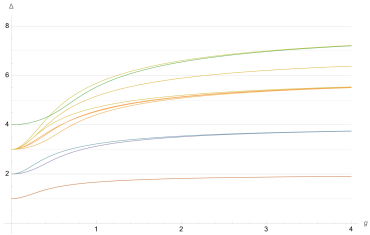

Figure: [Cavaglià, Gromov, Julius, Preti, '22]

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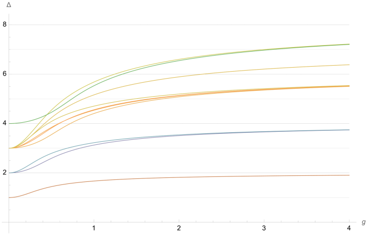

Figure: [Cavaglià, Gromov, Julius, Preti, '22]

- Add integrated 4pt correlators [Cavaglià, Gromov, Julius, Preti, '21 '22]

$$\int_0^1 dz\,(1+\log z)\frac{\delta G(z)}{z^2} = \frac{3\mathbb{C}-\mathbb{B}}{8\mathbb{B}^2}\,,\ \int_0^1 dz\,\frac{\delta f(z)}{z} = \frac{\mathbb{C}}{4\mathbb{B}^2}+\mathbb{F}-3\,.$$

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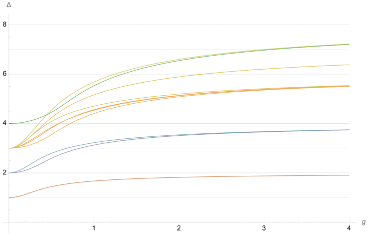

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- RHS can be computed exactly for any coupling with integrability ($\mathbb{C}$) [Gromov, Levkovich-Maslyuk '15] or localization ($\mathbb{B}$) [Correa, Henn, Maldacena, Sever '12, Fiol, Garolera, Lewcowycz '12].

Bounds on OPE coefficients

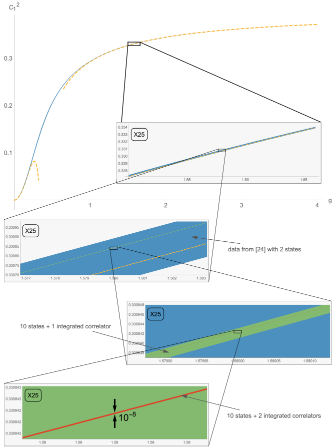

Figure: Bounds on OPE coefficient $\hat{c}_{\Delta_0}$ [Cavaglià, Gromov, Julius, Preti, '22]

Compared with analytic bootstrap results at

- Strong coupling [Ferrero, Meneghelli '21]
- Weak coupling [Cavaglià, Gromov, Julius, Preti, '22]

Sum rules with bulk stress-tensor multiplet

- Back to the bulk-defect correlators:

$$\int_{-\infty}^{\infty} dx \, \langle \mathcal{D}_1(x) \, \mathcal{O}_{20'}(x_1, y_1) \rangle \sim \langle \mathcal{O}_{20'}(x_1, y_1) \rangle$$

- RHS now known [Semenoff, Zarembo '01]:

$$\langle \mathcal{O}_{20'}(x, y) \rangle = \frac{1}{|y|^2} \, \frac{1}{2\,N} \sqrt{\frac{\lambda}{2}} \, \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}$$

- leading to [Giombi, Pestun '12]

$$\mu_t^{\mathcal{O}_{20'}} = \frac{\pi}{\sqrt{2}\,N} \; .$$

- For 3-point function checked with weak coupling perturbative results [Artico, Barrat, Xu '24]

$$\int_0^1 \frac{d\omega}{\omega^{3/2}\sqrt{1-\omega}} \, \frac{1}{2} \, \mathcal{A}_T(\omega) = \mu_t^{\mathcal{O}_{20'}} \; .$$

Outlook: bootstrapping bulk operators

Similar setup as [Ghosh, Paulos, Suchel '25]

1d crossing, bootstrability

soft sum rules

[Girault, Paulos, PvV '25] [Belton, Kong '25]

$$\begin{pmatrix} \hat{c}_{\hat{\Delta}} & \mu_2^{\hat{\Delta}} \end{pmatrix} \begin{pmatrix} \langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1 \rangle & \langle \mathcal{O}_{20'}\mathcal{D}_1\mathcal{D}_1 \rangle \\ \langle \mathcal{O}_{20'}\mathcal{D}_1\mathcal{D}_1 \rangle & \langle \mathcal{O}_{20'}\mathcal{O}_{20'} \rangle \end{pmatrix} \begin{pmatrix} \hat{c}_{\hat{\Delta}} \\ \mu_2^{\hat{\Delta}} \end{pmatrix}$$

locality

[Levine, Paulos '23]

NO crossing, integrated correlators

[Billò, Galvagno, Frau, Lerda '24]

+ Quantum Spectral Curve for spectrum.

Bootstrap for finite N

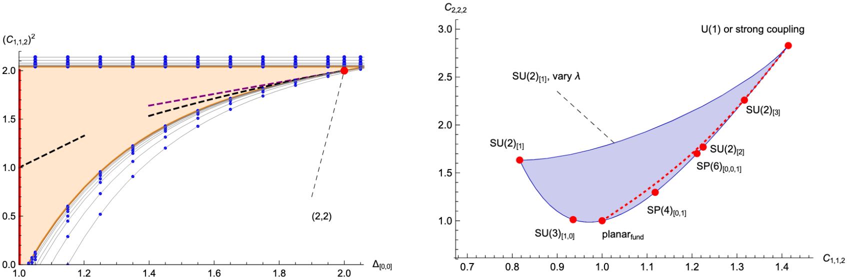

Figure: [Liendo, Meneghelli, Mitev '18]

For $\mathcal{N} = 4$ SYM without a defect, see e.g. [Chester, Dempsey, Pufu '21]

Bootstrap for finite $N = 2$

- Look at Wilson line in fundamental representation of $SU(2)$.
- $\hat{c}^2_{\mathcal{D}_2}, \mathbb{B}$ from localization [Liendo, Meneghelli, Mitev '18],[Correa, Henn, Maldacena, Sever '12] [Fiol, Garolera, Lewkowycz '12]:

$$\hat{c}^2_{\mathcal{D}_2} = 2 - \frac{3072}{(\lambda + 48)^2}\,, \quad \mathbb{B} = \frac{\lambda(\lambda + 48)}{64\pi^2(\lambda + 16)}\,.$$

- Use combination of sum rules [Drukker, Kong, Sakkas '22, Cavaglia,Gromov,Julius,Preti '22]:

$$-2 \times \int_0^1 dx \frac{1 + \log \chi}{\chi^2} \delta G(\chi) + 3 \times \int_0^1 dx \frac{\delta f(\chi)}{\chi} = \frac{1}{4\mathbb{B}(g)} + 3\mathbb{F}(g) - 9\,.$$

- Numerics in progress.

Generalization to higher p

- 3-point bulk-defect correlators:

$$\int_0^1 \frac{dw}{w^{1+\frac{p}{2}}(1-w)^{1-\frac{p}{2}}}\mathcal{H}^{\Phi}_{tt}(w) = -\mu^{\Phi}_t \frac{\Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}}} \, .$$

- Tilt soft sum rule

$$\iint_{\mathrm{Im}\, z>0} \frac{dz\, d\bar{z}}{|z|^{p-\Delta_{\hat{\mathcal{O}}}}} (\mathrm{Im}\, z)^{p-2}\, \mathcal{G}^{JijI}(z,\bar{z}) = 0 \, .$$

- Tilt double-soft sum rule

$$\iint_{\mathrm{Im}\, z>0,\, |z|<1} \frac{dz\, d\bar{z}}{|z|^{p-\Delta_{\hat{\mathcal{O}}}}} (\mathrm{Im}\, z)^{p-2} \log|z|\, \mathcal{G}^{I[ij]J}(z,\bar{z}) = Q^{[ij]}_{IJ} \frac{\Gamma(p-1)}{(2\pi)^{p-1}} \, .$$

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- Combine with analytic functionals [Mazac, Paulos '18] or local blocks [Levine, Paulos '23] to obtain ready-to-use sum rules.

Conclusion and further outlook

- Combine with numerical bootstrap, extend to multicorrelator setup

$$\langle tt\hat{\mathcal{O}}\hat{\mathcal{O}}\rangle\,,\langle DD\hat{\mathcal{O}}\hat{\mathcal{O}}\rangle\,,\langle ttDD\rangle\,.$$

and carve out space of defect CFTs.

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- Look at defects which completely break SUSY $\rightarrow$ displacino.

Thank you!

Backup

Example I: Pinning field in $O(N)$ model

- The action of the localized magnetic field line defect is given by:

$$S = \int dx\, d^q y \left(\frac{1}{2}(\partial_\mu \phi_a)^2 + \frac{\lambda}{4!}(\phi_a \cdot \phi^a)^2 \right) + h \int_{-\infty}^{\infty} dx\, \phi_1(x)\,,$$

- Fixed point in $d = 4 - \varepsilon$ [Allais, Sachdev '14]

$$\frac{\lambda_*}{2\pi} = \frac{3\varepsilon}{N+8} + \frac{9(3N+14)\varepsilon^2}{(N+8)^3} + \mathcal{O}(\varepsilon^3),$$

$$h_*^2 = (N+8) + \frac{4N^2+45N+170}{2(N+8)}\varepsilon + \mathcal{O}(\varepsilon^2)\,.$$

- Breaking $O(N) \to O(N-1)$ introduces tilt $t_{\tilde{a}}$ and a scalar $\hat{\phi}_1$ with dimension [Cuomo, Komargodski, Mezei '22]

$$\Delta_{\hat{\phi}_1} = 1 + \varepsilon - \frac{184}{121}\varepsilon^2 + O(\varepsilon^3) \quad \xrightarrow{\text{Padé}} \quad 1.55\,.$$

- and displacement

$$\mathrm{D} = h_\star \nabla \hat{\phi}_1\,.$$

Example I: Pinning field in $O(N)$ model

- Use double-soft sum rule at $\mathcal{O}(\varepsilon)$ to obtain higher-order data:

$$\underbrace{\left(\lambda^{(1)}_{tt\hat{\phi}_1}+\lambda^{(2)}_{tt\hat{\phi}_1}\right)^2}_{\mathcal{O}(\varepsilon^3)}\ \underbrace{f^{t,\text{dsoft}}_S[\Delta_{\hat{\phi}_1}]}_{\mathcal{O}(\frac{1}{\varepsilon^2})}+$$

$$+\underbrace{\left(\left(\lambda^{(0)}_{tt\hat{T}}\right)^2 f^{t,\text{dsoft}}_T[2+\gamma_{\hat{T}}]+\sum_{i=\pm}\left(\lambda^{(0)}_{tt\hat{s}_i}\right)^2 f^{t,\text{dsoft}}_S[2+\gamma_{\hat{s}_i}]\right)}_{\mathcal{O}(\varepsilon)}=\underbrace{\frac{2}{C_t}}_{\mathcal{O}(\varepsilon)},$$

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- Same for displacement [Gabai, Sever, Zhong '25]

$$\lambda^{(2)}_{DD\hat{\phi}_1} = -\frac{\pi\left(29N^2+413N+1610\right)}{12(N+8)^{5/2}}.$$

Displacement correlators

- Basis of broken generators satisfies algebra

$$[K^a, P^b] = 2\delta^{ab} D - 2\, M^{ab}\,, \qquad [M^{ma}, M^{nb}] = -\delta^{mn} M^{ab} - \delta^{ab} M^{mn}\,,$$
$$[M^{ma}, P^b] = \delta^{ab} P^m\,, \qquad [M^{ma}, K^b] = \delta^{ab} K^m\,.$$

- Can compute sum rules for each generator individually, or use shadow:

$$\tilde{\mathcal{O}}(x_0) = \int \mathrm{d}^p x\, \frac{1}{(x-x_0)^{2(p-\Delta)}} \hat{\mathcal{O}}_\Delta(x)\,.$$

- leading to the soft identity

$$\langle \tilde{D}^a(x_0)\mathcal{X}\rangle = \int \mathrm{d}^p x\ (x-x_0)^2 \langle D^a(x)\mathcal{X}\rangle = \langle \delta_{K^a_{x_0}}\mathcal{X}\rangle\,.$$

- We can now expand around x_0.

Defect four-point functions

- 4-pt correlator of displacement multiplet [Liendo, Meneghelli, Mitev '18]:

$$\langle \mathcal{D}_1(\tau_1)\mathcal{D}_1(\tau_2)\mathcal{D}_1(\tau_3)\mathcal{D}_1(\tau_4)\rangle = \frac{G(\chi)}{\tau_{12}^2\tau_{34}^2}\,, \quad \chi \equiv \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}\,, \quad \tau_{ij} = \tau_i - \tau_j\,,$$

$$G(\chi) = \mathbb{F}\chi^2 + (2\chi^{-1} - 1)f(\chi) - (\chi^2 - \chi + 1)f'(\chi)$$

- $\mathbb{F} = 1 + \hat{c}^2_{\mathcal{D}_2}$ is topological term.

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- Decomposition in superconformal blocks:

$$f(\chi) = F_{\mathcal{I}}(\chi) + \underbrace{\hat{c}^2_{\mathcal{D}_2}}_{\text{localization:[Giombi, Komatsu '18]}} F_{\mathcal{D}_2}(\chi) + \sum_{\hat{\Delta}} \hat{c}^2_{\hat{\Delta}} F_{\hat{\Delta}}(\chi)\,.$$

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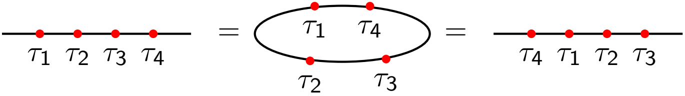
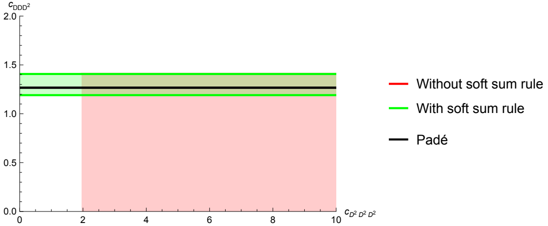
- 1d crossing symmetry for $f(\chi)$:



The importance of putting gaps

- What if we lower the gaps? $\rightarrow \Delta_{++} = 4.5$, $\Delta_{+-} = 4.5$, $\Delta_{--} = 5$.



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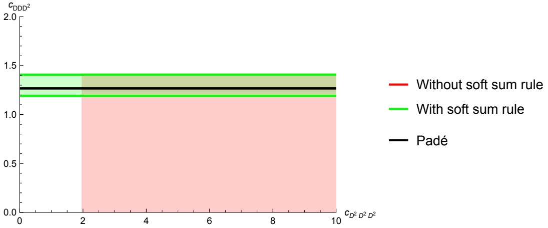
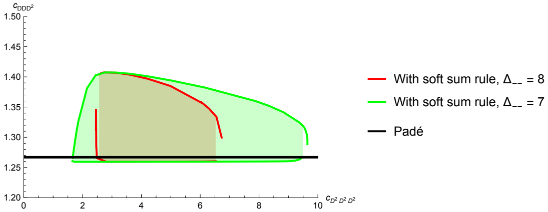


- And if we raise the gaps? $\rightarrow \Delta_{++} = 5.5,\ \Delta_{+-} = 5.5,\ \Delta_{--} = 8.$



Bootstrapping bulk operators

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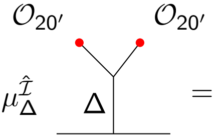
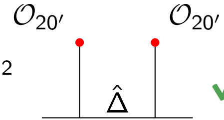
✗ $\sum_{\Delta} c_{\Delta}\, \mu^{\hat{\mathcal{I}}}_{\Delta}$



$= \sum_{\hat{\Delta}} (\mu^{\hat{\Delta}}_{2})^2$



✓

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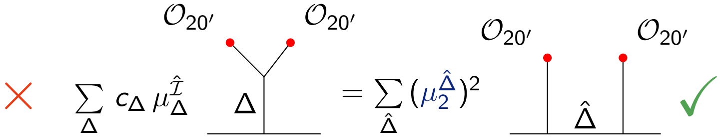

- Need to include 3-pt function for numerical conformal bootstrap:

$$\langle \mathcal{O}_{20'}(y_1, \tau_1) \mathcal{D}_1(\tau_2) \mathcal{D}_1(\tau_3) \rangle = \frac{G_{\text{3pt}}(z)}{y_1^2\, \tau_{23}^2}\,, \quad z = \frac{y_1^2 \tau_{23}^2}{x_{12}^2\, x_{13}^2}\,,$$

$$G_{\text{3pt}}(z) = F_{\mathcal{I}}(z) + \mu_2^2\, \hat{c}_{\mathcal{D}_2} F_{\mathcal{D}_2}(z) + \sum_{\hat{\Delta}} \mu_2^{\hat{\Delta}}\, \hat{c}_{\hat{\Delta}} F_{\hat{\Delta}}(z)\,,$$

Bootstrap for finite N

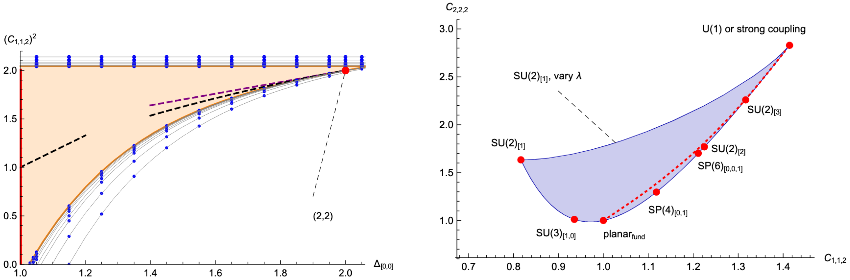

Figure: [Liendo, Meneghelli, Mitev '18]

For $\mathcal{N} = 4$ SYM without a defect, see e.g. [Chester, Dempsey, Pufu '21]

Bootstrap for finite $N = 2$

- Look at Wilson line in fundamental representation of $SU(2)$.
- $\hat{c}^2_{\mathcal{D}_2}, \mathbb{B}$ from localization [Liendo, Meneghelli, Mitev '18],[Correa, Henn, Maldacena, Sever '12] [Fiol, Garolera, Lewkowycz '12]:

$$\hat{c}^2_{\mathcal{D}_2} = 2 - \frac{3072}{(\lambda + 48)^2}\,, \quad \mathbb{B} = \frac{\lambda(\lambda + 48)}{64\pi^2(\lambda + 16)}\,.$$

- Use combination of sum rules [Drukker, Kong, Sakkas '22, Cavaglia,Gromov,Julius,Preti '22]:

$$-2 \times \int_0^1 dx \frac{1 + \log\chi}{\chi^2} \delta G(\chi) + 3 \times \int_0^1 dx \frac{\delta f(\chi)}{\chi} = \frac{1}{4\mathbb{B}(g)} + 3\mathbb{F}(g) - 9\,.$$

Inserting defect operators

To insert operators on defect, we write:

$$\mathcal{D}_1 = \frac{1}{\sqrt{\hat{n}_1}} \mathcal{W}_\ell[v \cdot \hat{\phi}] = \frac{1}{N} \operatorname{Tr} \mathcal{P}[v \cdot \hat{\phi} \exp \int d\tau \, (iA_0 + \Phi^6)] \, .$$

Correlation functions are then written as

$$\begin{aligned} &\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_i(x_i) \hat{\mathcal{O}}_{i+1}(\tau_{i+1}) \dots \hat{\mathcal{O}}_{i+j}(\tau_{i+j}) \rangle \\ &= \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_i(x_i) \mathcal{W}_\ell[\hat{\mathcal{O}}_{i+1}(\tau_{i+1}) \dots \hat{\mathcal{O}}_{i+j}(\tau_{i+j})] \rangle \, . \end{aligned}$$

Bounds on stress-tensor in AdS

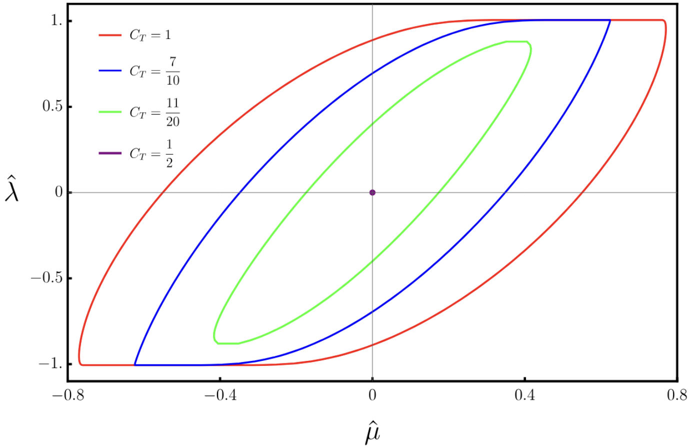

Figure: [Ghosh, Paulos, Suchel '25]