Extensions of topological symmetries

Ingo Runkel (Hamburg Univ.)

joint work with

Federico Ambrosino (Perimeter Institute, Waterloo)
Anatoly Konechny (Heriot-Watt, Edinburgh)
Gérard Watts (King's College London)

Outline:

1) topological & transl. inv. defects
2) perturbative & algebraic description in chiral TFT
3) defect flows in 2d CFT

Categories, categories, categories!

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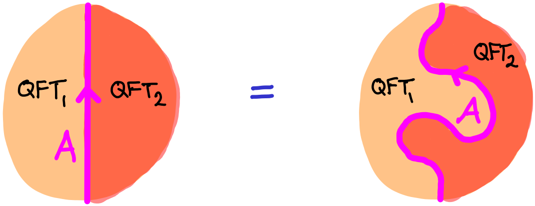
1) topological & transl. inv. defects (categories)
2) perturbative & algebraic description in chiral TFT (categories)
3) defect flows in 2d CFT (categories)

Topological defects

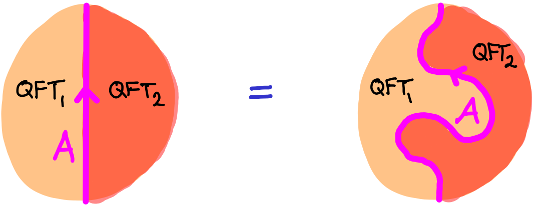


For now: $QFT_1 = QFT_2$

The symmetries of a quantum mech. system form a group.

The topol. line defects of a 2d QFT form a $\boxed{\quad\quad}$.

Topological defects



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The symmetries of a quantum mech. system form a group.

The topol. line defects of a 2d QFT form a $\boxed{}$.

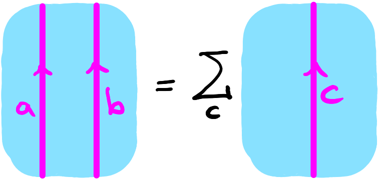
pivotal monoidal category

Fuchs, Schweigert, IR '02
Fröhlich, Fuchs, Schweigert, IR '09
Davydov, Kong, IR '11
Thorngren, Wang '19

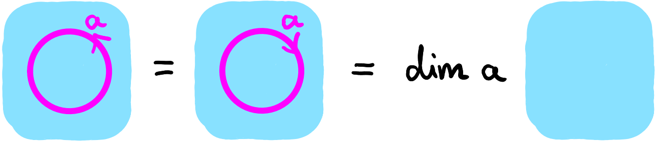
Fusion categories of topological defects

- finite list of elementary top. def. $\{1, a_2, \dots, a_n\}$
- closed under fusion

$$a \otimes b \simeq \bigoplus c$$



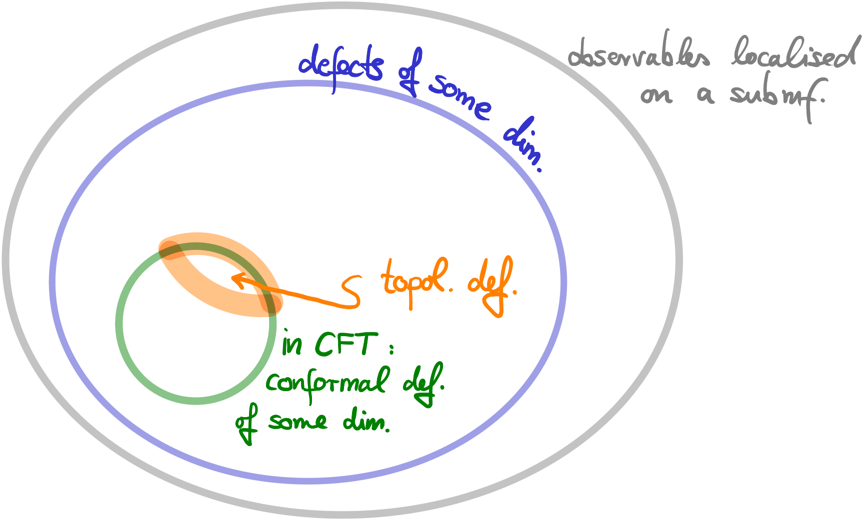
- quantum dimensions



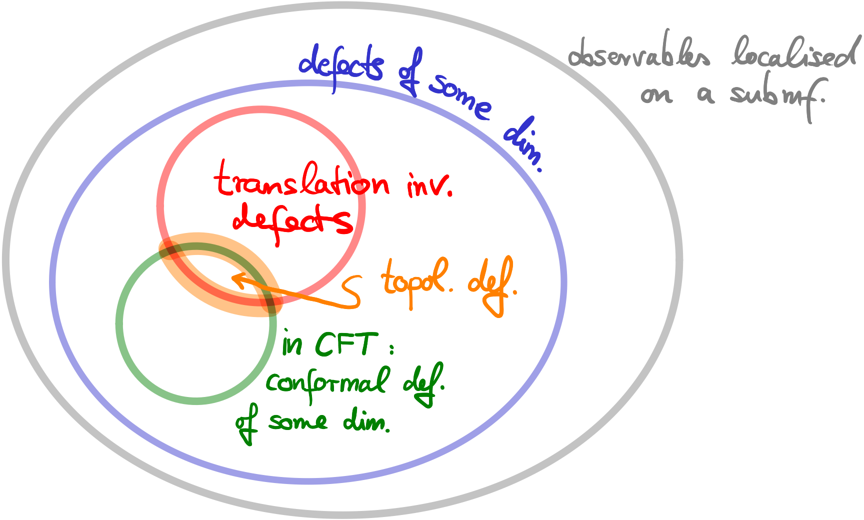
(spherical)

fusion category

Defects


observables localised on a submf.
defects of some dim.
topol. def.
in CFT :
conformal def.
of some dim.

Defects



Here: only 2d QFT and line defects.

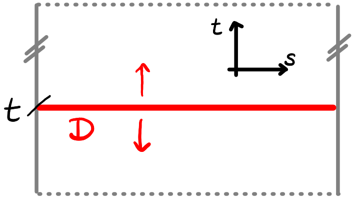
Translation invariant defects

QFT on a cylinder

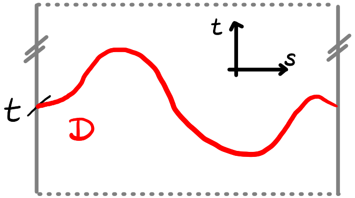
$$[H, \mathcal{D}] = 0$$

Hamiltonian

defect operator

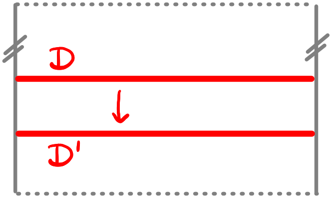


$\neq$ (in general)



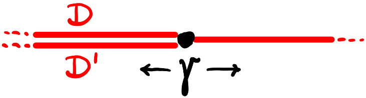
… transl. invariant defects

Manolopoulos, IR '09, Bücher, IR '12, Ambrosino, Watts, IR '25



Properties:

- well-defined fusion



- topological defect junction fields

→ $\boxed{\text{monoidal category of transl. inv. defects}}$

$$\{\text{topol. defects}\} \subset \{\text{transl. inv. defects}\}$$

full monoidal subcat.

... transl. invariant defects

Ambrosino, Watts, IR '25

A defect which is both transl. inv. & conformal is topological.

$\hat{D}$: defect operator

transl. inv. : $[L_0 + \bar{L}_0 , \hat{D}] = 0$ (& anyway rot. inv. $[L_0 - \bar{L}_0, \hat{D}] = 0$)

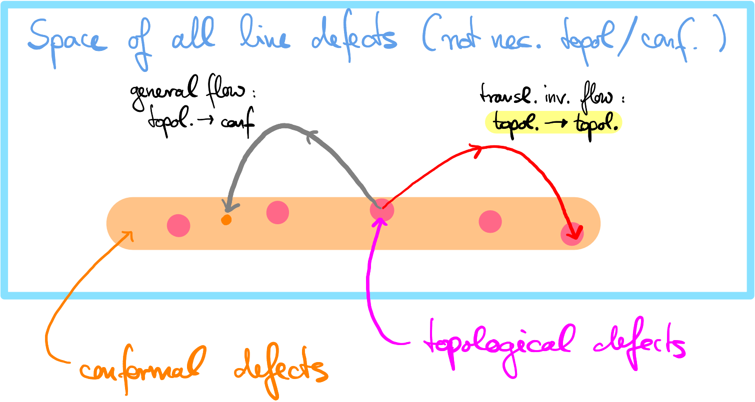
conformal : $[L_n - \bar{L}_{-n} , \hat{D}] = 0$

topological : $[L_n , \hat{D}] = 0 = [\bar{L}_n , \hat{D}]$

for all $n \in \mathbb{Z}$

Two applications to renormalisation group flows

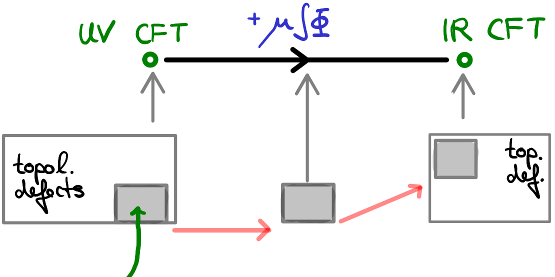
① Defect flows within a given 2d CFT



Question: Can monoidal category structure on transl. inv. defects constrain the RG flow endpoint?

... two applications to RG flows

② Bulk flow between two different CFTs



subset of topol. defects preserved along flow

... two applications to RG flows

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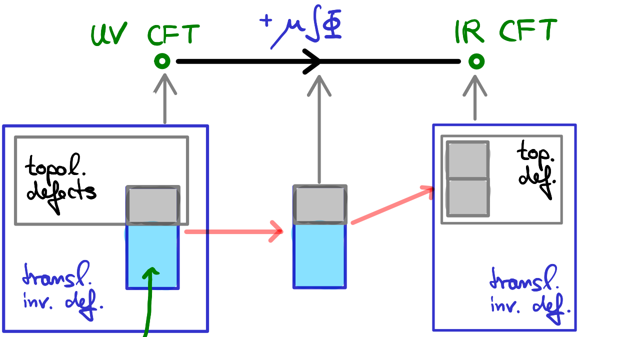
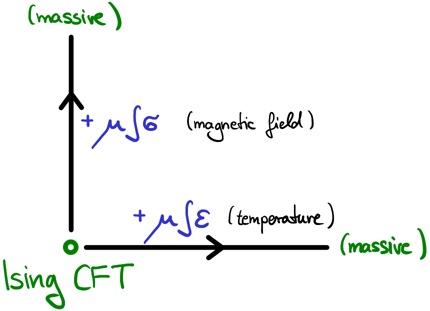


subset of transl. inv. defects preserved along flow

② Bulk flows ... Ising CFT

Primary fields	$\mathbb{1}$	σ	ε
$(h,\bar{h})$	$(0,0)$	$(\frac{1}{16},\frac{1}{16})$	$(\frac{1}{2},\frac{1}{2})$

Topol. defects $\underbrace{\overbrace{\text{id} \quad S}^{\mathbb{Z}_2} \quad KW}_{KW \otimes KW \simeq \text{id} \oplus S}$



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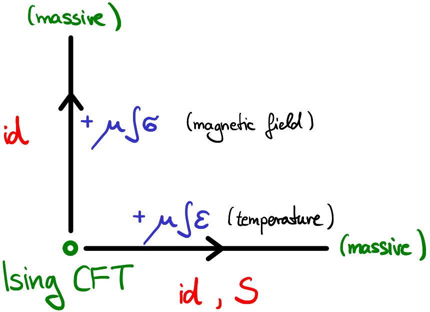
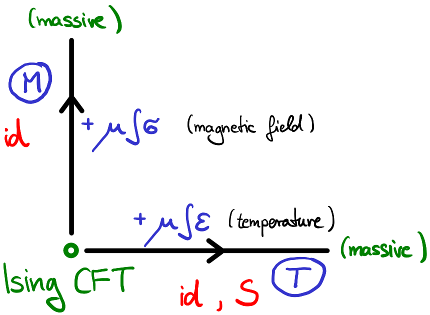


topological defects preserved along flow

② Bulk flows ... Ising CFT

Primary fields $(h,\bar h)$: $\mathbb{1}$ $(0,0)$, σ $\left(\tfrac{1}{16},\tfrac{1}{16}\right)$, ε $\left(\tfrac12,\tfrac12\right)$

Topol. defects $\underbrace{\overbrace{\mathrm{id}\quad S}^{\mathbb{Z}_2}\quad KW}_{KW \otimes KW \,\simeq\, \mathrm{id} \oplus S}$



Transl. inv. defects

M perturbation of $D = \mathrm{id} \oplus KW$

T perturbation of $D = KW$

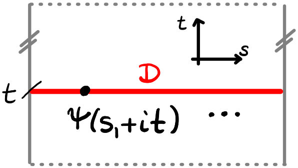
"The $\underset{\text{M}}{(1,2)}$, $\underset{\text{T}}{(1,3)}$ and $(1,5)$ perturbations of a minimal model CFT are integrable."

① Defect flows within a given 2d CFT

Bazhanov, Lukyanov, Zamolodchikov '94, Konik, LeClair '97,
Bachas, Gaberdiel '04, IR '09, Ambrosino, Watts, IR '25

A topological defect, perturbed by a holomorphic defect field, is translation invariant.

$\hat{D}(\lambda\psi)$ = " D with $\exp \lambda \int \psi$ on it "



Then

$$[H_{CFT}, \hat{D}(\lambda\psi)] = 0$$

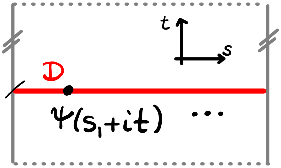
since $\bar{\partial}\psi = 0$, i.e.:

$$\partial_t \psi(s+it) = i\,\partial_s \psi(s+it)$$

① Defect flows within a given 2d CFT

A topological defect, perturbed by a holomorphic defect field, is translation invariant.

$\hat{D}(\lambda\psi)$ = " D with $\exp \lambda \int \psi$ on it "



Questions : 1) monoidal category of ?

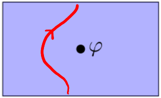
2) constrain flow endpoint ?

$$D \xrightarrow{\;+\lambda\psi\;} D_\infty$$

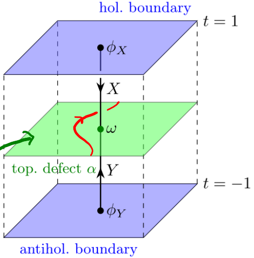
Topological line defects & TFT

Fuchs, Schweigert, IR '02

2d CFT with topological line defects

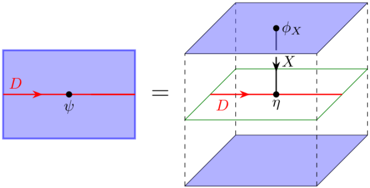


Chiral TFT



here : choose surf. defect trivial

holomorphic defect field



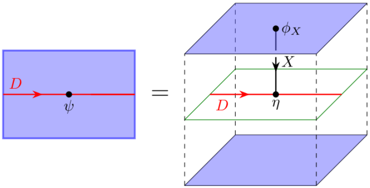
Aim: find flow endpoints of chirally perturbed defects

Ambrosino, Konechny, Watts, IR - in prep

Ingredients

1) Category $\mathcal{C}_X$ of chirally perturbed defects
2) Exact sequences give sum relations for defect operators
3) Asymptotic semiring

Category $\mathcal{C}_X$ of chirally perturbed defects



Data :

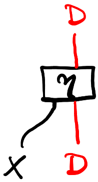
$\mathcal{C}$ – braided monoidal cat.

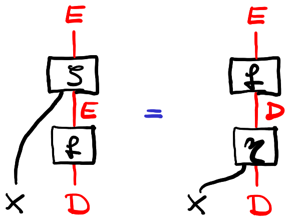
$X \in \mathcal{C}$ – rep^n of perturbing chiral defect field

Category of chirally perturbed defects (for trivial surface defect)

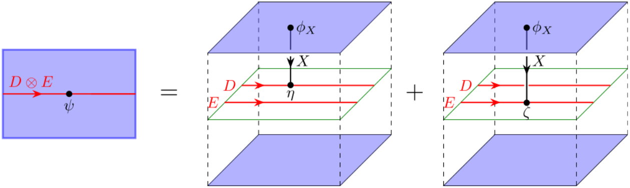
$\mathcal{C}_X$ obj : $D(\eta)$ morph : $f : D(\eta) \longrightarrow E(\zeta)$

$D \in \mathcal{C}$,



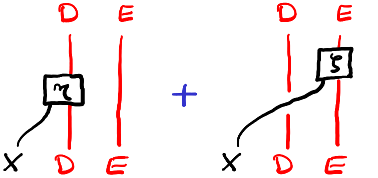


... $\mathcal{C}_X$: chirally perturbed defects



Tensor product in $\mathcal{C}_X$: fusion of chirally pert. defects :

$$D(\eta) \otimes E(\xi) = \Big(\underbrace{D \otimes E}_{\text{fusion of topol. def}} ,$$



$\Big)$

... $\mathcal{C}_X$: chirally perturbed defects

$\mathcal{C}_X$ is (for $\mathcal{C}$ k-linear abelian ribbon)

- abelian
- monoidal & rigid (= has left & right duals)

Even for $\mathcal{C}$ finitely semisimple, typically $\mathcal{C}_X$

- is not semisimple
- is not pivotal (left dual $\neq$ right dual)
- has an infinite number of simples
- simple $\otimes$ simple is generically simple

E.g. Lee-Yang minimal model

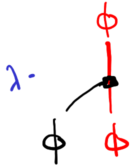
$M(2,5)$, $c = -22/5$

conf. weights of primaries: $h_1 = 0$
$h_\phi = -1/5$

$\mathcal{C}$: braided fusion cat.

simples $1, \phi$ $\quad \phi \otimes \phi = 1 \oplus \phi$

Take $X = \phi \rightsquigarrow \phi(\lambda)$



fixed basis choice, $\lambda \in \mathbb{C}$

Prop. $\phi(\lambda) \otimes \phi(\mu)$ is simple in $\mathcal{C}_X$ for all $\lambda, \mu \in \mathbb{C}$, except

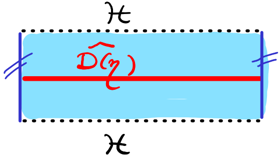
- $\lambda = e^{4\pi i/5}\mu$: $\quad \mathbb{1} \longrightarrow \phi(e^{4\pi i/5}\lambda) \otimes \phi(\lambda) \longrightarrow \phi(e^{2\pi i/5}\lambda)$
- $\lambda = e^{-4\pi i/5}\mu$: $\quad \phi(e^{-2\pi i/5}\lambda) \longrightarrow \phi(e^{-4\pi i/5}\lambda) \otimes \phi(\lambda) \longrightarrow \mathbb{1}$

Exact sequences give sum relations for defect operators

For $D(\eta) \in \mathcal{C}_x$ get defect operator

$$\widehat{D(\eta)} : \mathcal{H} \longrightarrow \mathcal{H}$$

cylinder with pert. defect:



If $A(\alpha) \longrightarrow B(\beta) \longrightarrow C(\gamma)$ is short exact seq. in $\mathcal{C}_x$ then

$$\widehat{B(\gamma)} = \widehat{A(\alpha)} + \widehat{C(\gamma)}$$

"The map $\mathcal{C}_x \longrightarrow \mathrm{End}\,\mathcal{H}$ factors through $Gr(\mathcal{C}_x)$"

E.g. Lee-Yang model: $\widehat{\phi(e^{2\pi i/5}\lambda)}\,\widehat{\phi(e^{-2\pi i/5}\lambda)}$

"T-system functional rel." $= \mathrm{id} + \widehat{\phi(\lambda)}$

Asymptotic semiring

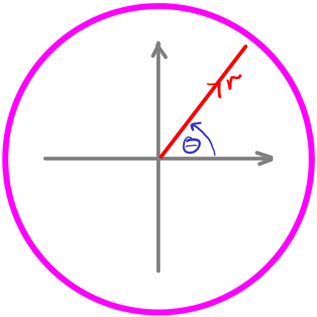
Ambrosino, Konechny, Watts, IR - in prep

$\mathbb{C}^\times$ action on $\mathcal{C}_X$: $\quad \mathbb{C}^\times \longrightarrow \mathrm{Aut}_\otimes(\mathcal{C}_X)\,, \quad z \longmapsto \mathcal{U}_z$

$$\mathcal{U}_z(D(\eta)) = D(z\eta)$$

$\mathcal{U}_z$ preserves functional relations

"circle at infinity"



Idea Take $z = re^{i\theta}$ and

try to take limit $r \to \infty$;

use functional rel. "at infinity"

Input from physics: Assume leading contribution

$$\widehat{D}(re^{i\theta}\eta) \underset{r\to\infty}{\sim} D_\infty \, e^{-f_D \cdot \ell} \qquad \ell = \left(re^{i\theta}\right)^{\frac{1}{1-h}}$$

Sum of defect ops

$$\widehat{D}(\cdots) + \widehat{E}(\cdots) \underset{r\to\infty}{\sim} D_\infty e^{-f_D \ell} + E_\infty e^{-f_E \ell}$$

$\underset{r\to\infty}{\sim}$ either or dep. on $\mathrm{Re}(-f_{D/E})$

Product

$$\widehat{D}(\cdots)\,\widehat{E}(\cdots) \underset{r\to\infty}{\sim} D_\infty E_\infty \, e^{-(f_D+f_E)\ell}$$

Max-Plus semiring $\mathbb{R}_{max}$: $a \oplus b = \max(a,b)$

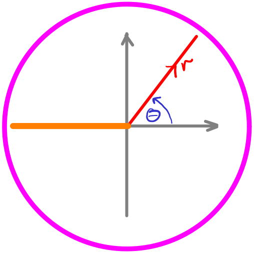
$a \odot b = a + b$

Expect semiring hom.

$$\mathrm{Gr}(\mathcal{C}_X)_{\geq 0} \longrightarrow \text{asymptotic semiring } \text{``}\mathrm{Fun}(U(1),\ \underbrace{\mathrm{Fus}_{\mathrm{top}}}_{\text{fusion semiring of topol. defects}} \times \mathbb{R}_{\max})\text{''}$$

E.g. Lee-Yang model

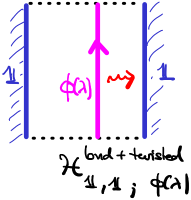
- <u>assume</u>: one anti-Stokes line
- then fun. rel. has <u>unique soln</u> in asymp. semiring



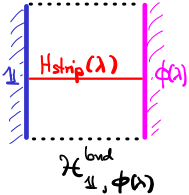
$$\widehat{\phi(re^{i\theta})} \sim \mathrm{id}\ e^{(\mathrm{const})\, r^{5/6} e^{i\frac{5}{6}\theta}}$$

But is that what happens? $\rightsquigarrow$ Truncated conformal space approach

1) Put system on a strip with Cardy boundary condition $\mathbb{1}$ and defect $\phi(\lambda)$



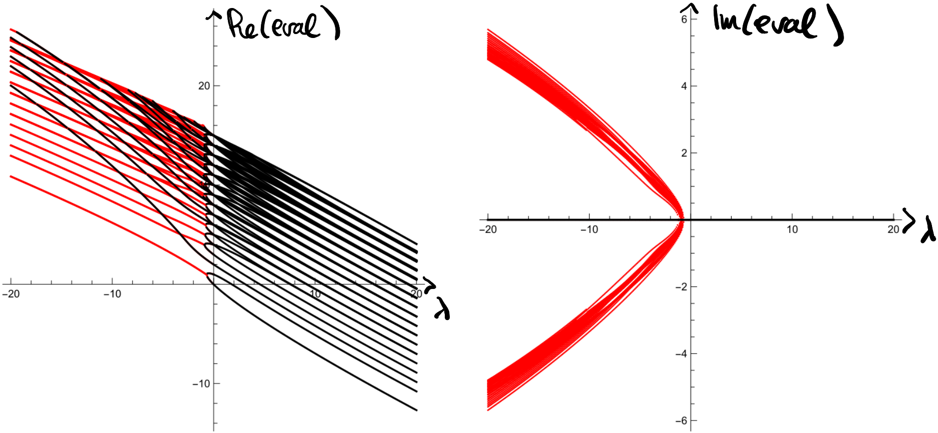
2) Push transl. inv. defect into bnd. to get a bnd. perturbation



3) Diagonalise $H_{\text{strip}}(\lambda) = (\text{const})\left(L_0 - \frac{c}{24} + \lambda\, \phi_{\text{bnd}}\right)$ on $\mathcal{H}_{\mathbb{1},\phi}$ (unperturbed) up to some energy $E_{\max}$, numerically.

... TCSA

Spectrum of $H_{strip}(\lambda)$, $\lambda \in \mathbb{R}$

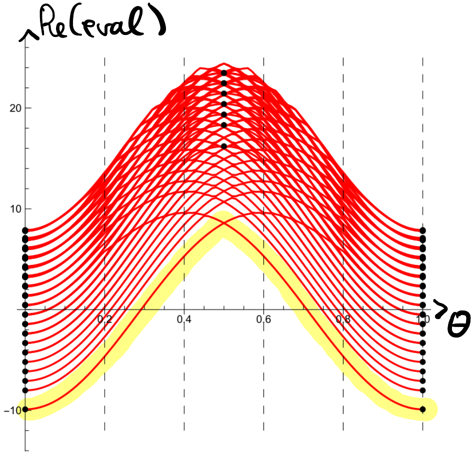


(a) The real parts.

(b) The imaginary parts.

... TCSA

Spectrum of $H_{strip}(r e^{i\theta})$, r fixed "large"



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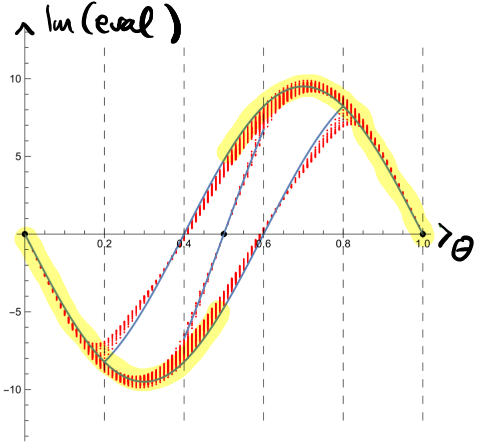

(b) The imaginary parts.