Defects, flat gauging, T-duality:
at any radius, in the continuum and on the lattice

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Defects and Extended Excitations in Quantum Field Theory,
Quantum Matter and Statistical Models
Galileo Galilei Institute

Based on:
2409.11822 with Andrés Collinucci, Giovanni Galati, Ondrej Hulik and Elise Paznokas
2604.09126 with Giovanni Galati and Nathan Godechal

Motivation

Many QFTs have continuous symmetries and continuous parameters

Topological manipulations of QFTs (such as gauging and pasting SPTs) are usually discrete

If we are to reach any point in field theory space (e.g. conformal manifold) we need to define continuous topological manipulations

The trouble with such manipulations is that they will generate topological interfaces with infinite quantum dimension

Motivation

Our aim is to discuss such interfaces and defects, both in the continuum and on the lattice

We will take as a case study a simple theory:
the compact scalar in 1+1 dimensions, a.k.a. the $c = 1$ CFT

We will find interfaces between any two points of its conformal manifold, and a T-duality symmetry defect at any point as well

Interfaces and defects in the continuum

Compact boson in 1+1d

The theory

$$S = \frac{R^2}{4\pi}\int d\Phi \wedge \star d\Phi \qquad \text{with} \qquad \Phi \sim \Phi + 2\pi$$

is a CFT and has two global continuous 0-form symmetries:

- Momentum symmetry $U(1)^{\mathrm{m}}$ from current $j^{\mathrm{m}} = -i\frac{R^2}{2\pi}d\Phi$
- Winding symmetry $U(1)^{\mathrm{w}}$ from current $j^{\mathrm{w}} = \frac{1}{2\pi} * d\Phi$

R^2 is a marginal coupling $\Rightarrow$ it defines a conformal manifold

[add the action of $\mathbb{Z}_2 : \Phi \to -\Phi$ to get also the orbifold branch]

Compact boson in 1+1d: T-duality

An important property of

$$S = \frac{R^2}{4\pi} \int d\Phi \wedge \star d\Phi \qquad \text{with} \qquad \Phi \sim \Phi + 2\pi$$

is T-duality:

It maps the theory with coupling R^2 to the one with coupling $1/R^2$

At the same time it maps states $e^{in\Phi}$ charged under momentum symmetry to states charged under winding symmetry,
i.e. such that $\int_{\Sigma_1} d\Phi = 2\pi w$

Finite topological manipulations

Familiar topological manipulations are the following:

[see e.g. Fuchs, Gaberdiel, Runkel, Schweigert 07; Thorngren, Wang 21]

- gauge $\mathbb{Z}_q \in U(1)^{\mathrm{w}}$: throws away some winding states
  $\Rightarrow$ rescales the radius $R \to qR$

- gauge $\mathbb{Z}_p \in U(1)^{\mathrm{m}}$: throws away some momentum states
  $\Rightarrow$ rescales the radius $R \to R/p$

Implementation in the Lagrangian?

Finite topological manipulations

Gauging a $\mathbb{Z}_q$ subgroup of the winding symmetry:

$$S = \frac{R^2}{4\pi}\int d\Phi \wedge *d\Phi + \frac{i}{2\pi}\int C \wedge d\Phi - \frac{iq}{2\pi}\int C \wedge d\Psi$$

C is a $U(1)$ gauge field and $\Psi \sim \Psi + 2\pi$

Integrating out C gives

$$d\Phi = q d\Psi$$

and

$$S = \frac{q^2R^2}{4\pi}\int d\Psi \wedge *d\Psi$$

Finite topological manipulations

Gauging a $\mathbb{Z}_p$ subgroup of the momentum symmetry:

$$S = \frac{R^2}{4\pi} \int (d\Phi - B) \wedge *(d\Phi - B) + \frac{ip}{2\pi} \int B \wedge d\tilde{\Psi}$$

B is a $U(1)$ gauge field and $\tilde{\Psi} \sim \tilde{\Psi} + 2\pi$
Integrating out B gives

$$R^2 * (d\Phi - B) = ipd\tilde{\Psi}$$

and

$$S = \frac{p^2}{4\pi R^2} \int d\tilde{\Psi} \wedge *d\tilde{\Psi}$$

We have also implemented T-duality! [and only that for $p = 1$]

More topological manipulations

We would like not to restrict to finite subgroups of $U(1)^{\mathrm{m}}$ or $U(1)^{\mathrm{w}}$

What if we want to throw away all winding states?
(or all momentum states?)

Take the winding states, we would like to gauge all of $U(1)^{\mathrm{w}}$, but without gapping the theory

$\Rightarrow$ Gauge it with flat connections

Continuous topological manipulations

Flat gauging of the winding symmetry:

$$S = \frac{R^2}{4\pi}\int d\Phi \wedge *d\Phi + \frac{i}{2\pi}\int C \wedge d\Phi - \frac{i}{2\pi R}\int C \wedge d\psi$$

C is a $U(1)$ gauge field and ψ is non-compact: $\int d\psi = 0$ on any cycle
Integrating out C gives

$$d\Phi = \frac{1}{R} d\psi$$

and

$$S = \frac{1}{4\pi}\int d\psi \wedge *d\psi$$

This theory has a $\mathbb{R}^{\mathrm{m}}$ momentum symmetry and no winding symmetry

Discrete topological manipulations

Gauging a $\mathbb{Z}$ subgroup of the $\mathbb{R}$ momentum symmetry:

$$S = \frac{1}{4\pi}\int (d\psi - b)\wedge *(d\psi - b) + \frac{iR'}{2\pi}\int b\wedge d\Psi$$

b is a $\mathbb{R}$ gauge field and $\Psi \sim \Psi + 2\pi$

Integrating out b gives

$$*(d\psi - b) = iR' d\Psi$$

and

$$S = \frac{{R'}^2}{4\pi}\int d\Psi \wedge *d\Psi + \frac{iR'}{2\pi}\int d\psi \wedge d\Psi$$

Note: the momentum states of ψ become winding states of Ψ

A topological rescaling

To sum up: rescaling the radius from R to R' can be made in two steps:

- Gauge $U(1)^{\mathrm{w}}$ with flat connections

This decompactifies the boson, which now has just an $\mathbb{R}$ momentum symmetry

- Gauge $\mathbb{Z} \subset \mathbb{R}^{\mathrm{m}}$ with a periodicity given by R'

This recompactifies the boson at the new radius.

For $R' = R$ we are implementing a symmetry that exchanges momentum and winding at any fixed radius

Interface between R and R'

We split spacetime in two: $X_2 = \Gamma_\mathrm{l} \cup \Gamma_\mathrm{r}$, with $\partial\Gamma_\mathrm{l} = -\partial\Gamma_\mathrm{r} = I$

$$S = S_l + S_r + S_I = \sum_{i=\mathrm{l,r}} \frac{R^2}{4\pi} \int_{\Gamma_i} d\Phi_i \wedge * d\Phi_i + \frac{i}{2\pi} \int_I (\Phi_\mathrm{l} - \Phi_\mathrm{r}) d\Phi$$

Then we follow all the steps of the topological rescaling only on Γ_r

$$\begin{aligned} S = & \frac{R^2}{4\pi} \int_{\Gamma_\mathrm{l}} d\Phi_\mathrm{l} \wedge * d\Phi_\mathrm{l} + \frac{R'^2}{4\pi} \int_{\Gamma_\mathrm{r}} d\Phi'_\mathrm{r} \wedge * d\Phi'_\mathrm{r} \\ & + \frac{i}{2\pi} \int_I (\Phi_\mathrm{l} - \frac{1}{R}\psi) d\Phi + R'\psi d\Phi'_\mathrm{r} \end{aligned}$$

with $\psi \in \mathbb{R}$ and $\Phi \in U(1)$ two edge modes on I

Interface between R and R'

Let us focus on the interface action

$$S_{R|R'} = \frac{i}{2\pi}\int_I (\Phi_1 - \frac{1}{R}\psi)d\Phi + R'\psi d\Phi'_{\rm r}$$

For $R' = R$ this is the T-duality symmetry defect

$$S_{R|R} = S_T$$

Integrating out Φ we have an alternative presentation

$$S_T = \frac{iR}{2\pi}\int_I \psi d\Phi_{\rm r} \qquad \text{with} \qquad \Phi_1|_I = \frac{1}{R}\psi$$

Note that we should avoid the wrongly quantized

$$S_T = \frac{iR^2}{2\pi}\int_I \Phi_1 d\Phi_{\rm r}$$

Interface between R and R'

$$S_{R|R'} = \frac{i}{2\pi}\int_I (\Phi_l - \frac{1}{R}\psi)d\Phi + R'\psi d\Phi'_r$$

The non-compact edge mode ψ gives an infinite quantum dimension to this interface/defect [cfr. finite version: Niro, Roumpedakis, Sela 22]

This can be seen because neglecting the coupling to the bulk modes we are left with a mixed $\mathbb{R}/U(1)$ BF theory on I

$$\frac{i}{2\pi R}\int_I \psi d\Phi$$

It has the (infinite dimensional) Hilbert space of a particle on a ring

$\Rightarrow$ Interpretation? Tamba$\mathbb{R}$a-Yamagami?

SymTFT interlude

The topological manipulation changing the radius has a nice interpretation in the SymTFT framework [Freed et al, Kong et al, Apruzzi et al]

The SymTFT for the 2d compact boson is of $\mathbb{R}\times\mathbb{R}$ type: [Antinucci, Benini 24]

$$S_{3d}=\frac{i}{2\pi}\int b^{+}db^{-}$$

with lines

$$U_x=\exp\left(ix\int b^{+}\right)\quad,\quad V_y=\exp\left(iy\int b^{-}\right)\quad \text{with}\quad x,y\in\mathbb{R}$$

and braiding

$$\langle U_x(\gamma)V_y(\gamma')\rangle=\exp\left(2\pi ixy\mathrm{Link}(\gamma,\gamma')\right)$$

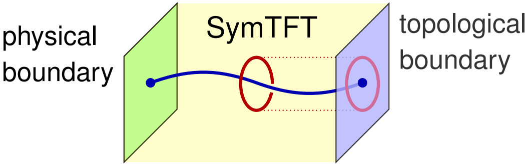
SymTFT: Topological and physical boundaries

The radius is fixed by the b.c. at the topological boundary:

$$\int b^+ \in 2\pi R\mathbb{Z} \qquad , \qquad \int b^- \in 2\pi R^{-1}\mathbb{Z}$$

At the physical boundary one imposes conformal b.c.: [Antinucci, Benini, Rizi 24]

$$b^- = i \star b^+$$



Changing the radius is just changing the boundary conditions in this SymTFT $\Rightarrow$ It is then a topological manipulation in the 2d CFT.

Interfaces and defects in a Euclidean lattice

Modified Villain model for compact boson

It is by now known that in order to have a Euclidean (square) lattice model with both momentum and winding symmetry, the modified Villain prescription is needed

[Sulejmanpasic, Gattringer 19; Gorantla, Lam, Seiberg, Shao 21; Choi, Cordova, Lam, Hsin, Shao 21]

It prescribes that there are no dynamical vortices, hence the winding symmetry is exact

$$S = \frac{R^2}{4\pi} \sum_l (d\phi - 2\pi n)_l^2 - i \sum_p \phi'_p dn_p$$

with

- $\phi_v \in \mathbb{R}$ defined on vertices
- $n_l \in \mathbb{Z}$ defined on links
- $\phi'_p \in U(1)$ defined on plaquettes

Modified Villain as flat gauging

$$S = \frac{R^2}{4\pi} \sum_l (d\phi - 2\pi n)_l^2 - i \sum_p \phi'_p dn_p$$

- n compactifies ϕ to $U(1)$ because of the gauge redundancy

$$\phi \to \phi + 2\pi k_\phi \qquad n \to n + dk_\phi$$

- ϕ' imposes the absence of dynamical vortices $dn = 0$

We immediately observe that the modified Villain prescription is nothing else than the (flat) gauging of $\mathbb{Z} \in \mathbb{R}$ momentum symmetry $\phi \to \phi + \lambda$ as we implemented in the continuum

Modified Villain and winding

The $U(1)$ winding symmetry acts as $\phi' \to \phi' + \lambda'$

It can be entirely gauged by introducing a gauge field $a_l \in U(1)$

$$S \to \frac{R^2}{4\pi} \sum_l (d\phi - 2\pi n)_l^2 + i \sum_p (\phi')_p (dn)_p - i \sum_l a_l n_l + i \sum_l a_l (dm)_l$$

$m_v \in \mathbb{Z}$ keeps it flat $\delta a = 0$

Integrating out a one eventually finds the non-compact boson

$$S = \frac{1}{4\pi} \sum_l (d\phi)_l^2$$

Modified Villain and T-duality

T-duality on the square lattice is Poisson resummation

The sum over n_l in the partition function with action

$$S = \frac{R^2}{4\pi}\sum_l (d\phi - 2\pi n)_l^2 - i\sum_p \phi'_p dn_p$$

is traded with a sum over k_l in a partition function with the dual action

$$S = \frac{1}{4\pi^2 R^2}\sum_{\tilde{l}} (d\tilde{\phi}' - 2\pi\tilde{k})_{\tilde{l}}^2 + i\sum_{\tilde{p}} \tilde{\phi}_{\tilde{p}} d\tilde{k}_{\tilde{p}}$$

written in the dual lattice where $\tilde{\phi} = *\phi$, $\tilde{\phi}' = *\phi'$ and $\tilde{k} = *k$

Modified Villain and T-duality defect

The strategy to build a T-duality interface/defect is the same as in the continuum

- Cut the lattice in two at the interface $\Lambda = \Lambda_{\mathrm{left}} \cup I \cup \Lambda_{\mathrm{right}}$
- Decompactify on the right
- Compactify to $1/R$ on the right
- Poisson resum on the right

The defect action is

$$S_I = \frac{iR^2}{2\pi} \sum_{v \in I} \psi_v (d\phi_{\mathrm{r}} - 2\pi n_{\mathrm{r}})_{l_v} \quad \text{with} \quad (\phi_{\mathrm{l}})_{v \in I} = \psi_v \qquad (n_{\mathrm{l}})_{l \in I} = 0$$

ψ is a non-compact scalar degree of freedom on I

[cfr. finite version: Choi, Cordova, Lam, Hsin, Shao 21]

Modified Villain and T-duality defect

$$S_T^{\text{Eucl.latt.}} = \frac{iR^2}{2\pi} \sum_{v \in I} \psi_v (d\phi_{\mathrm{r}} - 2\pi n_{\mathrm{r}})_{l_v} \quad \text{with} \quad (\phi_{\mathrm{l}})_{v \in I} = \psi_v \quad (n_{\mathrm{l}})_{l \in I} = 0$$

To be confronted with

$$S_T^{\text{cont.}} = \frac{iR}{2\pi} \int_I \psi d\Phi_{\mathrm{r}} \qquad \text{with} \qquad \Phi_{\mathrm{l}}|_I = \frac{1}{R}\psi$$

The map is

$$\psi^{\text{cont.}} \leftrightarrow \psi^{\text{Eucl.latt.}} \qquad \Phi_{\mathrm{l,r}}^{\text{cont.}} \leftrightarrow (\phi_{\mathrm{l,r}}, n_{\mathrm{l,r}})^{\text{Eucl.latt.}}$$

The lattice defect can be thought as a 1d state sum over $\mathbb{Z}$
$\rightarrow$ infinite quantum dimension: $Z = |\mathbb{Z}|$

Interfaces and defects in a Hamiltonian chain

Hamiltonian chain and modified Villain

Start from a chain of non-compact bosonic degrees of freedom $q, p \in \mathbb{R}$ with nearest-neighbor coupling

$$H_{R=\infty} = \pi \sum_j p_j^2 + \frac{1}{4\pi} \sum_j (\phi_j - \phi_{j+1})^2 \qquad [\phi_i, p_j] = i\delta_{ij}$$

- At each site we have $\dim \mathcal{H}_{p,q} = \infty$

Introducing then on each link a pair $n \in \mathbb{Z}, E \in U(1)$

$$[E_{i+1/2}, n_{j+1/2}] = i\delta_{ij} \qquad \text{with constraint} \qquad G_{i+1/2} \equiv e^{2\pi i n_{i+1/2}} = 1$$

- It is a particle on a ring (also $\dim \mathcal{H}_{n,E} = \infty$)

Hamiltonian chain and modified Villain

Compactification of the boson is achieved by coupling the two systems through a gauge invariance $k \in \mathbb{Z}$

$$\phi_i \to \phi_i + 2\pi k_i \qquad n_{i+1/2} \to n_{i+1/2} + k_i - k_{i+1}$$

It is implemented by the constraint

$$G_i \equiv e^{i\left(E_{i-1/2} - E_{i+1/2} - 2\pi p_i\right)} = 1$$

The Hamiltonian becomes

$$H_R = \frac{\pi}{R^2} \sum_j p_j^2 + \frac{R^2}{4\pi} \sum_j \left(\phi_j - \phi_{j+1} - 2\pi n_{j+1/2}\right)^2$$

The absence of E in H makes it modified Villain

[Cheng, Seiberg 22; Fazza, Sulejmanpasic 22; Seifnashri 26]

Hamiltonian chain and modified Villain

Momentum symmetry generated by

$$j_i^{\mathrm{m}} = p_i$$

Winding symmetry generated by

$$j_{i+1/2}^{\mathrm{w}} = \frac{1}{2\pi}(\phi_j - \phi_{j+1} - 2\pi n_{j+1/2})$$

Realized due to presence of n!

(De)compactification interface on the chain

Radius R on the left, ∞ radius on the right, interface at $j = I$

$$H^{(I)}_{R|\infty} = \frac{\pi}{R^2} \sum_{j \le I-1} p_j^2 + \pi \sum_{j \ge I} \bar{p}_j^2 + \frac{R^2}{4\pi} \sum_{j \le I-2} (\phi_j - \phi_{j+1} - 2\pi n_{j+1/2})^2$$

$$+ \frac{R^2}{4\pi} \left(\phi_{I-1} - \frac{1}{R} \bar{\phi}_I - 2\pi n_{I-1/2} \right)^2 + \frac{1}{4\pi} \sum_{j \ge I} (\bar{\phi}_j - \bar{\phi}_{j+1})^2$$

It can be moved to another site just by some redefinition of variables
$\Rightarrow$ topological defect

An interface between R and R' is obtained by introducing two defects $H^{(I)}_{R|\infty}$ and $H^{(J)}_{\infty|R'}$ and then sending $I \leftarrow J$

The defect $H^{(I)}_{R|R'}$ includes a non-compact pair $\bar{p}_I, \bar{\phi}_I \Rightarrow$ non-invertible

T-duality on the chain

We can redefine the variables

$$
\begin{aligned}
\tilde{p}_{i+1/2} &= \frac{1}{2\pi}(\phi_i - \phi_{i+1} - 2\pi n_{i+1/2}) \\
\tilde{\phi}_{i+1/2} &= E_{i+1/2} \\
\tilde{n}_i &= \frac{1}{2\pi}(E_{i-1/2} - E_{i+1/2} - 2\pi p_i) \\
\tilde{E}_i &= \phi_i
\end{aligned}
$$

It exchanges

- momentum and winding
- R and $1/R$
- chain and dual chain

T-duality defect on the chain

We can perform this redefinition only on part of the chain

$$H^{(I)}_{T(R|R^{-1})} = \frac{\pi}{R^2}\sum_{j\leq I} p_j^2 + \frac{1}{4\pi R^2}\sum_{j\geq I+1}(\tilde{\phi}_{j-1/2} - \tilde{\phi}_{j+1/2} - 2\pi\tilde{n}_j)^2$$
$$+ \frac{R^2}{4\pi}\sum_{j\leq I-1}(\phi_j - \phi_{j+1} - 2\pi n_{j+1/2})^2 + \pi R^2 \sum_{j\geq I}\tilde{p}^2_{j+1/2}$$

Two constraints couple the two sides of the interface

$$G_I = e^{i(E_{I-1/2} - \tilde{\phi}_{I+1/2} - 2\pi p_I)} \qquad G_{I+1/2} = e^{i(\phi_I - \tilde{E}_{I+1} - 2\pi \tilde{p}_{I+1/2})}$$

This defect as well can be shifted to a different site by redefinitions

Non-invertible symmetry defect on the chain

Fusing $H^{(I)}_{R|R^{-1}}$ with $H^{(J)}_{T(R^{-1}|R)}$ we get a symmetry defect for any R

$$
\begin{aligned}
H^{(I)}_{T\mathrm{sym}} =& \frac{\pi}{R^2}\sum_{j\leq I-1} p_j^2 + \pi \bar{p}_I^2 + \frac{\pi}{R^2}\sum_{j\geq I}\tilde{p}^2_{j+1/2} \\
&+ \frac{R^2}{4\pi}\sum_{j\leq I-2}(\phi_j - \phi_{j+1} - 2\pi n_{j+1/2})^2 + \frac{R^2}{4\pi}\left(\phi_{I-1} - \frac{1}{R}\bar{\phi}_I - 2\pi n_{I-1/2}\right)^2 \\
&+ \frac{R^2}{4\pi}\sum_{j\geq I+1}(\tilde{\phi}_{j-1/2} - \tilde{\phi}_{j+1/2} - 2\pi \tilde{n}_j)^2
\end{aligned}
$$

This defect also includes the non-compact pair $\bar{p}_I, \bar{\phi}_I \Rightarrow$ non-invertible

Infinite quantum dimension $\Leftrightarrow$ continuous spectrum

Outlook

We have constructed in a straightforward way topological defects that have infinite quantum dimension

These defects are robust under discretization of space and time

Higher dimensional generalizations should follow quite directly

More generally TQFTs with non-compact fields (with ∞ objects) need a more rigorous definition
$\Rightarrow$ state sums based on non-compact groups?

Homework: What are the physical consequences of the 'irrational' non-invertible symmetries such as the ones discussed here?

Thank you!