

# Superconductors with dynamical $\text{Spin}_c$ connection

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M. Berg, A. Cappelli, R. Villa, arXiv:2604.18733 [hep-th]

# Superconductors (SCs): U(1) Higgs phase

All the defining features of a SC are a consequence of higgsing  $U(1)$ : the system is effectively described by a gauge invariant action  $S[d\phi - qa]$ ,  $\phi$  the Higgs field phase, with charge  $q$ , and  $a$  the EM gauge field

- ✓ Zero resistivity
- ✓ Meissner effect
- ✓ Half-flux quantization (and similarly for any  $\mathbb{Z}_q$  subgroup)
- ✓ Josephson effect

Weinberg (1986)

- SC can be either gapless or gapped and either a bosonic or fermionic system
- A gapped superconductor is a  $\mathbb{Z}_2$  topologically ordered phase (known since 1911): low energy excitations are fermionic quasiparticle and half-flux ANO vortex, with non trivial  $-1$  braiding phase
- Effective field theory for gapped s-wave SC: Abelian Higgs model

Krauss, Wilczek (1989), Reznik, Aharonov (1989), Wen (1991), Hansson, Oganessian, Sondhi (2004)

Higgs condensate = Cooper pair field,  $q = 2$  → IR  $\mathbb{Z}_2$  gauge theory

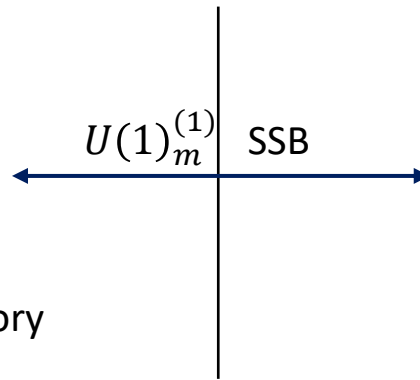
# Abelian Higgs model: s-wave SC

Abelian Higgs model picture: complex scalar field  $\phi$  charge 2 coupled to dynamical EM field  $a$  (say  $d = 4$ )

- Electric and magnetic symmetry:  $\mathbb{Z}_{2,e}^{(1)} \times U(1)_m^{(1)}$  (backgrounds  $B_e$  and  $B_m$ ) with mixed anomaly  $i\pi \int_Y B_e \cup \frac{dB_m}{2\pi}$
- Two phases, distinguished by symmetry realization:

Coulomb phase  $\int_X \frac{1}{2} da \wedge * da$

- Both symmetries SSB
- $\mathbb{Z}_{2,e}^{(1)}$  enhanced to  $U(1)_e^{(1)}$
- Gapless phase (photon)
- Anomaly matched by Maxwell theory



SC phase  $\int_X \frac{2}{2\pi} adb$   $b$  dual field of  $\phi$

- $U(1)_m^{(1)}$  confined: gapped phase
- $\mathbb{Z}_{2,e}^{(1)}$  still SSB and emergent  $\mathbb{Z}_{2,w}^{(2)}$  symmetry
- Topological order
- Symmetry fractionalization:  $\pi$ -flux vortices (anomaly matching)

$C = \frac{dB_m}{2\pi} = c_1(B_m)$  ( $C$  background for  $\mathbb{Z}_{2,w}^{(2)}$ )

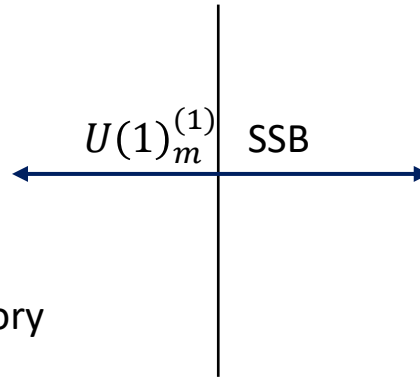
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## Issues of the Abelian Higgs model

- Effective mean field theory (and it specifically applies only to the s-wave case)
- Dependence on the electric symmetry (absent in the microscopic theory)
- It does not distinguish if Higgs field is fundamental or bound state of fermions
- It misses a fermionic quasiparticle in its spectrum [Thorngren \(2014\)](#)

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We want to characterize SCs more generally under two assumptions:

- $Spin_c(d)$  is a good symmetry for electronic systems (nucleii are neglected) ( $\mathbb{Z}_2^f = \text{fermion parity}$ )

[Seiberg, Witten \(2016\)](#)

$$Spin_c(d) = \frac{Spin(d) \times U(1)_f}{\mathbb{Z}_2} \quad \int \frac{dA_f}{2\pi} = \int \frac{w_2}{2} \text{ mod } 1 \quad \text{spin}_c \text{ connection}$$

*Spin-charge relation:* (integer spin, even charge), (half-integer spin, odd charge) since  $\mathbb{Z}_2^f \subset U(1)_f$

- SCs are Higgs phases for  $U(1)_f, A_f \rightarrow a_f$  dynamical field

→ These basic features should apply to any superconducting phase. Q: how much can we say from this?

# Dynamical $Spin_c$ connection

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Every system with a dynamical  $spin_c$  connection has a gravitational-magnetic mixed anomaly:

$$A_c = i\pi \int_Y w_2 \cup \frac{dB_m}{2\pi} \quad \text{Thorngren, Rakovszky, Verresen, Vishwanath (2023)}$$

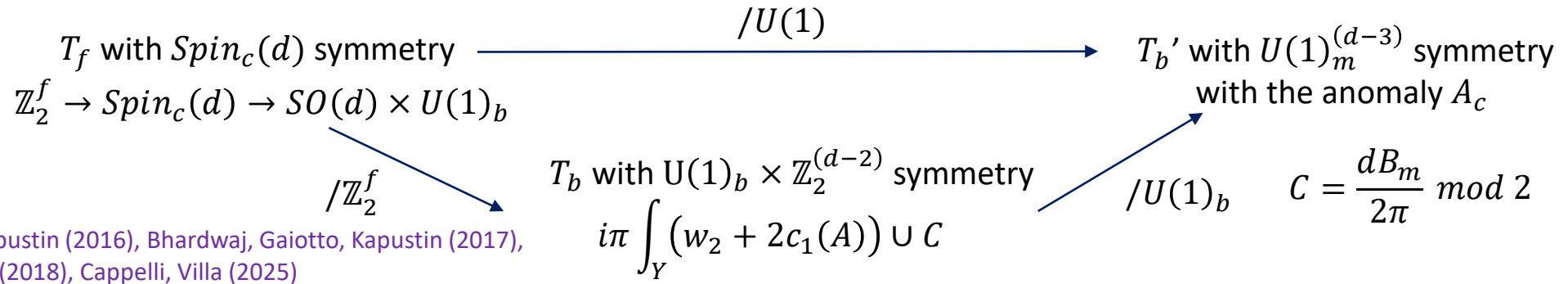
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- It can be seen either directly from the magnetic coupling or by bosonization.  
 Bosonization picture: since  $\mathbb{Z}_2^f \subset U(1)$ , the final theory is bosonic (gauge field for  $\mathbb{Z}_2^f$  is (twisted) spin structure)



Gaiotto, Kapustin (2016), Bhardwaj, Gaiotto, Kapustin (2017), Thorngren (2018), Cappelli, Villa (2025)

- Note: in 3d  $A_c$  is a mixed anomaly with time reversal, which must be assumed in the microscopic theory

# Higgs phase for a $\text{Spin}_c$ gauge theory

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Assume there is a Higgs mechanism that higgses the gauge group down to a subgroup  $\mathbb{Z}_q \subset U(1)$ :

- By  $\text{spin}_c$  symmetry,  $q$  must be even (only bosonic states could condense, so  $q$  cannot be 1)
- The mixed magnetic anomaly prevents trivially gapped phase by anomaly matching, there should be non-trivial degrees of freedom in the IR (even only topological)
- Assuming that the phase is gapped and  $q = 2$  (i.e. ordinary SC), the only option is [Johnson-Freyd \(2020\)](#)

$$i\pi \int b \cup (da + w_2) + C \cup a \quad C = \frac{dB_m}{2\pi} \text{ mod } 2$$

- $\mathbb{Z}_2$  gauge theory with a particular choice of symmetry fractionalization ( $w_2$ ):  $a$  Wilson line is fermionic
  - On-shell  $da = w_2$ :  $\mathbb{Z}_2$  gauge theory for the IR emergent spin structure =  $\mathbb{Z}_2^f$  gauge theory (the  $\text{spin}_c$  structure is higgsed to a spin structure, quasiparticles seem neutral)
  - This is forced by symmetry arguments, it does not rely on a mean-field description nor on electric symmetry
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# Ordinary superconductors

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Low energy theory of an ordinary SC :

$$2 + 1d \quad i\pi \int b \cup (da + w_2) \quad \longleftrightarrow \quad i\pi \int b \cup da \quad \text{Hsin, Shao (2020)}$$

$$3 + 1d \quad i\pi \int b \cup (da + w_2) \quad \longleftrightarrow \quad i\pi \int b \cup da + b \cup b + b \cup_1 db \quad \text{Kapustin, Thorngren (2017)}$$

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Twisted BF theory  $\int \frac{2}{2\pi} bda + \frac{4}{4\pi} b \wedge b$

Dijkgraaf, Witten (1990)

- Twisted BF theory for  $q = p = 2$ . Generically, DW theory for a two-form  $\mathbb{Z}_q$  gauge field in four-dimensions:

$$\int \frac{q}{2\pi} bda + \frac{qp}{4\pi} b \wedge b \sim \int \frac{2\pi p}{2q} P(b) \quad q \in \mathbb{Z}, p \in \mathbb{Z}_{2q}, qp \in 2\mathbb{Z} \quad \mathbb{Z}_{\gcd(p,q)}^{(1)} \times \mathbb{Z}_{\gcd(p,q)}^{(2)} \text{ symmetry}$$

Kapustin, Seiberg (2014)

$$\curvearrowright P: H^2(X; \mathbb{Z}_q) \rightarrow H^4(X; \mathbb{Z}_{q\gcd(q,2)})$$

- On spin manifolds,  $p = q$  is almost equivalent to  $p = 0$   
On non-spin manifolds (e.g.  $\mathbb{C}P^2$ ) ground-state half-flux vortex on the non-trivial cycle dual to  $w_2$

# Higher-charge superconductors

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The discussion of the Higgs phase can be generalized when the Higgs field has charge  $q = 2n$  and represents a bound state of  $2n$  electrons ( $q = 4$  and  $q = 6$  are realistic). If gapped:

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- $3 + 1d$ . Equivalent to twisted BF theory with  $q = p = 2n$  with  $\mathbb{Z}_{2n}$  topological order.
  - $q = p = 4$ : only possibility by anomaly matching, so  $\mathbb{Z}_4$  topological order as expected (i.e.  $p = 0$ )
  - $q = 6$ : there are three possible choices for anomaly matching:  $p = 2, 6, 10$ . The  $\mathbb{Z}_6$  topological order of  $p = 6$  (as  $p = 0$ ) is reduced to  $\mathbb{Z}_2$  for  $p = 2, 10$
- $2 + 1d$ . Equivalent to vanilla  $\mathbb{Z}_{2n}$  BF theory with  $\mathbb{Z}_{2n}$  topological order. For realistic  $q = 4$  case there is another possibility for anomaly matching according to DW, but always  $\mathbb{Z}_4$  topological order

# Topological superconductors (TSCs)

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Observations:

- For ordinary SCs, our reasoning leads us to  $\mathbb{Z}_2^f$  gauge theory: this is the result of gauging (summing) the spin structures  $\eta$  in the trivial spin TQFT  $1_\eta$
- The ordinary SC is the trivial phase in the space of the topological superconductors
- Every theory obtained by gauging  $\eta$  has a dual symmetry  $\mathbb{Z}_2^{(d-2)}$  symmetry with anomaly:  $i\pi \int_Y w_2 \cup C$   
 Fractionalization  $C = c_1(B_m)$  ensures both
  - ✓  $\mathbb{Z}_2^{(d-2)}$  charged objects are suitable ANO vortices
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→ Generalization to TSCs: take as spin TQFT not the trivial one, but the *invertible TQFT for the defining fermionic anomaly*  $T_\eta^{inv}$ ; the IR TQFT for the TSCs is given by its bosonization:

$$T_{TSC} = \sum_{\eta} T_{\eta}^{inv}$$

Read, Green (2000), Bernevig, Neupert (2015)

*Example:*  $3d p + ip$  TSC.  $T_\eta^{inv} = SO(1)_1$  and therefore  $T_{TSC} = Spin(1)_1$  (Ising TQFT)

- $Spin(1)_1$  reproduces the correct anyons with their fusion rules:  $\psi \times \psi = 1, \psi \times \sigma = \sigma, \sigma \times \sigma = 1 + \psi$
- Kitaev 16-fold way:  $Spin(16)_1 \sim Spin(0)_1 = \mathbb{Z}_2$  gauge theory, i.e. ordinary SC Kitaev (2006), Hsin, Shao (2020)

# Summary and outlook

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- The combination dynamical  $\text{spin}_c$  connection + Higgsing allows to infer some general properties of gapped superconductive phases
- For ordinary superconductors, the IR theory is fixed by symmetry arguments
- More complicated SCs should still comply with these properties: TSCs have non-trivial topological order when the gauge field is properly taken into account; 'extrinsic' defects become part of the IR TQFT
- It would be nice to see if the mixed magnetic anomaly could be connected to some observable properties in the SC phase
- We have not yet discussed boundary conditions (relevant for topological phases)
- What is the status of the Majorana fermion in the chiral  $p + ip$  TSC? A natural boundary condition for  $\text{Spin}(1)_1$  is the Ising CFT, not the Majorana one

Thank you!