Superconductors with dynamical $\mathrm{Spin_c}$ connection

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Superconductors (SCs): U(1) Higgs phase

All the defining features of a SC are a consequence of higgsing $U(1)$: the system is effectively described by a gauge invariant action $S[d\phi - qa]$, ϕ the Higgs field phase, with charge q, and a the EM gauge field

- ✓ Zero resistivity
- ✓ Meissner effect
- ✓ Half-flux quantization (and similarly for any $\mathbb{Z}_q$ subgroup)
- ✓ Josephson effect

Weinberg (1986)

- SC can be either gapless or gapped and either a bosonic or fermionic system
- *A gapped superconductor is a* $\mathbb{Z}_2$ *topologically ordered phase* (known since 1911): low energy excitations are fermionic quasiparticle and half-flux ANO vortex, with non trivial -1 braiding phase

Krauss, Wilczek (1989), Reznik, Aharonov (1989), Wen (1991), Hansson, Oganesyan, Sondhi (2004)

- Effective field theory for gapped s-wave SC: Abelian Higgs model

Higgs condensate = Cooper pair field, $q = 2$ $\xrightarrow{\text{IR}}$ $\mathbb{Z}_2$ gauge theory

Abelian Higgs model: s-wave SC

Abelian Higgs model picture: complex scalar field ϕ charge 2 coupled to dynamical EM field a (say $d=4$)

- Electric and magnetic symmetry: $\mathbb{Z}_{2,e}^{(1)} \times U(1)_m^{(1)}$ (backgrounds B_e and B_m) with mixed anomaly $i\pi \int_Y B_e \cup \frac{dB_m}{2\pi}$
- Two phases, distinguished by symmetry realization:

Coulomb phase $\int_X \frac{1}{2} da \wedge * \, da$

- ○ Both symmetries SSB
- ○ $\mathbb{Z}_{2,e}^{(1)}$ enhanced to $U(1)_e^{(1)}$
- ○ Gapless phase (photon)
- ○ Anomaly matched by Maxwell theory

$U(1)_m^{(1)}$ SSB

SC phase $\int_X \frac{2}{2\pi} adb$ b dual field of ϕ

- ○ $U(1)_m^{(1)}$ *confined: gapped phase*
- ○ $\mathbb{Z}_{2,e}^{(1)}$ still SSB and emergent $\mathbb{Z}_{2,w}^{(2)}$ symmetry
- ○ Topological order
- ○ *Symmetry fractionalization: π-flux vortices* (anomaly matching)

(acts on)

$$C = \frac{dB_m}{2\pi} = c_1(B_m)$$ (C background for $\mathbb{Z}_{2,w}^{(2)}$)

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Enter the $\mathbf{Spin}_c$ symmetry

Issues of the Abelian Higgs model

- Effective mean field theory (and it specifically applies only to the s-wave case)
- Dependence on the electric symmetry (absent in the microscopic theory)
- It does not distinguish if Higgs field is fundamental or bound state of fermions
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We want to characterize SCs more generally under two assumptions:

- $Spin_c(d)$ is a good symmetry for electronic systems (nucleii are neglected) ($\mathbb{Z}_2^f$ = fermion parity)

Seiberg, Witten (2016)

$$Spin_c(d) = \frac{Spin(d) \times U(1)_f}{\mathbb{Z}_2} \qquad \int \frac{dA_f}{2\pi} = \int \frac{w_2}{2} \ mod\ 1 \quad \text{spin}_\text{c}\ \text{connection}$$

Spin-charge relation: (integer spin, even charge), (half-integer spin, odd charge) since $\mathbb{Z}_2^f \subset U(1)_f$

- SCs are Higgs phases for $U(1)_f, A_f \to a_f$ dynamical field

⟶ These basic features should apply to any superconducting phase. Q: how much can we say from this?

Dynamical $\mathbf{Spin_c}$ connection

Every system with a dynamical $spin_c$ connection has a gravitational-magnetic mixed anomaly:

$$A_c = i\pi \int_Y w_2 \cup \frac{dB_m}{2\pi}$$

Thorngren, Rakovszky, Verresen, Vishwanath (2023)

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  Bosonization picture: since $\mathbb{Z}_2^f \subset U(1)$, the final theory is bosonic (gauge field for $\mathbb{Z}_2^f$ is (twisted) spin structure)

T_f with $Spin_c(d)$ symmetry

$\mathbb{Z}_2^f \to Spin_c(d) \to SO(d) \times U(1)_b$

$/U(1)$ →

$T_b{}'$ with $U(1)_m^{(d-3)}$ symmetry with the anomaly A_c

$/\mathbb{Z}_2^f$ →

T_b with $\mathrm{U}(1)_b \times \mathbb{Z}_2^{(d-2)}$ symmetry

$$i\pi \int_Y \left(w_2 + 2c_1(A)\right) \cup C$$

$/U(1)_b$ →

$$C = \frac{dB_m}{2\pi} \ mod\ 2$$

Gaiotto,Kapustin (2016), Bhardwaj, Gaiotto, Kapustin (2017), Thorngren (2018), Cappelli, Villa (2025)

- Note: in 3d A_c is a mixed anomaly with time reversal, which must be assumed in the microscopic theory

Higgs phase for a $\mathbf{Spin_c}$ gauge theory

Assume there is a Higgs mechanism that higgses the gauge group down to a subgroup $\mathbb{Z}_q \subset U(1)$:

- By $spin_c$ symmetry, q must be even (only bosonic states could condense, so q cannot be 1)
- The mixed magnetic anomaly prevents trivially gapped phase by anomaly matching, there should be non-trivial degrees of freedom in the IR (even only topological)
- Assuming that the phase is gapped and $q = 2$ (i.e. ordinary SC), the only option is Johnson-Freyd (2020)

$$i\pi \int b \cup (da + w_2) + C \cup a \qquad C = \frac{dB_m}{2\pi}\ mod\ 2$$

  - ➢ $\mathbb{Z}_2$ gauge theory with a particular choice of symmetry fractionalization (w_2): a Wilson line is fermionic
  - ➢ On-shell $da = w_2$: $\mathbb{Z}_2$ gauge theory for the IR emergent spin structure = $\mathbb{Z}_2^f$ *gauge theory* (the $\mathrm{spin_c}$ structure is higgsed to a spin structure, quasiparticles seem neutral)
- This is forced by symmetry arguments, it does not rely on a mean-field description nor on electric symmetry

Ordinary superconductors

Low energy theory of an ordinary SC :

$2+1d \qquad i\pi \int b \cup (da + w_2) \longleftrightarrow i\pi \int b \cup da$ Hsin, Shao (2020)

$3+1d \qquad i\pi \int b \cup (da + w_2) \longleftrightarrow i\pi \int b \cup da + b \cup b + b \cup_1 db$ Kapustin, Thorngren (2017)

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Twisted BF theory $\int \frac{2}{2\pi} bda + \frac{4}{4\pi} b \wedge b$

Dijkgraaf, Witten (1990)

- Twisted BF theory for $q = p = 2$. Generically, DW theory for a two-form $\mathbb{Z}_q$ gauge field in four-dimensions:

$$\int \frac{q}{2\pi} bda + \frac{qp}{4\pi} b \wedge b \sim \int \frac{2\pi p}{2q} P(b) \qquad q \in \mathbb{Z},\ p \in \mathbb{Z}_{2q},\ qp \in 2\mathbb{Z}$$

$\mathbb{Z}^{(1)}_{\gcd(p,q)} \times \mathbb{Z}^{(2)}_{\gcd(p,q)}$ symmetry

Kapustin, Seiberg (2014)

$P: H^2(X; \mathbb{Z}_q) \to H^4(X; \mathbb{Z}_{q\gcd(q,2)})$

- On spin manifolds, $p = q$ is almost equivalent to $p = 0$
  On non-spin manifolds (e.g. $\mathbb{C}P^2$) ground-state half-flux vortex on the non-trivial cycle dual to w_2

Higher-charge superconductors

The discussion of the Higgs phase can be generalized when the Higgs field has charge $q = 2n$ and represents a bound state of $2n$ electrons ($q = 4$ and $q = 6$ are realistic). If gapped:

Berg, Fradkin, Kivelson (2009)

$$\frac{2\pi i}{2n} \int b \cup (da + n w_2) + \frac{dB_m}{2\pi} \cup a$$

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- $3 + 1d$. Equivalent to twisted BF theory with $q = p = 2n$ with $\mathbb{Z}_{2n}$ topological order.
    - $q = p = 4$: only possibility by anomaly matching, so $\mathbb{Z}_4$ topological order as expected (i.e. $p = 0$)
    - $q = 6$: there are three possible choices for anomaly matching: $p = 2,6,10$. The $\mathbb{Z}_6$ topological order of $p = 6$ (as $p = 0$) is reduced to $\mathbb{Z}_2$ for $p = 2,10$

- $2 + 1d$. Equivalent to vanilla $\mathbb{Z}_{2n}$ BF theory with with $\mathbb{Z}_{2n}$ topological order. For realistic $q = 4$ case there is another possibility for anomaly matching according to DW, but always $\mathbb{Z}_4$ topological order

Topological superconductors (TSCs)

Observations:

- For ordinary SCs, our reasoning leads us to $\mathbb{Z}_2^f$ gauge theory: this is the result of gauging (summing) the spin structures η in the trivial spin TQFT 1_η
- The ordinary SC is the trivial phase in the space of the topological superconductors
- Every theory obtained by gauging η has a dual symmetry $\mathbb{Z}_2^{(d-2)}$ symmetry with anomaly: $i\pi \int_Y w_2 \cup C$
  Fractionalization $C = c_1(B_m)$ ensures both
  - ✓ $\mathbb{Z}_2^{(d-2)}$ charged objects are suitable ANO vortices
  - ✓ Anomaly matching for a UV $spin_c$ gauge theory

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⟶ Generalization to TSCs: take as spin TQFT not the trivial one, but the *invertible TQFT for the defining fermionic anomaly* T_η^{inv}; the IR TQFT for the TSCs is given by its bosonization:

$$T_{TSC} = \sum_\eta T_\eta^{inv}$$

Read, Green (2000), Bernevig, Neupert (2015)

Example: $3d\ p + ip$ TSC. $T_\eta^{inv} = \mathrm{SO}(1)_1$ and therefore $T_{\mathrm{TSC}} = \mathrm{Spin}(1)_1$ (Ising TQFT)

- $\mathrm{Spin}(1)_1$ reproduces the correct anyons with their fusion rules: $\psi \times \psi = 1, \psi \times \sigma = \sigma, \sigma \times \sigma = 1 + \psi$
- Kitaev 16-fold way: $\mathrm{Spin}(16)_1 \sim \mathrm{Spin}(0)_1 = \mathbb{Z}_2$ gauge theory, i.e. ordinary SC

Kitaev (2006), Hsin, Shao (2020)

Summary and outlook

- The combination dynamical spin_c connection + Higgsing allows to infer some general properties of gapped superconductive phases
- For ordinary superconductors, the IR theory is fixed by symmetry arguments
- More complicated SCs should still comply with these properties: TSCs have non-trivial topological order when the gauge field is properly taken into account; 'extrinsic' defects become part of the IR TQFT

- It would be nice to see if the mixed magnetic anomaly could be connected to some observable properties in the SC phase
- We have not yet discussed boundary conditions (relevant for topological phases)
- What is the status of the Majorana fermion in the chiral $p + ip$ TSC? A natural boundary condition for $\mathrm{Spin}(1)_1$ is the Ising CFT, not the Majorana one

Thank you!