Statistics of Elementary Excitations with Mixed Dimensionalities

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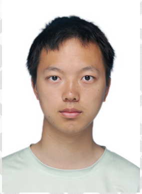


Based on:
Hanyu Xue and Xiao-Gang Wen
(In preparation)

Workshop on Defects and Extended Excitations
in Quantum Field Theory, Quantum Matter and Statistical Models

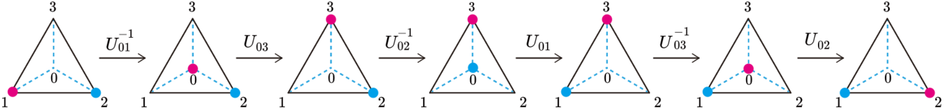
Quantum Statistics in Physics

- Quantum statistics is a fundamental property of physical systems.
- For point-like particles in $(3+1)$-dimensional spacetime, locality permits only **Bose** and **Fermi** statistics.
- In lower spacetime dimensions, the topology of configuration space is richer:
  - In $(2+1)$D, particles can obey **Abelian statistics** (braiding yields phase factors).
  - Or **non-Abelian statistics** (braiding operations act as matrices).

What about extended excitations (strings, membranes)?
What if particles and strings coexist?

Fermi Statistics from Link-Operator Algebra

- On a $3+1$D discrete lattice, for particle with $\mathbb{Z}_2$ conservation, its statistics is purely determined by the **algebra of local particle hopping operators**.



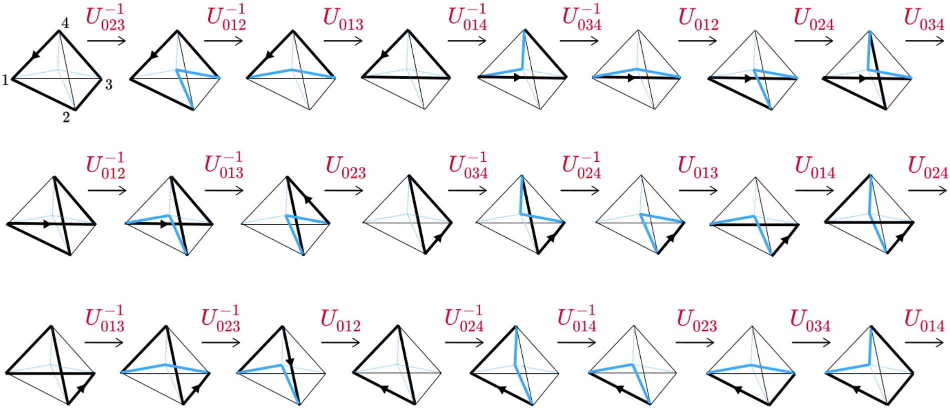
$$U_{02}\,U_{03}^{-1}\,U_{01}\,U_{02}^{-1}\,U_{03}\,U_{01}^{-1} = -1$$

Levin Wen, cond-mat/0302460

- **Local particle hopping operators**: far away hopping operators (**link-operators**) commute.

String Statistics from Triangle-Operator Algebra

- On a $3+1$D discrete lattice, $\mathbb{Z}_2$-string statistics are purely determined by the **algebra of local string hopping operators**.



- **Local string hopping operators**: far away **triangle operators** commute

Fidkowski Haah Hastings, 2110.14654

Kobayashi Li Xue Hsin Chen, 2412.01886

Two Kinds of Excitations

To study statistics more generally and more systematically, it is crucial to distinguish between two types of excitations in a many-body system.

In a D-dimensional spacetime complex M^D with degrees of freedom on vertices, edges, and higher simplexes.

1. Elementary Excitations

- Directly described by worldlines, worldsheets, and higher-dimensional worldvolumes formed by physical degrees of freedom on vertices, edges, triangles, *etc* of triangulated spacetime: **qubit-0 represent vacuum, qubit-1 form world trajectories**.
- Their world trajectories can be cycles, and may not be boundaries. Their Poincaré-dual currents are cocycles.
- Example: A fundamental fermionic electron.

Two Kinds of Excitations (cont.)

2. Bounding Excitations

- Arise as the boundaries of **condensed** elementary excitations.
- Their spacetime trajectories form the boundaries of one-higher-dimensional chains. Their Poincaré-dual currents are coboundaries.
- Example: Emergent fermions f and bosons e, m in $\mathbb{Z}_2$ gauge theory.

- Most previous literature mainly concerns about **bounding excitations** – emergent topological excitations in qubit systems, whose statistics is described by **braided fusion higher categories with trivial center**, which classify bosonic topological orders and their topological excitations (bounding excitations).
- Here, we focus on the statistics of **elementary excitations**.

Statistics and Conservation Law

- Non-trivial statistics requires non-trivial conservation. No conservation → trivial bosonic statistics.
- Conservation law is described by fusion rule

$$\text{spin-}\frac{1}{2} \otimes \text{spin-}\frac{1}{2} = \text{spin-0} \oplus \text{spin-1} \qquad \text{Rep}_{SU(2)}$$

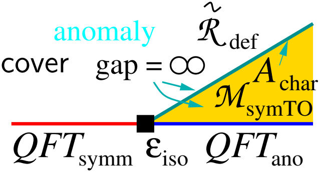
$$g \otimes h = gh, \quad g, h, gh \in G \qquad \text{Group-like, pointed}$$

$$\phi \otimes \phi = \mathbf{1} \oplus \phi \qquad \text{Fibonacci}$$

- **Symmetry-topological-order (Sym/TO) correspondence**: the most general conservation laws (*ie* the most general fusion rules) are descried by **topological orders** (*ie* **fully extended TQFTs**) in **one-higher dimension**, provided that the conservation laws are finite (the number of superselection sectors are finite). This is also called **categorical symmetry** Ji Wen, 1912.13492, Kong *etal* , 2005.14178
symTFT Apruzzi Bonetti Etxebarria Hosseini Schafer-Nameki 2112.02092
topological holography Moradi Moosavian Tiwari, 2207.10712

One-slide introduction to Sym/TO correspondence

Kong Wen Zheng arXiv:1502.01690



Freed Telemen Moore 2209.07471

We view a global symmetry within its symmetric sub-Hilbert space $\mathcal{V}_{\text{sym}}$, where the symmetry transformation acts trivially. To discover a symmetry in this setting is to discover a gravitational anomaly (*ie* a lacking of tensor product decomposition $\mathcal{V}_{\text{sym}} \neq \otimes_i \mathcal{V}_i$).

- **Non-invertible gravitational anomaly $\cong$ generalized symmetry (up to holo-equivalence)** (anomalous, higher-form, higher-group, non-invertible, *etc*). Ji Wen, 1912.13492
- **Non-invertible gravitational anomaly $\xlongequal{\text{1-to-1}}$ topological order in one-higher dimension** from anomaly inflow Kong Wen 1405.5858
- **Generalized symmetry (up to holo-equivalence) $\xlongequal{\text{1-to-1}}$ Topological order in trivial Witt class in one-higher dim**

Kong Lan Wen Zhang Zheng, 2005.14178

Boundary of bulk topological order simulates the symmetric sector of the corresponding symmetric system exactly.

Invertible Excitations and Abelian Statistics

- Here we consider a special conservation law described by **pointed** fusion rule $g \otimes h = gh$. The corresponding elementary excitations are called **invertible** (or **pointed**), satisfying the defining property

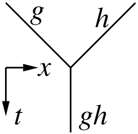
$$a \otimes \bar{a} = \mathbf{1}$$

- The statistics of invertible excitations are called **Abelian statistics**, because braiding operations act via scalar $U(1)$ phases rather than non-Abelian matrices.

- If all elementary excitations are invertible, their fusion rules are group-like (or higher-group-like), *ie* they fuse like the **symmetry defects** of **group** or **higher group**.

- Higher group can capture **conservation laws of mixed dimensionalities**: Intersections or junctions of higher-dimensional excitations can act as sources for lower-dimensional ones.

Cochain fields for mixed conserved currents



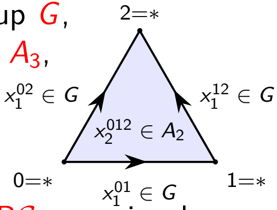
We trianglate D-dimensional spacetime M^D.

- Conservation law, fusion rules, and conserved currents:
  - **codimension-1 conserved objects (domain-walls)** labeled by g in group G: Domain-wall fusion $g \otimes h = gh$ (G-0-symmetry defects). Poincaré dual of G-valued $(D-1)$-dim worldvolume → G-valued 1-cochain current f_1.
  - **codimension-2 conserved objects** labeled by a_2 in Abelian group A_2. Dual of A_2-valued $(D-2)$-dimensional worldvolume → A_2-valued 2-cochain current $f_2 = A_2$-1-symmetry defects.
  - **Mixed conservation law**: describes how conservation laws of different dimensionalities couple to one another.

$$\mathrm{d} f_1 = 1, \qquad \mathrm{d} f_2 = k_3(f_1) \qquad \mathrm{d} f_j = k_{j+1}(f_1, \cdots, f_{j-1})$$

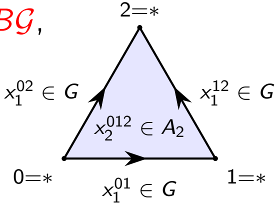
How do the conserved currents f_j and their mixed conservation law arise from a higher group $\mathcal{G}$?

Understand Higher Group from its classifying space



- A higher group $\mathcal{G}$ contains layers of 0-form group G, 1-form Anelian group A_2, 2-form Anelian group A_3,
- The classifying space $B\mathcal{G}$ encode those groups via $\pi_1(B\mathcal{G}) = G$, $\pi_2(B\mathcal{G}) = A_2$, $\pi_3(B\mathcal{G}) = A_3$,
- We build the classifying space $B\mathcal{G}$ as a **one-vertex trianglation**: All the simplices in $B\mathcal{G}$ are uniquely labeled by the labels on they sub-faces:
  - $B\mathcal{G}$ has only a single vertex $*$ (0-simplex, the base point).
  - An edge (1-simplex) is labeled by its two vertices $*$ (no information), and by an element of $\pi_1(B\mathcal{G}) = G$.
  - A triangle (2-simplex) is labeled by its three edges (labeled by $\pi_1(B\mathcal{G})$), together with an element of $\pi_2(B\mathcal{G}) = A_2$.
- **Canonical cochains** $x_1, x_2, \ldots$ on $B\mathcal{G}$ are defined by their evaluation on the simplices: evaluating x_1 on an edge returns its G label, evaluating x_2 on a triangle simply returns its A_2 label, *etc*

Understand Higher Group from its classifying space



- To form a valid higher-group classifying space $B\mathcal{G}$, these labels (the canonical cochains) cannot be arbitrary. They must satisfy structural equations encoding the **Postnikov data**.

- For example, three edges labled by $x_1^{01}, x_1^{12}, x_1^{02} \in G$ bound a triangle (012) if and only if $x_1^{01}(x_1^{02})^{-1}x_1^{12} \equiv (\mathrm{d}x_1)^{012} = 1$.

- In general, we have

$$\mathrm{d}x_1 = 1, \qquad \mathrm{d}x_2 = k_3(x_1) \qquad \mathrm{d}x_j = k_{j+1}(x_1, \cdots, x_{j-1})$$

- **From canonical cochains x_j to conserved currents f_j**: Let ϕ be a simplicial map from triangulated spacetime M^D to triangulated classifying space $B\mathcal{G}$. The pull back of x_j on $B\mathcal{G}$ give rise to cochains $f_j = \phi^* x_j$ on spacetime, satisfying the mixed conservations

$$\mathrm{d}f_1 = 1, \qquad \mathrm{d}f_2 = k_3(f_1) \qquad \mathrm{d}f_j = k_{j+1}(f_1, \cdots, f_{j-1})$$

Warm-up: Bosonization of Fermions

- Consider a fermion in D-dimensional spacetime. It is a point-like elementary excitation with $\mathbb{Z}_2$ conservation → $\mathbb{Z}_2$ 0-form symmetry or $\mathbb{Z}_2^{(d-1)}$ $(d-1)$-form symmetry ($d = D - 1 =$ spatial dimension)

- The dual of a fermion world line is a $\mathbb{Z}_2$-valued d-cocycle f_d satisfying $\mathrm{d} f_d = 0$. The $\mathbb{Z}_2$ conservation of a particle is described by a **higher group symmetry** $\mathbb{Z}_2^{(d-1)}$ with classifying space $B\mathbb{Z}_2^{(d-1)} = K(\mathbb{Z}_2, d)$ – Eilenberg-MacLane space with only $\pi_d = \mathbb{Z}_2$

- A path integral for the conserve bosonic field f_d is:

$$Z(M^D) = \sum_{f_d \in Z^d(M^D;\mathbb{Z}_2)} \mathrm{e}^{-\int_{M^D} \mathcal{L}(f_d)}$$

  which describe a bosonic particle with conserved current f_d.

- **How do we endow it with Fermi statistics?**

Warm-up: The WZW Term

- We can make f_d fermionic by appending a WZW topological term

$$Z(M^D) = \sum_{f_d} \mathrm{e}^{-\int_{M^D} \mathcal{L}(f_d)} \underbrace{\mathrm{e}^{\mathrm{i}\pi \int_{N^{D+1}} \mathrm{Sq}^2 f_d}}_{\text{WZW}=\pm 1}$$

(N^{D+1} is an extension of M^D: $\partial N^{D+1} = M^D$, Sq^n Sreenod squre)

- In order for this to be a consistent D-dimensional theory, the action amplitude must be independent of how f_d is extended into N^{D+1}. The WZW term $\mathrm{e}^{\mathrm{i}\pi \int_{N^{D+1}} \mathrm{Sq}^2 f_d}$ fails this condition

- But we can rewrite the path integral as

$$Z(M^D, s) = \sum_{f_d} \mathrm{e}^{-\int_{M^D} \mathcal{L}(f_d)} \mathrm{e}^{\mathrm{i}\pi \int_{M^D} f_d s} \mathrm{e}^{\mathrm{i}\pi \int_{N^{D+1}} (\mathrm{Sq}^2 f_d + f_d[\mathrm{w}_2 + \mathrm{w}_1^2])}$$

$\mathrm{d}s = \mathrm{w}_2 + \mathrm{w}_1^2 =$ spin structure, and introduce a modified WZW term $\mathrm{e}^{\mathrm{i}\pi \int_{N^{D+1}} \mathrm{Sq}^2 f_d + f_d(\mathrm{w}_2 + \mathrm{w}_1^2)}$, which can satisfy this condition on a spin manifold, due to Wu's formula $\mathrm{Sq}^2 f_d + f_d(\mathrm{w}_2 + \mathrm{w}_1^2) \overset{2,\mathrm{d}}{=} 0$.

WZW term requires spacetime to be spin → Fermi statistics

The Holographic Sym/TO Perspective

- The pure bulk action amplitude:

$$\mathrm{e}^{\mathrm{i}\,\pi \int_{N^{D+1}} \mathrm{Sq}^2 f_d}$$

  defines a $(D+1)$-dimensional **twisted higher gauge theory**, which describes the higher group symmetry $B\mathcal{G} = K(\mathbb{Z}_2, d)$ in one-lower dimension (the $\mathbb{Z}_2$ particle conservation).

- **Key idea:** When this higher gauge theory is untwisted, the $\mathbb{Z}_2$ conserved particles on the boundary described by current f_d are bosonic.

  Introducing the WZW cocycle twist $\mathrm{e}^{\mathrm{i}\,\pi \int_{N^{D+1}} \mathrm{Sq}^2 f_d}$ makes the $\mathbb{Z}_2$ conserved particles on the boundary described by current f_d to be fermions.

  This holographic Sym/TO perspective leads to a general framework for both mixed and unmixed conservation laws.

Main Result: Holographic Statistics

For **invertible elementary excitations** in D-dimensional spacetime with mixed-dimensionalities, their mixed conservation laws are described by a higher group symmetry $\mathcal{G}$ (*ie* the fusion of $\mathcal{G}$-symmetry defects describe the fusion of those excitations). Their Abelian statistics are classified by a $(D+1)$-cocycle:

$$\omega \in H^{D+1}(B\mathcal{G};\mathbb{R}/\mathbb{Z})$$

- The dynamics of those excitations is simulated holographically by the boundary of a $\mathcal{G}$ higher-group gauge theory in $(D+1)$-dimensional spacetime, twisted by ω:

$$Z = \sum_{\phi: M^D \to B\mathcal{G}} \mathrm{e}^{-\int_{M^D} \mathcal{L}(\phi)} \mathrm{e}^{2\pi \mathrm{i} \int_{N^{D+1}} \phi^* \omega}, \quad M^D = \partial N^{D+1}$$

The map $\phi : M^D \to B\mathcal{G}$ describes the current fluctuations f_j while preserving their mixed conservation.

Example 1: Particles in $2+1$D

- Let us apply this framework to point-like elementary excitations in $2+1$-dimensional spacetime M^3.

- The Poincaré dual of particle worldlines is a 2-cocycle f_2 on M^3. Qubits live on triangles in spacetime complex M^4.

- Assume a pointed fusion rule described by a finite Abelian group A. f_2 is an A-valued cocycle, viewed as a pullback:

$$f_2 = \phi^* x_2, \quad \phi : M^3 \to K(A,2)$$

- The Abelian statistics are classified by:

$$H^4(K(A,2), \mathbb{R}/\mathbb{Z})$$

Classification via Quadratic Functions

- Mathematically, this cohomology group is known to be:

$$H^4(K(A,2),\mathbb{R}/\mathbb{Z}) \cong \mathrm{Quad}(A,\mathbb{R}/\mathbb{Z})$$

the group of quadratic functions $q : A \to \mathbb{R}/\mathbb{Z}$.

- A function is quadratic if $q(-a) = q(a)$ and

$$b_q(a,b) := q(a+b) - q(a) - q(b)$$

forms a symmetric bilinear pairing.

- Agrees w/ result from **pointed braided tensor category (BTC)**

\- $q(a)$ determines the topological spin (self-statistics) of particle a.

\- $b_q(a,b)$ gives the mutual braiding phase between a and b.

$$\text{Pointed BFC}(A) \cong H^3_{\text{grp}}(A;\mathbb{R}/\mathbb{Z}) \cong \text{Quad}(A,\mathbb{R}/\mathbb{Z}) \cong H^4(K(A,2),\mathbb{R}/\mathbb{Z})$$

- For $A = \mathbb{Z}_2$ conserved particle, $H^4(K(\mathbb{Z}_2,2),\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_4$.
  $\omega = \frac{1}{4}(x_2^2 + x_2 \smile_1 \mathrm{d}x_2) \to$ semion. $\omega = \frac{1}{2}x_2^2 \to$ fermion.

Example 2: Strings in $3+1$D

- Next, consider elementary strings in $3+1$-dimensional spacetime M^4.

- The Poincaré dual of the string worldsheets is a 2-cocycle f_2. Quibts live on triangles in spacetime complex M^4.

- Notice that the current of a string in 4D and the current of a particle in 3D are both described by 2-cocycles.

- However, the spacetime dimension is now $D=4$. Thus, the statistics are classified by:

$$H^5(K(A,2),\mathbb{R}/\mathbb{Z})$$

instead of $H^4(K(A,2),\mathbb{R}/\mathbb{Z})$ for $D=3$.

Cohomology Classification for Strings

- For a generic finite Abelian group $A = \bigoplus_I \mathbb{Z}_{N_I}$, we find:

$$H^5(K(A,2), \mathbb{R}/\mathbb{Z}) \cong \left(\bigoplus_I \mathbb{Z}_{\gcd(N_I,2)} \right) \oplus \left(\bigoplus_{I<J} \mathbb{Z}_{N_{IJ}} \right)$$

- The first set of terms describes the **self-statistics** of strings (only exists for even N_I).
- The second set describes **mutual braiding statistics** between strings carrying different charges.

- Representative cocycle uses the Bockstein operation $\mathcal{B}_N f = \frac{1}{N}\mathrm{d}f$:

$$\omega = \sum_I \frac{p_I}{N_I} f_{2,I} \mathcal{B}_{N_I} f_{2,I} + \sum_{I<J} \frac{p_{IJ}}{N_{IJ}} f_{2,I} \mathcal{B}_{N_{IJ}} f_{2,J}$$

$\mathbb{Z}_2$ String Statistics and w_3 Structure

- For an elementary string with $\mathbb{Z}_2$ conservation, the nontrivial statistics are given by:

$$\omega(f_2) = \frac{1}{2} f_2 \mathcal{B}_2 f_2$$

$\to$ fermionic string studied by Thorngren 1404.4385; Wang Wen Witten, 1810.00844; Fidkowski Haah Hastings, 2110.14654; Wan Wang Wen, 2112.12148; Kobayashi Li Xue Hsin Chen, 2412.01886

- When is the corresponding WZW amplitude $\mathrm{e}^{\mathrm{i}2\pi \int_{N^5} \omega(f_2)}$ independent of the 5D extension? We require on any closed W^5:

$$\int_{W^5} f_2 \mathcal{B}_2 f_2 = 0 \pmod 2$$

- Applying Wu's formula on a 5-manifold:

$$\int_{W^5} f_2 \mathcal{B}_2 f_2 = \int_{W^5} f_2 \mathrm{w}_3(TW) \pmod 2$$

The $\mathrm{w}_3 = 0$ Geometric Requirement

- The ambiguity vanishes for all background string configurations f_2 if and only if $\mathrm{w}_3(TW) = 0$.

- This is the direct string-analogue of the spin condition for fermionic particles!

- To support nontrivial Abelian statistics for $\mathbb{Z}_2$ strings, spacetime must admit a w_3-**structure**.

- While a full spin structure ($\mathrm{w}_1 = \mathrm{w}_2 = 0$) implies $\mathrm{w}_3 = \mathcal{B}_2\mathrm{w}_2 = 0$, the $\mathrm{w}_3 = 0$ condition is strictly weaker and represents the minimal tangential condition detected by string statistics.

Example 3: Mixed Dimensionalities in $(d+1)$D

- Consider a system containing **both** particles and strings in $(d+1)$D spacetime ($d \geq 3$), both with $\mathbb{Z}_2$ conservation.

- Dual fields (the currents):
  - f_{d-1}: string worldsheets
  - f_d: particle worldlines

- Their conservation laws are governed by a higher group space $B\mathcal{G}_{k_{d+1}}$ with $\pi_{d-1}(B\mathcal{G}_{k_{d+1}}) = \mathbb{Z}_2$ and $\pi_d(B\mathcal{G}_{k_{d+1}}) = \mathbb{Z}_2$.

- The Postnikov class $k_{d+1} \in H^{d+1}(K(\mathbb{Z}_2, d-1), \mathbb{Z}_2)$ has two choices:

$$k_{d+1} = \rho \mathrm{Sq}^2 x_{d-1}, \quad \rho \in \{0, 1\}$$

Untwisted vs. Twisted Conservation Laws

- The spacetime conservation laws take the form:

$$\mathrm{d}f_{d-1} = 0, \quad \mathrm{d}f_d = \rho \mathrm{Sq}^2 f_{d-1} \pmod 2$$

Untwisted case ($\rho = 0$):

- Particles and strings are independently conserved: $\mathrm{d}f_{d-1} = 0, \quad \mathrm{d}f_d = 0 \pmod 2$.
- Statistics classified by $\omega \in H^{d+2}(B\mathcal{G}_0, \mathbb{R}/\mathbb{Z})$.
- Gives $(\mathbb{Z}_2)^2$ for $d > 3$ or $(\mathbb{Z}_2)^3$ for $d = 3$ independent self/mutual statistics.

- What happens when $\rho = 1$? Specific self-intersections of the string worldsheets act as sources for the particle current.

Statistics for Twisted Conservation $\to \mathbb{Z}_4$ Group

Twisted case ($\rho = 1$, mixed conservation):

- The Abelian statistics are classified by $H^{d+2}(B\mathcal{G}_{\mathrm{Sq}^2 f_{d-1}}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_4$.
- In the Serre spectral sequence, the string and particle self-statistics $\mathbb{Z}_2$ classes combine into a single, nonsplit $\mathbb{Z}_4$ **extension**.
- The generator is a secondary-operation cocycle $\omega(x_2, x_3)$ for $D = 4$:

$$\omega(x_2, x_3) = \frac{1}{2}\mathbb{S}\mathrm{q}^2 x_3 + \frac{1}{8}\mathbb{S}\mathrm{q}^3 x_2 + \frac{1}{2}\Delta_5(x_2)$$

$$\mathrm{d}\Delta_5(x_2) = \mathbb{S}\mathrm{q}^2\mathbb{S}\mathrm{q}^2 x_2 + \mathbb{S}\mathrm{q}^3\mathbb{S}\mathrm{q}^1 x_2$$

where the particle is a fermion.

- *Twice* this generator yields the nontrivial string self-statistics term:

$$2\omega(x_2, x_3) = \frac{1}{4}\mathbb{S}\mathrm{q}^3 x_2 = \frac{1}{4} x_2 \mathrm{d} x_2 = \frac{1}{2} x_2 \mathcal{B}_2 x_2 \quad (\text{mod } 1, \mathrm{d})$$

where the particle is a boson.

Summary

1. **Elementary Excitations:** We study excitations with strictly closed (but not exact) currents, in contrast to emergent anyons (bounding excitations).
2. **Holographic Sym/TO Description:** The conservation laws and Abelian statistics of invertible elementary excitations are classified by a twisted higher-group gauge theory in one higher dimension. The higher group $\mathcal{G}$ encode the conservation laws and the cocyle in $H^{D+1}(B\mathcal{G}, \mathbb{R}/\mathbb{Z})$ encode statiatics.
3. **Geometric Constraints:** Nontrivial statistics necessitate generalized spacetime structures (e.g., spin structures for fermions, $\mathrm{w}_3 = 0$ structures for fermionic $\mathbb{Z}_2$ strings).
4. **Mixed Dimensionalities:** Higher groups elegantly capture systems where particles and strings interact, converting $\mathbb{Z}_2$ statistics into unified $\mathbb{Z}_4$ topological structures.

Contrast with Chern-Simons Theory

- The statistics generated by this WZW term on N^4 fundamentally differ from those generated by a Chern-Simons term on M^3.

- The WZW term describes elementary conserved currents f_2 which are strictly closed but not required to be exact.
  - It is anomaly-like.
  - Depends on the 4D extension.

- A Chern-Simons term applies when the current is a coboundary ($f_2 = \mathrm{d}a_1$).
  - If we assume exactness, WZW descends to a boundary term, but lacks the correct normalization for CS statistics.
  - WZW statistics $\neq$ CS bounding statistics!

Physical Realization: Topological Superconductors

- For the $p = 1$ or $p = 3$ cases of this $\mathbb{Z}_4$ statistics group, the system perfectly matches a p-**wave topological superconducting (TS) string**.

- In such TS models:
  - Particles carry Fermi statistics and $\mathbb{Z}_2$ fusion.
  - Fermions can condense along a string to form a TS string, which possesses $\mathbb{Z}_2$ conservation.
  - TS strings and $\mathbb{Z}_2$ fermions exhibit the exact same mixed conservation law: $\mathrm{d}f_d \equiv \mathrm{Sq}^2 f_{d-1} \pmod 2$.

- **This strongly suggests that the string defects in the $p = 1, 3$ cases are physically realized as TS strings!**

Beyond Abelian: Non-Abelian Statistics

- This holographic framework naturally extends to **non-Abelian** statistics.

- A $(D+1)$-dimensional state-sum theory provides the bulk description for the boundary excitations' conservation laws and statistics.
  - Local amplitude dictated by a tensor on top-dimensional simplices.
  - Retriangulation invariance ensures the theory is topological.

- For non-invertible excitations, braiding acts via matrices, and the underlying algebraic structure shifts from a simple higher group to a general **fusion** $(D-1)$-**category**.

Higher groups represent the pointed (invertible) case of this broader categorical framework.