<u>Topological order enriched by non-invertible symmetry via anyon condensation</u>

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Introduction

- Topological order (TO) in (2+1)d: theory of anyons

- In the presence of global symmetry given by group G (finite): symmetry enriched topological order (SET)

- For both of these well-established mathematical formulation:

  - TO $\Longleftrightarrow$ Unitary modular tensor category (UMTC) $\mathscr{B}$
  - G-SET $\Longleftrightarrow$ G-crossed braided fusion category $\mathscr{B}_G^{\times}$

- How do we extend this when the enriching symmetry is *non-invertible*?

  [See also Balasubramanian, Buican, Delcamp, Radhakrishnan '25; Eck, Huston, Kawagoe, Penneys '26]

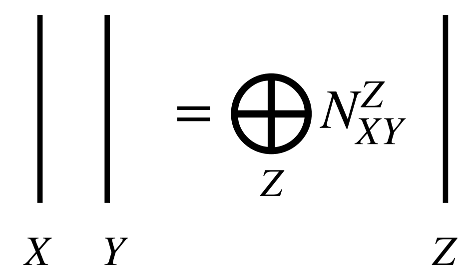
  - Notice this is a natural question coming from two perspectives: *condensation defects* and *anyon condensation*

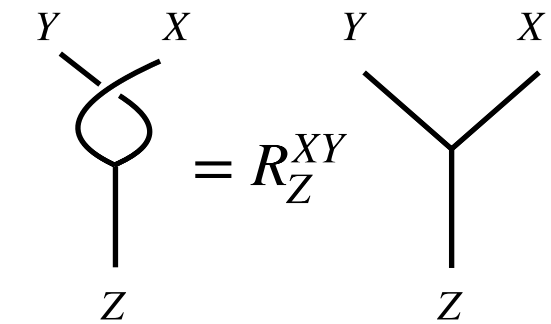
  [Roumpedakis, Seifnashri, Shao '23]

Symmetry enriched TO

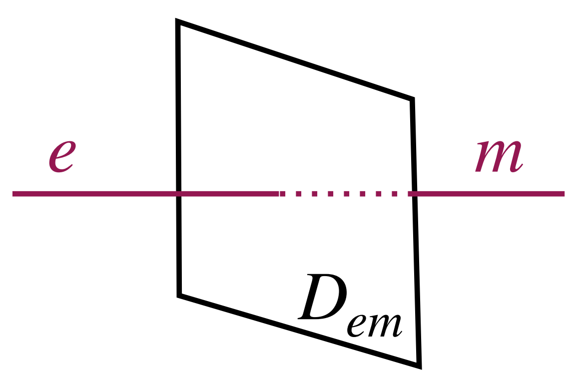
[Barkeshli, Bonderson, Cheng, Wang '19; Lan, Kong, Wen '17]

- Given an UMTC $\mathscr{B}$, simple objects $a \in \mathscr{B}$ correspond to anyon species



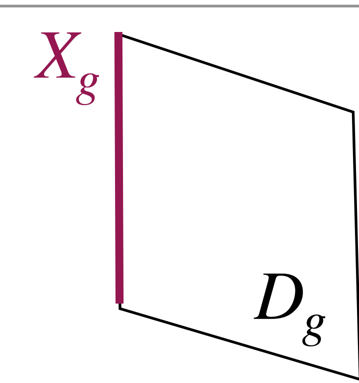


- In the presence of global symmetry G
  - Symmetry acts on anyons in a way compatible with fusion structure and braiding (*braided tensor auto-equivalences*)
  - Domain wall D_g for each element $g \in G$



- Example: *em* duality ($G = \mathbb{Z}_2$) in the toric code with anyons $\{1, e, m, f\}$

  - Moreover, *twisted sector* lines living at the end of a g-domain wall



G-crossed braided fusion categories $\mathscr{B}_G^{\times}$

[Etingof, Nikshych, Ostrik '10;
Etingof, Gelaki, Nikshych, Ostrik '16]

- G-graded fusion category: $\mathscr{B}_G^{\times} = \bigoplus_{g \in G} \mathscr{B}_g$ with

$$\begin{cases} \mathscr{B}_1 = \mathscr{B} & \text{untwisted anyons (underlying topological order)} \\ \mathscr{B}_g\,, g \neq 1 & g\text{-twisted sector} \end{cases}$$

- G-grading compatible with tensor structure: $\mathscr{B}_{g_1} \boxtimes \mathscr{B}_{g_2} \to \mathscr{B}_{g_1 g_2}$
- G-action $T : G \to \mathrm{Aut}^{\mathrm{br}}_{\otimes}(\mathscr{B})$ such that: $T_g(\mathscr{B}_x) \subset \mathscr{B}_{gxg^{-1}}$
- Braiding twisted by the G-action: $X \in \mathscr{B}_g$ $\quad R_{X,Y} : X \otimes Y \to T_g(Y) \otimes X$

☆ *G-equivariantisation* (physically: gauging the 0-form symmetry G)

- Once you have $\mathscr{B}_G^{\times}$, can consider $(\mathscr{B}_G^{\times})^G$
- Objects in $(\mathscr{B}_G^{\times})^G$ are labelled by $(X, \{\lambda_g\}_{g \in G})$, with $X \in \mathscr{B}_G^{\times}$ and $\lambda_g : T_g(X) \cong X$

G-SET via anyon condensation

[Bais, Slingerland '09; Kong '13]

- G-crossed braided $\mathscr{B}_G^\times$ can be obtained via *anyon condensation*
- Patterns of anyon condensations classified by *condensable algebras* A in $\mathscr{B}$
  - Roughly, set of bosonic anyons that become identified with the identity

- Condensing A gives rise to a new topological order $\mathscr{B}_A$
  - *Deconfined* excitations $\mathscr{B}_A^{\mathrm{loc}}$ (local modules of A) braid trivially with A
  - The rest, braiding non-trivially with A, are *confined* excitations (non-local modules of A)

[Drinfeld, Gelaki, Nikshych, Ostrik '10]

- If A 'generates' Rep(G) in $\mathscr{B}$ then $\mathscr{B}_A$ is G-crossed braided
- Moreover $(\mathscr{B}_A)^G \simeq \mathscr{B}$

Example: S_3 quantum double model $D(S_3)$

$S_3 = \{r, s \mid r^3 = s^2 = 1,\ sr = r^2 s\}$

- $\mathscr{B} = \mathscr{Z}(\mathrm{Vec}_{S_3})$ (*Drinfel'd center*) contains $\mathrm{Rep}(S_3)$ as a subcategory (*pure charges*)
  - $\mathrm{Rep}(S_3) = \{1, e, \pi\}$ with e *sign irrep* and π *2d irrep* of S_3

$e^2 = 1,\ e \otimes \pi = \pi$

$\pi^2 = 1 \oplus e \oplus \pi$

- Condensable algebra $A(\mathbb{Z}_3) \cong 1 \oplus e$ with

$\mathscr{Z}(\mathrm{Vec}_{S_3})^{\mathrm{loc}}_{A(\mathbb{Z}_3)} \simeq \mathscr{Z}(\mathrm{Vec}_{\mathbb{Z}_3})$	Deconfined excitations
$\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_3)} \simeq \mathscr{Z}_{\mathrm{Vec}_{\mathbb{Z}_3}}(\mathrm{Vec}_{S_3})$	*Relative* Drinfel'd center: confined & deconfined

- $\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_3)}$ has a $\mathbb{Z}_2 = S_3/\mathbb{Z}_3$ *charge conjugation* symmetry: $e \leftrightarrow \bar{e},\ m \leftrightarrow \bar{m}$

- $\mathbb{Z}_2$-equivariantisation recovers $\mathscr{Z}(\mathrm{Vec}_{S_3})$

$$\mathscr{Z}(\mathrm{Vec}_{S_3}) \underset{S_3/\mathbb{Z}_3\text{-equivariantisation}}{\overset{A(\mathbb{Z}_3)\text{ condensation}}{\rightleftarrows}} \mathscr{Z}_{\mathrm{Vec}_{\mathbb{Z}_3}}(\mathrm{Vec}_{S_3})$$

Condensing more general algebras

- $\mathcal{Z}(\mathrm{Vec}_{S_3})$ has another condensable algebra $A(\mathbb{Z}_2) \cong 1 \oplus \pi$

  - $\mathcal{Z}(\mathrm{Vec}_{S_3})^{\mathrm{loc}}_{A(\mathbb{Z}_2)} \simeq \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2})$ Deconfined excitations $\{1, e, m, f\}$
  - $\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)} \simeq \mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_2}}(\mathrm{Vec}_{S_3})$ Confined excitations $\{c_1, c_2\}$

- Question: what is the symmetry enriching $\mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_2}}(\mathrm{Vec}_{S_3})$?

  - We propose it is a *hypergroup symmetry* associated with the double coset $S_3//\mathbb{Z}_2$

- $\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)}$ naturally decomposes over double cosets $[k] \in \mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2$

  - $S_3//\mathbb{Z}_2$ has 2 components $[1] = \{1, s\}$ and $[r] = \{r, r^2, rs, r^2 s\}$
  - $\{1, e, m, f\} \in (\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[1]}$ $\{c_1, c_2\} \in (\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[r]}$

Hypergroup grading

- Hypergroup: set $\{k_i\}$ with product $k_1 \star k_2 = \sum_{k_3} C_{k_1 k_2}^{k_3}\, k_3$ with $C_{k_1 k_2}^{k_3} \geq 0$ and $\sum_{k_3} C_{k_1 k_2}^{k_3} = 1$

  [Bischoff '16]

- A double coset $H\backslash G/H$ has the structure of an hypergroup

$$[k] \equiv HkH \in H\backslash G/H \qquad [k_1] \star [k_2] := \frac{1}{|H|} \sum_{h \in H} H(k_1 h k_2)H \equiv \sum_{[k_3]} C_{[k_1][k_2]}^{[k_3]} [k_3]$$

- For our example

$$[r] \star [r] := \frac{1}{2}[1] \oplus \frac{1}{2}[r] \quad \Longrightarrow \quad C_{[r][r]}^{[1]} = C_{[r][r]}^{[r]} = \frac{1}{2}$$

- Compatibility condition $X \in (\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[k_1]}$, $Y \in (\mathscr{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})_{[k_2]}$ (simple objects)

$$\frac{\mathrm{FPdim}(\Pi_{[k_3]}(X \otimes Y))}{\mathrm{FPdim}(X \otimes Y)} = C_{[k_1][k_2]}^{[k_3]}$$

[Hempel '23]

- Check with fusion rules

$$c_1 \otimes c_1 = c_1 \otimes c_1 = 1 \oplus e \oplus c_2$$

$$c_1 \otimes c_2 = m \oplus f \oplus c_1$$

Non-invertible symmetry action

- We define the *non-invertible hypergroup symmetry* as a composition of maps

$$T : \mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)} \longrightarrow \mathcal{Z}(\mathrm{Vec}_{S_3}) \longrightarrow \mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)}$$

- T decomposes as $T = \bigoplus_{[k] \in \mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2} T_{[k]}$ so $T = T_{[1]} \oplus T_{[r]}$ [Riesen '25]

- Compute $[r]$-action:
  - $T_{[r]}(1) = T_{[r]}(e) = 1 \oplus e$
  - $T_{[r]}(m) = T_{[r]}(f) = c_1$
  - $T_{[r]}(c_1) = m \oplus f \oplus c_1$
  - $T_{[r]}(c_2) = c_2 \oplus c_2$

- Compatibility condition (R *regular object*)
  - $\mathrm{Ad}_{[k_1]}([k_2]) := [k_1] \star [k_2] \star [k_1^{-1}]$
  - $\mathrm{Ad}_{[k_1]}([k_2]) \equiv \sum_{[k_3] \in S_3 // \mathbb{Z}_2} A^{[k_3]}_{[k_1][k_2]} [k_3]$
  - $$\frac{\mathrm{FPdim}(\Pi_{[k_3]}(T_{[k_1]}(R_{[k_2]})))}{\mathrm{FPdim}(T_{[k_1]}(R_{[k_2]}))} = A^{[k_3]}_{[k_1][k_2]}$$

- One can show taking *T-modules:* $(\mathcal{Z}(\mathrm{Vec}_{S_3})_{A(\mathbb{Z}_2)})^T \simeq \mathcal{Z}(\mathrm{Vec}_{S_3})$ [Cui, Zini, Wang '19]

- Generalised *fixed point theorem*

Beyond group case & final remarks

- Our construction for $\mathcal{Z}(\mathrm{Vec}_{S_3}) \to \mathcal{Z}_{\mathrm{Vec}_{\mathbb{Z}_2}}(\mathrm{Vec}_{S_3})$ relies on $\mathbb{Z}_2 \leq S_3$
- Generalise this to any $\mathcal{B} = \mathcal{Z}(\mathcal{C})$ related via anyon condensation to $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ with $\mathcal{D} \subseteq \mathcal{C}$
- Hypergroup symmetry given by $\mathcal{H} = \mathcal{Z}(\mathcal{D}) \backslash \mathcal{Z}_{\mathcal{D}}(\mathcal{C}) / \mathcal{Z}(\mathcal{D})$
- Summarising

	G-crossed braided	Hypergroup extension
Compatibility grading-fusion	$\mathcal{B}_{g_1} \boxtimes \mathcal{B}_{g_2} \to \mathcal{B}_{g_1 g_2}$	$\dfrac{\mathrm{FPdim}(\Pi_{[k_3]}(X \otimes Y))}{\mathrm{FPdim}(X \otimes Y)} = C^{[k_3]}_{[k_1][k_2]}$
Compatibility grading-action	$T_g(\mathcal{B}_x) \subset \mathcal{B}_{gxg^{-1}}$	$\dfrac{\mathrm{FPdim}(\Pi_{[k_3]}(T_{[k_1]}(R_{[k_2]})))}{\mathrm{FPdim}(T_{[k_1]}(R_{[k_2]}))} = A^{[k_3]}_{[k_1][k_2]}$
Equivariantisation	$(\mathcal{B}^{\times}_{G})^{G}$	T-modules in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$

Thank you!