Rényi Defect Criticality and Entropy in (2+1)D QCPs

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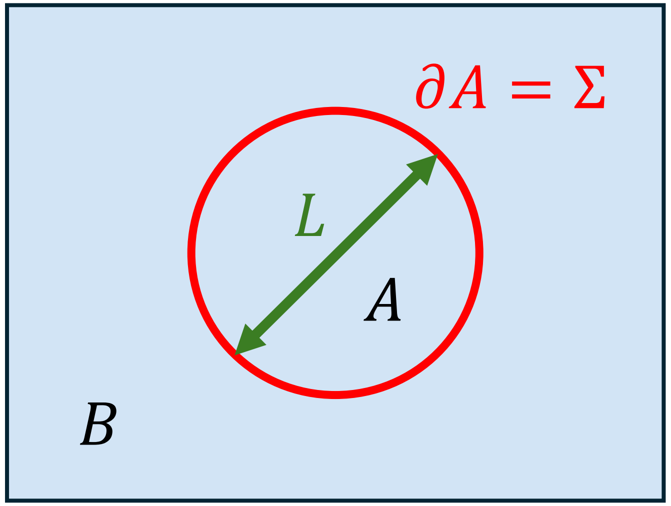
Partly based on W.I.P. with

G. Shankar (Yale) and Meng Cheng (Yale)

Entanglement Entropies in CFT

- Standard expectation of area law for Rényi entanglement entropies in CFT_D ($D = d + 1$)



$\rho_A = \mathrm{Tr}_B\, \rho$

$$S_{A,n} = \frac{\log \mathrm{Tr}_A\, \rho_A^n}{1-n} \sim g_n^{(d-1)} \left(\frac{L}{\epsilon}\right)^{d-1} + \cdots + g_n^{(0)} \log\left(\frac{L}{\epsilon}\right) + b_n$$

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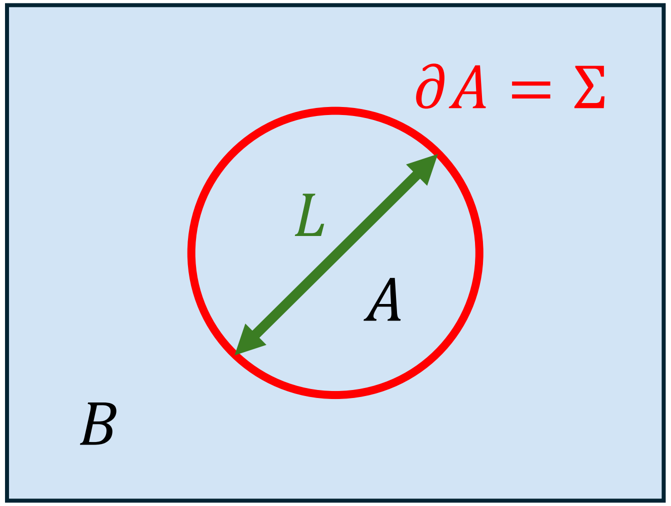

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$$D = 2:\ \ S_{A,n} = \frac{c}{6}\left(1 + \frac{1}{n}\right) \log\left(\frac{L}{\epsilon}\right)$$

0

$$D = 3:\ \ S_{A,n} \sim g_n^{(1)} \left(\frac{L}{\epsilon}\right) + g_n^{(0)} \log\left(\frac{L}{\epsilon}\right) + b_n$$

- Universal logarithmic contribution comes from Weyl anomaly, does not appear in odd D dimensions.
- Exception when Σ is singular, e.g. cusped in $D = 3$.

Entanglement Entropies in CFT

- In $D = 3$, computing entanglement entropies in (interacting) CFTs is difficult. One has to rely on numerics and lattice simulation when feasible.

Universal part of entropy should be independent of UV details of lattice model?

Entanglement Entropies in CFT

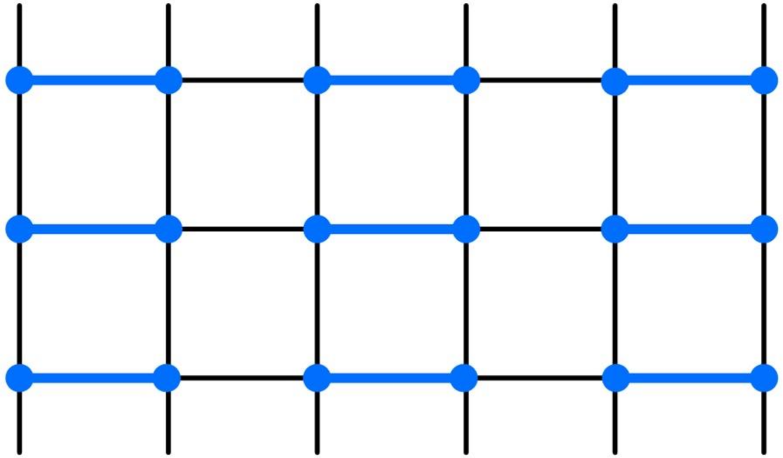
Universal part of entropy should be independent of UV details of lattice model?

- ***Not seen in practice: Heisenberg model with anisotropy of $\boldsymbol{O}(3)$ Wilson-Fisher.***

$$H = \sum_{\langle i,j \rangle} J_1 \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \sum_{\langle i,j \rangle'} J_2 \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

Trivial (Singlet) O(3) SSB

$K = J_1/J_2 \to \infty$ $K_c \approx 1.9$ $K \to 0$



(Matsumoto-Yasuda-Todo-Takayama, 2001)

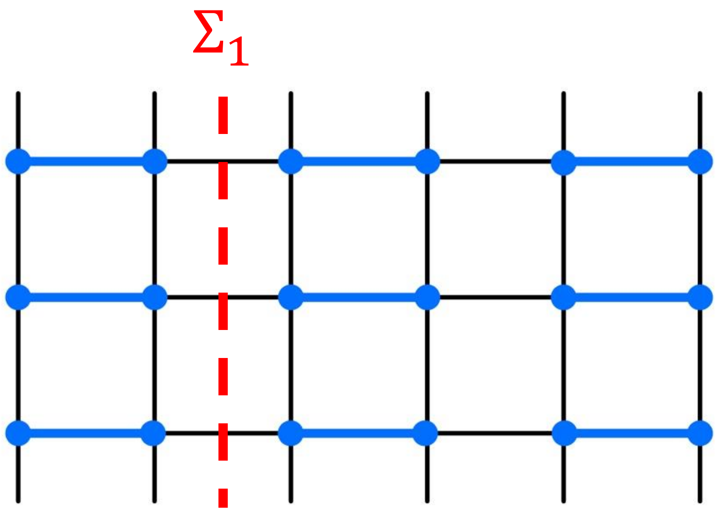
Entanglement Entropies in CFT

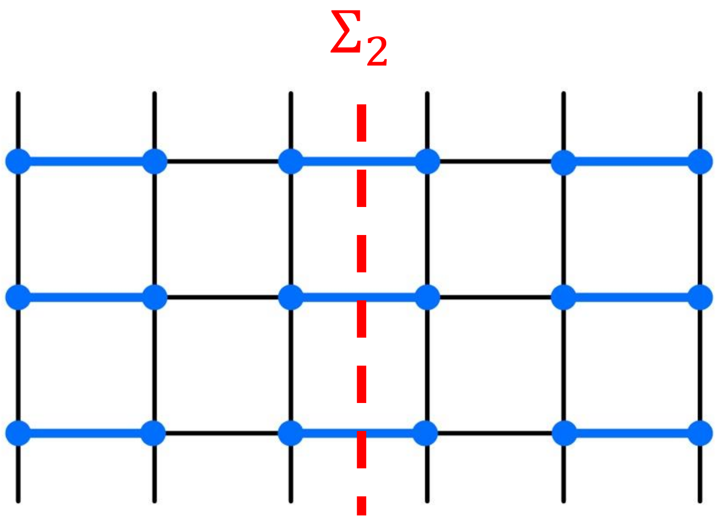
Not seen in practice: Heisenberg model with anisotropy of $O(3)$ Wilson-Fisher.

- Numerics sees different values for 2nd Rényi entropy (Zhao-Wang-Yan-Cheng-Meng 2022) and spin scaling dimensions (Zhu-Wang-Cheng-Yan, 2026).

$\hat{\Delta}_1(n=2) = 0.358$

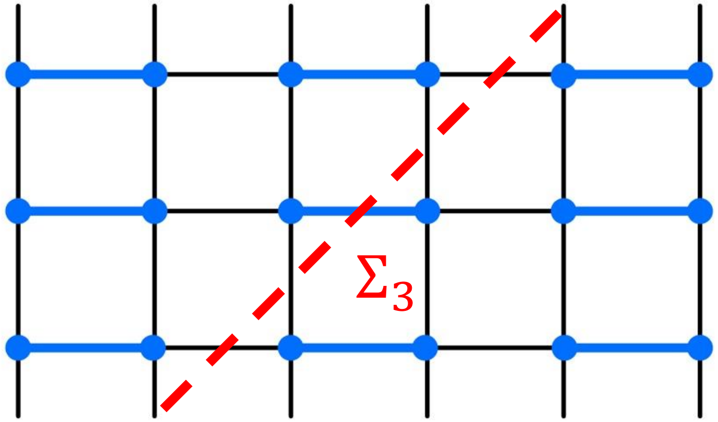
$\hat{\Delta}_1(n=3) = 0.324$





$\hat{\Delta}_2(n=2) = 0.278$

$\hat{\Delta}_2(n=3) = 0.154$



$\hat{\Delta}_3(n=2) = 0.317$

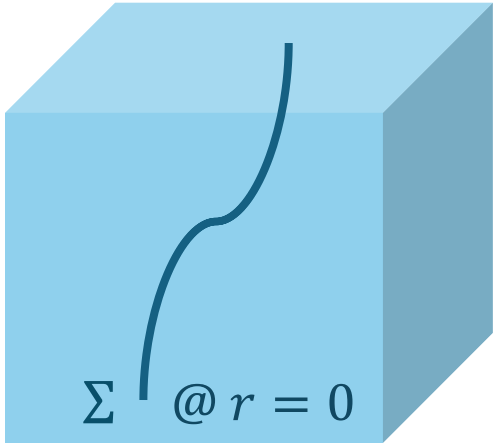
$\hat{\Delta}_3(n > n_c \approx 4) \to 0$

($\widehat{\Delta}$ values from table 1 in Zhu-Wang-Cheng-Yan, 2026, 2605.00104).

Entangling Surface and Conical Defect BCs

Possible choice of entangling surface (Σ) *boundary conditions may account for this puzzle.*

- Entangling surface manifests as (space-time) codimension-2 conical defect in path integral. Consider scalar field ϕ



$$\phi(\boldsymbol{x}, r, \theta + 2\pi n) = \phi(\boldsymbol{x}, r, \theta)$$

$$S_{n,A} = \frac{n\mathcal{F}_1 - \mathcal{F}_n}{1-n} \quad \text{with } \mathcal{F}_n = -\log Z_n$$

- Boundary conditions on conical defect Σ, can be made apparent after mapping the problem to AdS via Weyl transformation.

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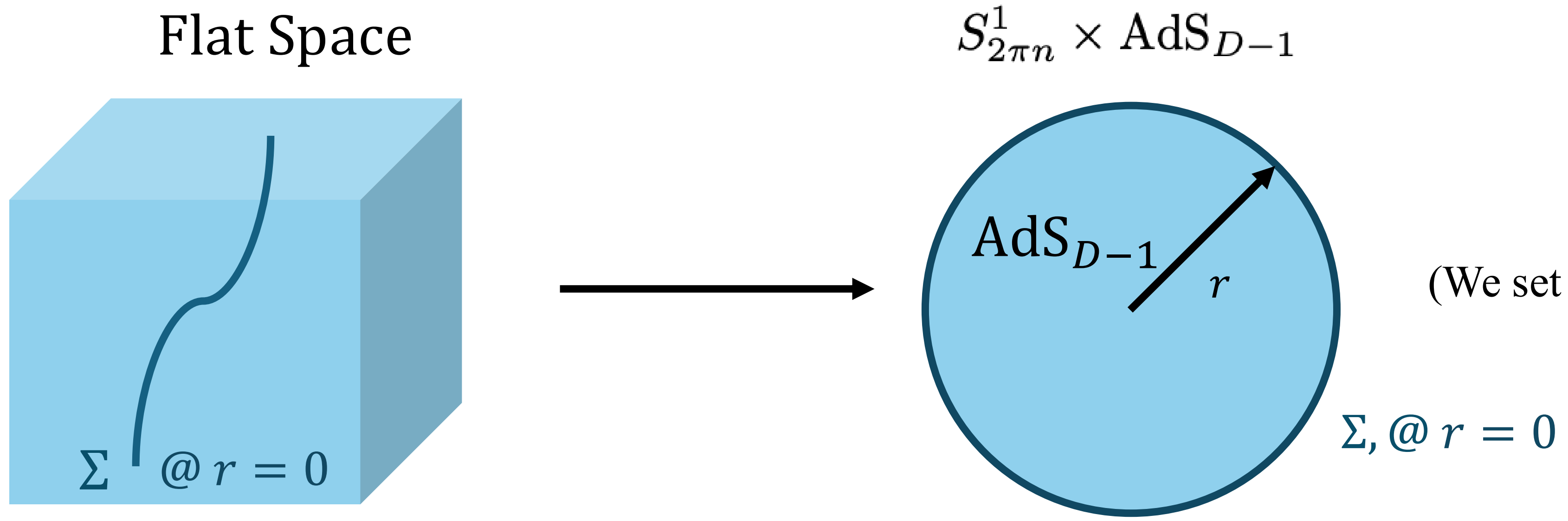

(We set $L_{AdS} = 1$)

$$\mathrm{d}s^2_{\text{flat}} = r^2\left(\frac{\mathrm{d}\boldsymbol{x}^2 + \mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right) = \Omega^{-2}\,\mathrm{d}s^2_{\text{hyp.}} \quad \text{w/}\; \mathrm{d}s^2_{\text{hyp.}} = L^2_{\text{AdS}}\left(\frac{\mathrm{d}\boldsymbol{x}^2 + \mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right) \text{ and } \Omega = \frac{L_{\text{AdS}}}{r}, \theta \sim \theta + 2\pi n$$

$$\phi_{\text{hyp.}}(\boldsymbol{x}, r, \theta) = \Omega^{-\Delta_\phi}\phi_{\text{flat}}(\boldsymbol{x}, r, \theta) = \left(\frac{r}{L_{\text{AdS}}}\right)^{\Delta_\phi}\phi_{\text{flat}}(\boldsymbol{x}, r, \theta).$$

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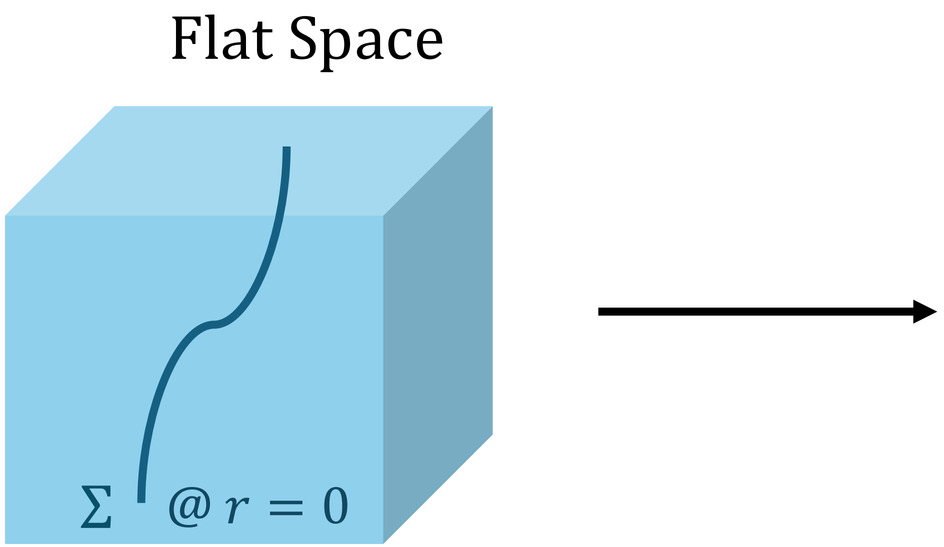

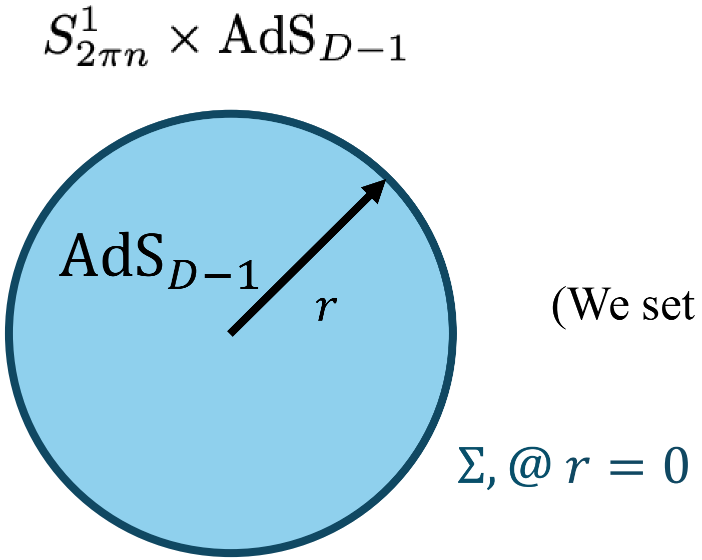

(We set $L_{AdS} = 1$)

- Used frequently: conical defects (Casini-Huerta-Myers 2011, Klebanov-Pufu-Sachdev-Safdi 2011…); monodromy (Giombi-Helfenberger-Ji-Khanchandani 2022…); pinning fields (Cuomo-Komargodski-Mezei 2022…) + many more.

Entangling Surface and Conical Defect BCs

- Asymptotic boundary conditions in AdS correspond to choice of boundary conditions on Σ and different conical defect fixed points. For ***free scalar***

$$S^1_{2\pi n} \times \mathrm{AdS}_{D-1}$$

$$S = \frac{1}{2}\int \mathrm{d}^D x\,\sqrt{g}\left[(\partial\phi)^2 - \frac{(D-2)^2}{4}\phi^2\right]$$

$$\text{KK mode: } \phi(\boldsymbol{x}, r, \theta) = \sum_{\omega\in\mathbb{Z}} e^{i\omega\theta/n}\phi_\omega(\boldsymbol{x}, r)$$

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$$\text{EOM: } \phi_\omega(\boldsymbol{x}, r\to 0) \sim r^{\hat{\Delta}_{\pm\omega}}\hat{\phi}_\omega(\boldsymbol{x}) + \dots$$

$$\text{Std. Quantization: } \hat{\Delta}_{+,\omega} = \frac{D-2}{2} + \frac{|\omega|}{n},$$

$$\text{Alt. Quantization: } \hat{\Delta}_{-,\omega} = \frac{D-2}{2} - \frac{|\omega|}{n}$$

- In free theory, KK zero mode only has one BC as $\widehat{\Delta}_{+,\omega=0} = \widehat{\Delta}_{-,\omega=0}$.

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Flat Space

$$S = \frac{1}{2}\int \mathrm{d}^D x\,(\partial\phi)^2$$

$$\text{KK mode: } \phi(\boldsymbol{x}, r, \theta) = \sum_{\omega\in\mathbb{Z}} e^{i\omega\theta/n}\phi_\omega(\boldsymbol{x}, r)$$

$$\text{EOM: } \phi_\omega(\boldsymbol{x}, r\to 0) \sim r^{\hat\Delta_{\pm\omega}}\hat\phi_\omega(\boldsymbol{x}) + \dots$$

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$$\text{Alt. Quantization: } \hat\Delta_{-,\omega} = \frac{D-2}{2} - \frac{|\omega|}{n}$$

$$\text{bulk-to-defect OPE: } \phi_\omega(\boldsymbol{x}, r\to 0) \sim r^{\hat\Delta_{\pm,\omega}-\Delta_\phi}\hat\phi_\omega(\boldsymbol{x}) + \dots$$

same $\hat\Delta_{\pm,\omega}$ but attained from requiring that

$$\langle \partial^2\phi_\omega(\boldsymbol{x}, r)\hat\phi_\omega(\boldsymbol{x}')\rangle = 0$$

- KK mode with lowest dimension dominates, will be the zero mode: $\widehat{\Delta}_\phi = \widehat{\Delta}_{\pm,\omega=0}$.

O(N) model in $S^1_{2\pi n} \times \mathrm{AdS}_{D-1}$ and Large-N

$$S = \frac{1}{2}\int \mathrm{d}^D x\,\sqrt{g}\left[(\partial\phi)^2 - \frac{(D-2)^2}{4}\phi^2 + \frac{\lambda}{4!}(\phi^2)^2\right],$$
$$\sim \frac{1}{2}\int \mathrm{d}^D x\,\sqrt{g}\left[(\partial\phi)^2 - \frac{(D-2)^2}{4}\phi^2 + \sigma\phi^2\right]$$

- At large-N, saddle-point approximation after integrating out ϕ tells us HS field is evaluated at constant value $\langle\sigma\rangle = \sigma^\star$ and there are new boundary operator scaling dimensions due to new AdS "mass".

$$\hat{\Delta}_{\pm,\omega} = \frac{D-2}{2} \pm \sqrt{\frac{\omega^2}{n^2} + \sigma^\star}$$

- $\langle\sigma\rangle = \sigma^\star$ has to solve saddle-point/gap equation from minimizing free energy.

$$\left.\frac{\partial\mathcal{F}}{\partial\sigma}\right|_{\sigma=\sigma^\star} = 0 \Rightarrow \sum_{\omega\in\mathbb{Z}} \frac{\Gamma(\hat{\Delta}_{\pm,\omega})}{\Gamma(3-D+\hat{\Delta}_{\pm,\omega})} = 0$$

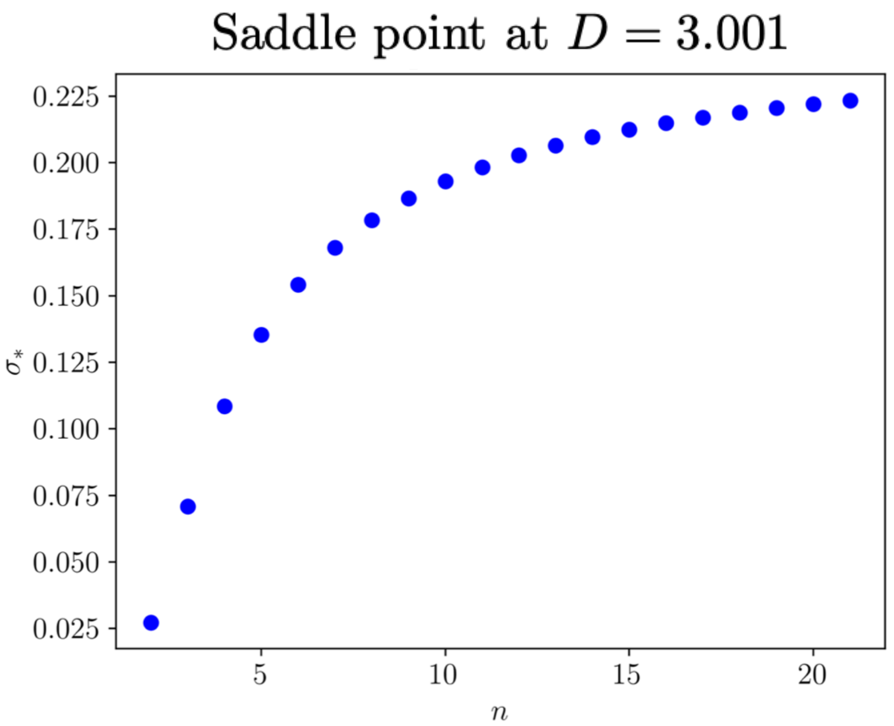
O(N) model in $S^1_{2\pi n} \times \text{AdS}_{D-1}$ and Large-N

- Gap equation has solutions for KK zero mode in *alternate quantization*, with the rest in *standard quantization* in $D \to 3$ [consistent with result (Metlitski-Fuentes-Sachdev, 2009)].

$$\hat{\Delta}_\phi = \hat{\Delta}_{-,\omega=0} = \frac{1}{2} - \sqrt{\sigma^\star} \text{ and } \hat{\Delta}_{+,\omega} = \frac{1}{2} + \sqrt{\frac{\omega^2}{n^2} + \sigma^\star}$$

- For example, for $n = 2$ one finds $\sigma^\star \approx 0.027$ and $\widehat{\Delta}_\phi = \widehat{\Delta}_{-,\omega=0} \approx 0.34$ for D close to $\widehat{\Delta}_\phi < \frac{1}{2}$ for $D = 3$ and decreases with n.

- Close to $D = 4$, there is an ordering transition where $\widehat{\Delta}_\phi = \widehat{\Delta}_{-,\omega=0}$ looks parametrically small.



Comparison with $D = 4 - \epsilon$ expansion results

- Calculation can be done in flat space and in $S^1_{2\pi n} \times \mathrm{AdS}_{D-1}$. With $\phi(\boldsymbol{x}, r, \theta) = \phi(\boldsymbol{x}, r, \theta + 2\pi n)$ and conical defect at $r = 0$.

$$\text{Flat Space:} \quad S = \int \mathrm{d}^D x \, \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}(\phi^2)^2 + \int_{\text{defect}} \mathrm{d}^{D-2}x \, \frac{c}{2}\hat{\phi}^2_{\omega=0}$$

$$S^1_{2\pi n} \times \mathrm{AdS}_{D-1}: \quad S = \int \mathrm{d}^D x \, \sqrt{g} \left[\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}(\phi^2)^2\right] + \int_\partial \mathrm{d}^{D-2}x \, \frac{c}{2}\hat{\phi}^2_{\omega=0}$$

- Zero mode defect/boundary mass $\hat{\phi}^2_{\omega=0}$ is classically marginal with $2\widehat{\Delta}_{\pm,\omega=0} = D - 2$, KK zero mode is renormalized.

- Anomalous dimension at 1-loop is $\hat{\gamma}_{\omega=0} = \frac{c^\star}{2\pi n}$ and β-function for c attained by requiring bulk-to-bulk zero mode propagator is finite (Metlitski-Fuentes-Sachdev, 2009).

Comparison with $D = 4 - \epsilon$ expansion results

$$\beta_c(c) = \frac{(N+2)\lambda}{144\pi}\left(n - \frac{1}{n}\right) + \frac{c^2}{2\pi n} + \frac{(N+2)\lambda c}{48\pi^2 n} + \frac{(N+2)\lambda c}{3} C'_{\phi^2}(0)$$

- $C_{\phi^2}(\epsilon)$ is normalization of bulk 1-pt function $\langle \phi^2 \rangle$ in $D = 4 - \epsilon$.

- β-function consistent in $n \to 1$ limit with (Metlitski-Fuentes-Sachdev, 2009) and solutions are qualitatively similar with: *two (alternate) fixed-points with $c^\star < 0$ and one runaway (ordered) fixed-point.*

- *Qualitatively similar results with large-N and lattice model.*

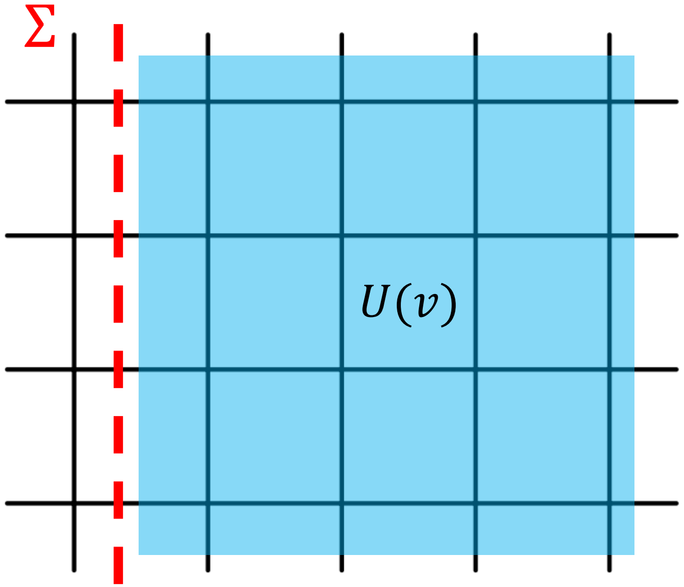
Rényi-Monodromy defects and Disorder Ops.

- KK zero mode can be regularized with $U(1)$ twist/monodromy along $S^1_{2\pi n}$ or around conical defect. For N even, or with complex scalars $\phi_i(\boldsymbol{x}, r, \theta + 2\pi n) = e^{2\pi i v}\phi_i(\boldsymbol{x}, r, \theta)$,

$$\omega \in \mathbb{Z} + v: \ \hat{\Delta}_{\pm,\omega} = \frac{D-2}{2} \pm \frac{|\omega|}{n} \Longrightarrow \hat{\Delta}_{\pm,\omega=v} = \frac{D-2}{2} \pm \frac{v}{n}$$

- The mass of lowest mode $\hat{\phi}^2_{\omega=v}$ is no longer classically marginal, $2\hat{\Delta}_{\pm,\omega=v} = \mathrm{D} - 2 \pm \frac{2v}{n}$.

Monodromy can be thought of coming from acting on state with $U(1)$ disorder (patch sym.) operator on the lattice.



$$U(v) = e^{2\pi i v \sum_{i \in \mathrm{patch}} \hat{n}_i}$$

Rényi-Monodromy defects and Disorder Ops.

$$\omega \in \mathbb{Z} + v: \;\; \hat{\Delta}_{\pm,\omega} = \frac{D-2}{2} \pm \frac{|\omega|}{n} \Longrightarrow \hat{\Delta}_{\pm,\omega=v} = \frac{D-2}{2} \pm \frac{v}{n}$$

- Still non-trivial correction to defect/boundary scaling dimension from bulk interaction depending on whether $\omega = v$ mode is in *standard* or *alternate*.

$$\hat{\gamma}_{\omega=v}(\epsilon) = \pm \frac{N+2}{N+8}\left[\frac{1}{12v}\left(\frac{1}{n} - n\right) + \frac{v \mp 1}{2n}\right]\epsilon$$

- Matches well with large-N results near $D = 4$.
- For $n = 2$, near $D = 3$ $(\epsilon \to 1)$ for some range of monodromy parameter v standard BC not allowed which is qualitatively similar to large-N results (when $v < v_c \approx 0.483$).

Rényi-Monodromy defects and Disorder Ops.

- When N is odd and not all fields can see $U(1)$ monodromy. Pair up $N-1$ fields into complex ψ_i and turn on $\psi_i(\boldsymbol{x}, r, \theta + 2\pi n) = e^{2\pi i v}\psi_i(\boldsymbol{x}, r, \theta)$. In flat space

$$S = \int \mathrm{d}^D x \, \frac{1}{2}|\partial\boldsymbol{\psi}|^2 + \frac{1}{2}(\partial\phi_N)^2 + \frac{\lambda}{4!}|\boldsymbol{\psi}^2|^2 + \frac{\lambda}{4!}\phi_N^4 + \frac{\lambda}{12}|\boldsymbol{\psi}|^2\phi_N^2 + \int_{\text{defect}} \mathrm{d}^{D-2}x \, \frac{c}{2}\hat{\phi}^2_{N,\omega=0}.$$

- Defect/boundary KK zero mode for $\hat{\phi}_{N,\omega=0}$ will still be renormalized, and fixed-point $c^\star$ will depend on monodromy parameter.

$$\beta_c(c) = \frac{\lambda(N+2)}{3(48\pi)}\left(n - \frac{1}{n}\right) + \frac{c^2}{2\pi n} + \frac{\lambda c}{16\pi^2 n} + \lambda c C'_{\phi^2}(0) - \frac{(N-1)\lambda v(v-1)}{24\pi n} + \frac{c\lambda(N-1)C'_{\psi^2}(0)}{6}$$

- Again fixed-points seem qualitatively similar to previous problem without monodromy, even when $n = 1$. Need to check if consistent with Large-N.

Still many things to do and in-progress…

Thank you!