From QED$_3$ to the multicritical point of the Fradkin-Shenker model

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2602.23420 by T. Dumitrescu, PN, R. Thorngren

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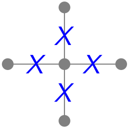
Toric code

Toric code Hamiltonian on a 2d square lattice of spin-1/2 dofs [Kitaev '97]

$$H_{\mathrm{TC}} = -\sum_v A_v - \sum_p B_p$$

$$A_v = \prod_{\partial\ell \ni v} X_\ell =$$



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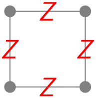

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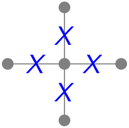

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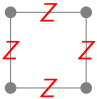

- Deconfined phase described by a (2+1)-d $\mathbb{Z}_2$ TQFT
- Gapped topological excitations = $\{1, e, m, f\}$
  $e \leftrightarrow A_v$ flips $\leftrightarrow \mathbb{Z}_2$ gauge charge (electric)
  $m \leftrightarrow B_p$ flips $\leftrightarrow \mathbb{Z}_2$ π flux (magnetic)
  $f = e \times m \leftrightarrow \theta = -1$ (fermion)
- $\mathbb{Z}_2^{\mathrm{D}}$ self-duality symmetry: $X_\ell \leftrightarrow Z_\ell$ and $v \leftrightarrow p \;\Rightarrow\; e \leftrightarrow m$

Fradkin-Shenker model

3d Euclidean $\mathbb{Z}_2$ gauge-Higgs lattice model by [Fradkin and Shenker '79]
$\equiv$ (2+1)-d Hamiltonian lattice model [Tupitsyn, Kitaev, Prokof'ev, Stamp '08]

$$H_{\mathrm{FS}} = H_{\mathrm{TC}} - h_e \sum_{\ell} Z_\ell - h_m \sum_{\ell} X_\ell$$

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- If $h_m = 0$: there is a critical point h_e^* in Ising* universality class, separating toric code phase and trivially gapped vacuum $\Rightarrow$
  Higgsing transition for Ising-like "order parameter" φ_e
  $\Rightarrow$ Condensation of e particle: Higgs phase
- Same if $h_e = 0$ by self-duality $h_e \leftrightarrow h_m$: a critical point h_m^*, where a dual magnetic Higgsing transition takes place
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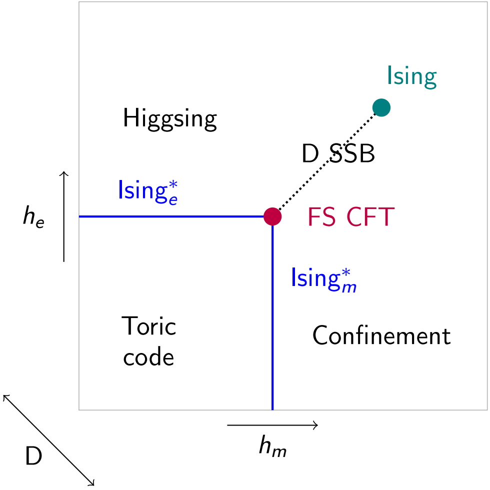
- Higgsing and confinement are continuously connected

Fradkin-Shenker model

[Vidal, Dusuel, Schmidt '08; Wu, Deng, Prokof'ev '12; Somoza, Serna, Nahum '20]



Multicritical self-dual CFT w/ two relevant deformations:

- D-even direction $h_m + h_e$ $\Delta_{\text{even}} \sim 1.51$
- D-odd direction $h_m - h_e$ $\Delta_{\text{odd}} \sim 1.22$

Self-dual line of first-order transitions with SSB of D, ending in standard Ising CFT

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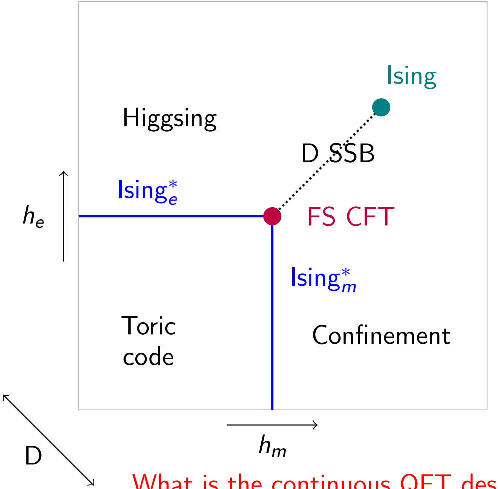

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What is the continuous QFT description of the FS CFT?

Mutually non-local e and m particles become simultaneously massless!

A $U(1)_e \times U(1)_m$ generalization of the FS model

$$H_{\mathrm{SFS}} = H_{\mathrm{TC}} - h_e \sum_{\ell} \widetilde{Z}_{\ell}^{+} - h_m \sum_{\ell} \widetilde{X}_{\ell}^{+}$$

with $(O(2)_e \times O(2)_m) \rtimes (\mathbb{Z}_2^{\mathrm{D}} \times \mathbb{Z}_2^{\mathrm{T}})$ global symmetry (if $h_e = h_m$)

- $U(1)_{e,m}$ charges: $\mathrm{Q}_e = \frac{1}{4}\sum_v (-1)^v A_v$ and $\mathrm{Q}_m = \frac{1}{4}\sum_p (-1)^p B_p$ under which e, m carry respective charge $1/2$

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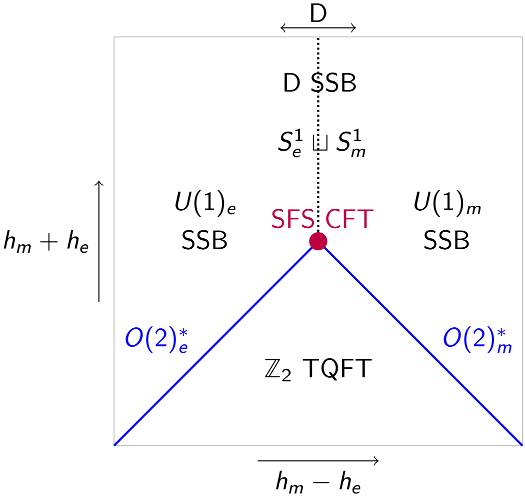
- With only one h-coupling, it has an $O(2)^*_{e,m}$ second-order transition, separating the toric code from an S^1 phase due to $U(1)_{e,m}$ SSB
- It admits a $U(1)_e \times U(1)_m$-breaking deformation ΔH to FS model
$$H_{\mathrm{FS}} = H_{\mathrm{SFS}} + \Delta H$$

A $U(1)_e \times U(1)_m$ generalization of the FS model

We conjecture the SFS phase diagram as the $U(1)_e \times U(1)_m$ uplift of the FS phase diagram (no trivially gapped phase: first-order line cannot end)



Higgs-Yukawa-QED in (2+1)-d

QED_3 w/ $N_f = 2$ charge-1 Dirac $\psi^{i=1,2}$ + charge-2 Higgs ϕ + Yukawa

$$\mathscr{L}_{\mathrm{HYQED}} = -\frac{1}{4e^2} f^{\mu\nu} f_{\mu\nu} - i\bar{\psi}_i \not{D}_a \psi^i - |D_{2a}\phi|^2 - \lambda|\phi|^4 + y\left(\phi^\dagger \psi^1\psi^2 + \text{ h.c.}\right)$$

$$\frac{U(1)_f \times U(1)_{\mathcal{M}}}{\mathbb{Z}_2} = U(1)_e \times U(1)_m \quad \text{with } q_{e,m} = \frac{1}{2}(q_{\mathcal{M}} \pm q_f)$$

Monopole operators : $\mathcal{M}^{i=1} \quad (q_e = 1,\ q_m = 0) \quad \& \quad \mathcal{M}^{i=2} \quad (q_e = 0,\ q_m = 1)$

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$$S(O(2)_e \times O(2)_m) \rtimes (\mathbb{Z}_2^{\mathrm{D}} \times \mathbb{Z}_2^{\mathrm{T}})$$

Mixed anomaly with T: $S_{4\mathrm{d}} = \pi \int \frac{dA_e}{2\pi} \wedge \frac{dA_m}{2\pi}$

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Mass deformations: $\mathscr{L}_{\text{masses}} = \underbrace{-m_\phi^2|\phi|^2}_{\text{D-even}} + \underbrace{im_3(\bar{\psi}_1\psi^1 - \bar{\psi}_2\psi^2)}_{\text{D-odd}}$

Phase diagram of Higgs-Yukawa-QED

Assume $N_f = 2$ QED_3 flows to S^3 sigma-model due to $\langle \mathcal{M}^i \rangle \neq 0$ and $U(2)_{\mathrm{QED}} \xrightarrow{\mathrm{SSB}} U(1)$ [Chester, Komargodski '24; Dumitrescu, PN, Thorngren '24]

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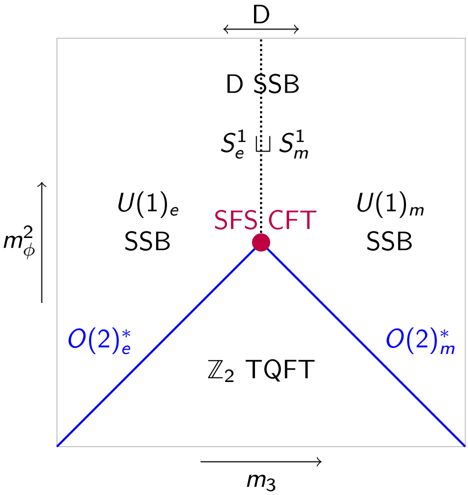

$m_\phi^2 > 0$:
deformed $N_f = 2$ QED$_3$
S^3 lifted to $S^1_e \sqcup S^1_m \Rightarrow \mathbb{Z}_2^{\mathrm{D}}$ SSB

$m_\phi^2 < 0$:
$\langle \phi \rangle \neq 0$ and Higgsing to $\mathbb{Z}_2$
gapped $\mathbb{Z}_2$ TQFT $=$ toric code

$|m_3| \neq 0$:
flow to charge-2 AH model
with $U(1)_{e,m}$ magnetic symmetry
$\xleftrightarrow{\mathrm{P/V}} O(2)^*_{e,m}$ model for $\varphi_{e,m}$
w/ $q_{e,m} = 1/2$ ($\mathcal{M}^{i=1,2} \leftrightarrow \varphi^2_{e,m}$)

From the SFS CFT to the FS CFT

Consistency of the proposal requires emergent $\mathbb{Z}_2^{\text{mirror}}$: $U(1)_{\mathcal{M}} \leftrightarrow U(1)_f$ at the multicritical point, implying enhancement to $O(2)_e \times O(2)_m$

Large-N_f expansion (for monopoles in [Dupuis, Boyack, Witczak-Krempa '21])

$$\Delta_{|\phi|^2} \sim 1.46 \qquad \Delta_{\bar{\psi}\sigma_3\psi} \sim 0.65 \qquad \Delta_{\mathcal{M}_{(1)}} \sim 0.63$$

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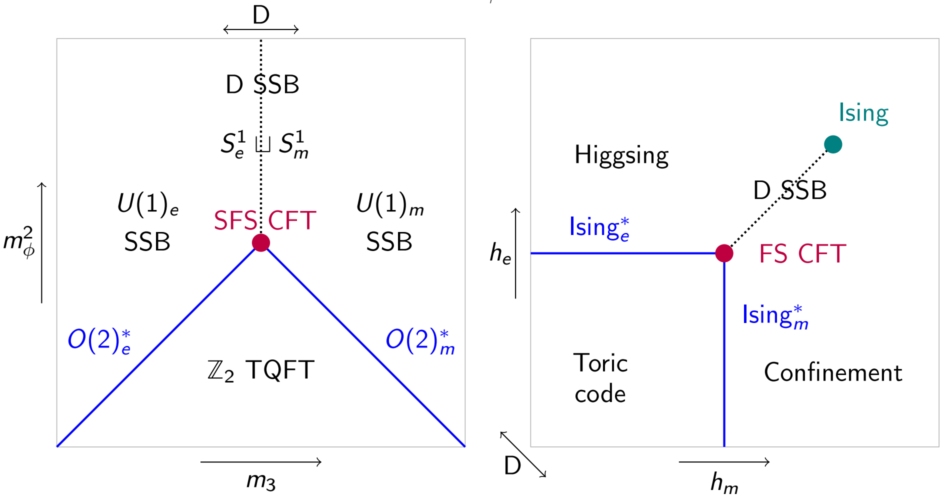
Unique, $\mathbb{Z}_2^{\mathsf{C}} \times \mathbb{Z}_2^{\mathsf{T}}$-invariant unit-charge (relevant) monopole deformation maps to the lattice deformation ΔH from SFS to FS:

- $\mathbb{Z}_2$ TQFT is unaffected
- $O(2)^*_{e,m}$ transition $\Rightarrow$ Ising$^*_{e,m}$ transition
- Each $S^1_{e,m}$ $\Rightarrow$ single trivial vacuum
- Competition monopoles vs fermion quartic on S^3 for $m_\phi^2 > 0$: Two (one) vacua for small (large) m_ϕ^2, separated by ordinary Ising

Phase diagrams

Identifying $m_3 \sim h_m - h_e$ and $m_\phi^2 \sim h_m + h_e$, the flow leads to



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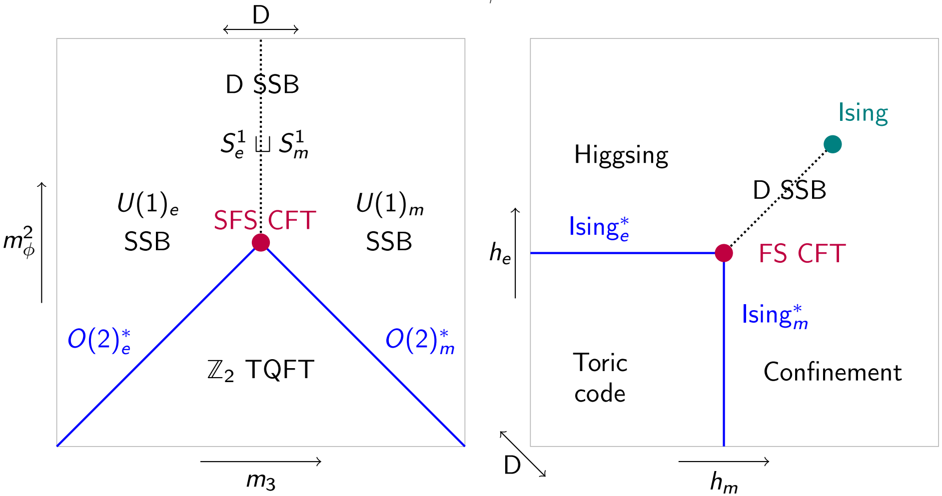

Δ's of the SFS CFT from numerics? Does $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$ act in the FS CFT?

Easy-Plane $\mathbb{CP}^1$ model in (2+1)-d

QED$_3$ w/ $N_f = 2$ charge-1 scalars $z^{i=1,2}$ + $V_{\rm EP}$ with $-2 < \lambda_{\rm EP}/\lambda < 2$

$$\mathscr{L}_{\rm EP\mathbb{CP}^1} = -\frac{1}{4e^2} f^{\mu\nu} f_{\mu\nu} - |D_b z_i|^2 - \lambda\left(|z_1|^4 + |z_2|^4\right) - \lambda_{\rm EP}|z_1|^2|z_2|^2 - m_z^2|z_i|^2$$

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$(O(2)_e \times O(2)_m) \rtimes \mathbb{Z}_2^{\mathsf{T}}$ with emergent $\mathbb{Z}_2^{\mathsf{D}} : U(1)_e \leftrightarrow U(1)_m$ by P/V

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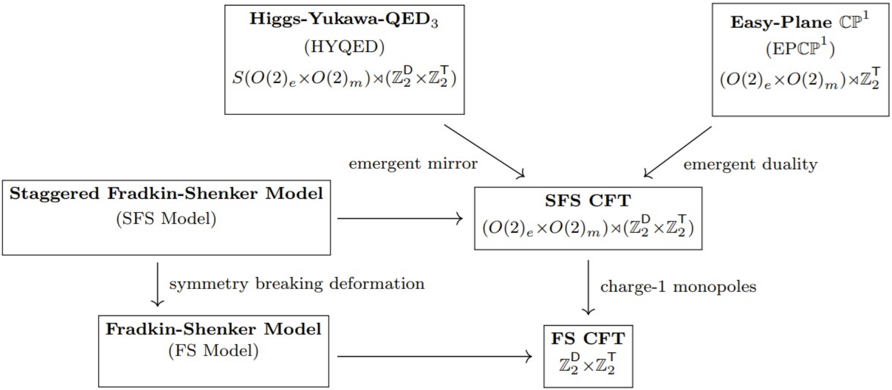
- D-odd m_z^2: S^1_m dual photon $(m_z^2 > 0)$ & S^1_e NGB $z_1 z_2^\dagger$ $(m_z^2 < 0)$
- D-even $\lambda_{\rm EP}$: must *also* be tuned to reach the critical point
  $\lambda_{\rm EP} < 0$: attractive $z_1 z_2$ interaction $\Rightarrow \langle z_1 z_2 \rangle \neq 0$ $\Rightarrow \mathbb{Z}_2$ TQFT
  $\lambda_{\rm EP} > 0$: generically a 1st-order transition [Desai, Kaul '19]

Outlook



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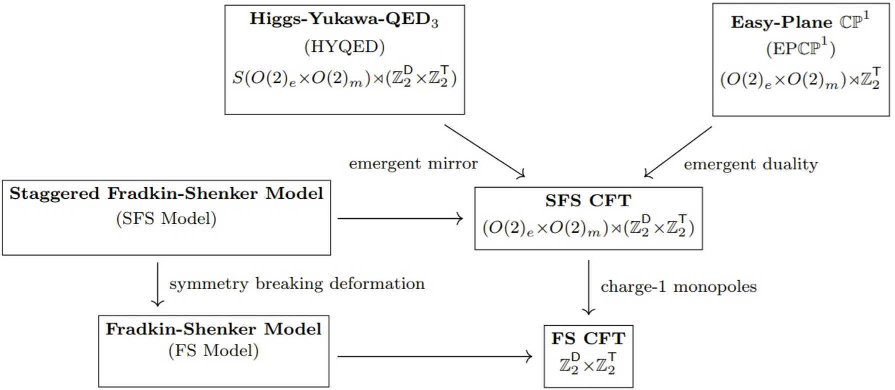

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THANK YOU FOR YOUR ATTENTION !