Superinflation in Loop Quantum Cosmology

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- **Slow-roll from LQC $k = 0$ and $k = 1$**
- **Superinflation in LQC**
- **Scalar and tensor power spectrum**

Lidsey, Mulryne, Nunes, Tavakol (2004)
Mulryne, Nunes, Tavakol, Lidsey (2004)
Mulryne, Nunes (2006)
Copeland, Mulryne, Nunes, Shaeri (2007) and (2008)

1. Loop Quantum Gravity

Strongest candidate to a quantum theory of gravity that is non-perturbative and background independent.
Based on Ashtekar's variables which bring GR into the form of a gauge theory.

- Densitized triad E^a_i and $E^a_i E^b_i = q^{ab} q$
- SU(2) connection $A^i_a = \Gamma^i_a - \gamma K^i_a$

Γ^i_a - spin connection; K^i_a - extrinsic curvature; γ - Barbero-Immirzi parameter.

Quantization proceeds by using as basic variables holonomies,

$$h_e = \exp \int_e \tau_i A^i_a \dot{e}^a dt$$

along curves e, and fluxes,

$$F = \int_S \tau^i E^a_i n_a d^2 y$$

in spacial surfaces S. Flux operators have a discrete spectrum.

2. Loop Quantum Cosmology

Focuses on minisuperspace settings with finite degrees of freedom (= homogeneous and isotropic spacetimes).

1. Inverse triad corrections:

   Based on the modification of the *inverse* scale factor below a critical scale a_*.

2. Holonomy corrections:

   Loops on which holonomies are computed have a non-vanishing minimum area. Leads to a ρ^2 modification in the Friedmann equation.

These corrections lead to interesting applications:

- Resolution of the initial singularity;
- Increase of the viability of the onset of inflation;
- Avoidance of a big crunch and oscillatory universes;

3. Key features of loop quantization

$$A_a^i = c\,\omega_a^i\,, \qquad c = \gamma \dot{a}$$

$$E_i^a = p\,e_i^a\,, \qquad p = a^2\,, \qquad \{c, p\} = \frac{8\pi G}{3}\gamma$$

$$\mathcal{H} = \frac{1}{8\pi G}\,\epsilon_{ijk}\frac{E_j^a E_k^b}{\sqrt{\det E}}\,F_{ab}^i + \frac{\pi_\phi^2}{2\sqrt{\det E}} + \sqrt{\det E}\,V(\phi)$$

We want to write this Hamiltonian in terms of holonomies
$h^{(\lambda)} = \exp(\lambda\,c\,\tau_i)$

1. Write Hamiltonian in terms of positive powers of the connection.
This can be done in several different ways $\Rightarrow$ ambiguity parameter ℓ

2. Write the connection in terms of holonomies. Need to take the trace over representation j of su(2) $\Rightarrow$ ambiguity parameter j $\Rightarrow$ critical scale a_*;

3. Key features of loop quantization (cont.)

3. Curvature component obtained by considering holonomies around closed square loop. Area is shrunk to the minimum eigenvalue of the area operator $\Delta \approx \ell_{\rm pl}^2 \quad \Rightarrow \quad \lambda \to \bar{\mu}$ and $\bar{\mu}^2 a^2 = \Delta$ [$\Rightarrow$ holonomy corrections];

4. Quantization proceeds by promoting triads and holonomies to operators (à la LQG);

5. Find eigenvalues of inverse triad operators such as $E^{ai}E^{bi}/\sqrt{\det E}$ and $1/\sqrt{\det E}$;

6. Spectrum of eigenvalues can be approximated by *continuous* correction functions $S(a)$ and $D_{l,j}(a)$ [inverse triad corrections];

7. Finally, Hamiltonian looks like this:

$$\mathcal{H} = -\frac{3}{8\pi G} S\, a \,\frac{\sin^2(\bar{\mu}\, c)}{\gamma^2 \bar{\mu}^2} + D_{l,j}\, a^{-3}\, \frac{\pi_\phi^2}{2} + a^3\, V(\phi)$$

8. $\quad \dot{p} = \{p, \mathcal{H}\} \qquad \Rightarrow \qquad$ Friedmann equation

4. Inverse volume operator

Classically: $d(a) = a^{-3}$

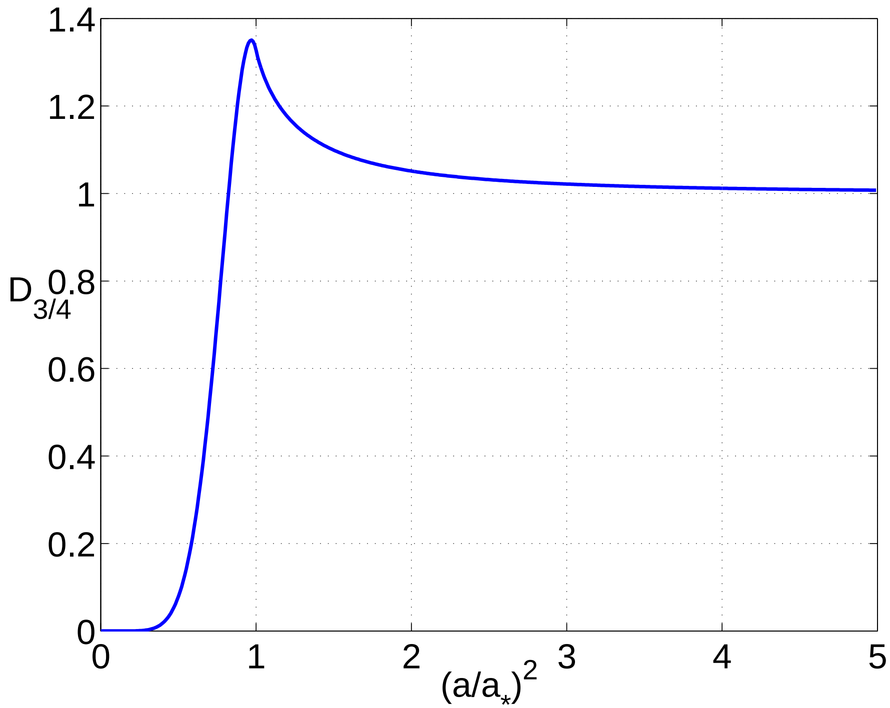
LQC: $d_{l,j}(a) = D_l(q)a^{-3}$ where $q = \left(\frac{a}{a_*}\right)^2$, $a_* \propto \sqrt{j}\,\ell_{\rm pl}$

semiclassical phase

for $a \ll a_*$, $D(q) \approx D_\star a^n$

classical phase

for $a \gg a_*$, $D(q) \approx 1$



5. Modified semi-classical equations

1. Modified Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{S}{3}\left(\frac{1}{2}\frac{\dot{\phi}^2}{D} + V(\phi)\right) - \frac{S^2}{a^2}$$

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2. Modified Klein-Gordon equation

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\left(1 - \frac{1}{3}\frac{d\ln D}{d\ln a}\right)\dot{\phi} + D\frac{dV}{d\phi} = 0$$

When $d\ln D/d\ln a > 3$: antifriction in expanding Universe and friction in contracting universe.

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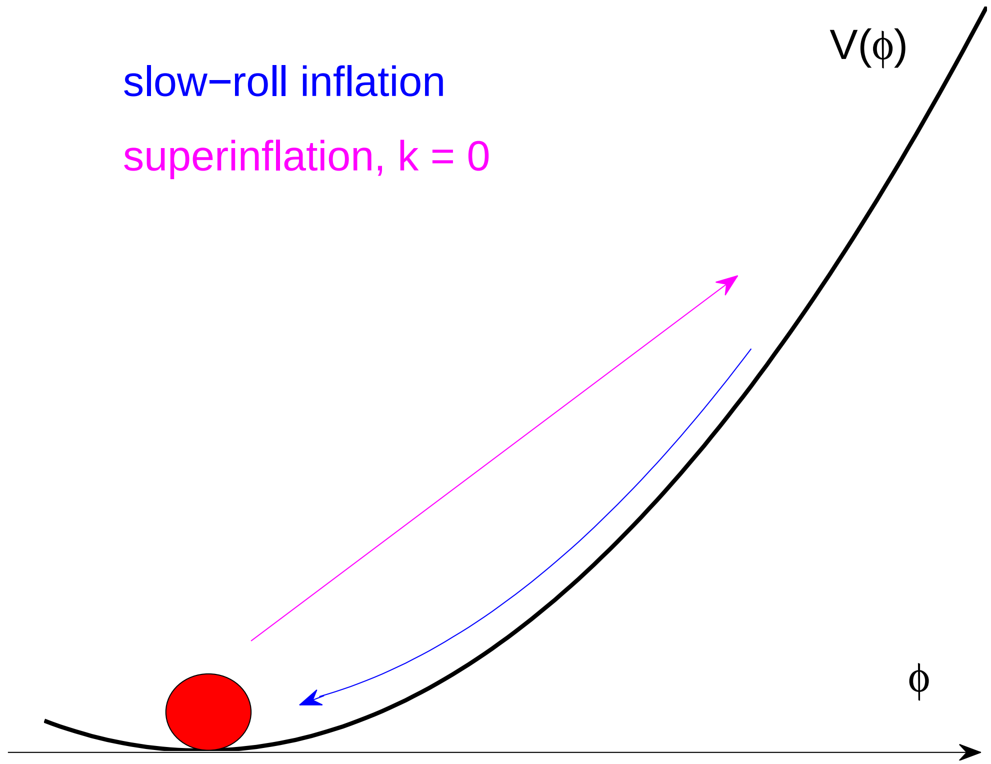
3. Variation of the Hubble rate

$$\dot{H} = -\frac{S\dot{\phi}^2}{2D}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a} - \frac{1}{6}\frac{d\ln S}{d\ln a}\right) + \frac{S}{6}\frac{d\ln S}{d\ln a}V + \left(1 - \frac{d\ln S}{d\ln a}\right)S^2\frac{1}{a^2}$$

Superinflation for $n + r = d\ln D/d\ln a + d\ln S/d\ln a > 6$.

6. Consequences for inflation ($k = 0$)



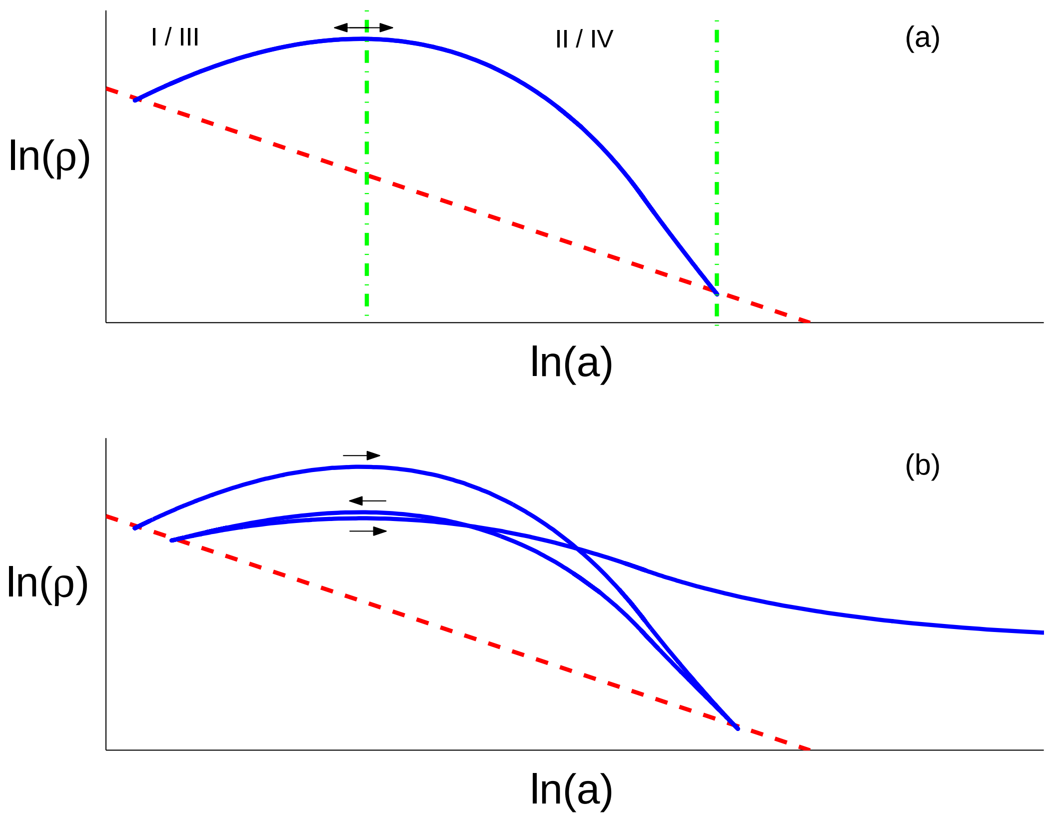
Tsujikawa and Singh (2003)

1. Super-inflation is brief;

2. $\phi_t \propto \dot{\phi}_{\rm init} q_{\rm init}^{-6} \exp(-q_{\rm init}^{15/4})$, independent of j;

3. $\phi_t < 2.4 \ell_{\rm pl}^{-1}$ if Hubble bound ($1/H > \ell_{\rm pl}$) is satisfied $\Rightarrow$ not enough slow-roll inflation!

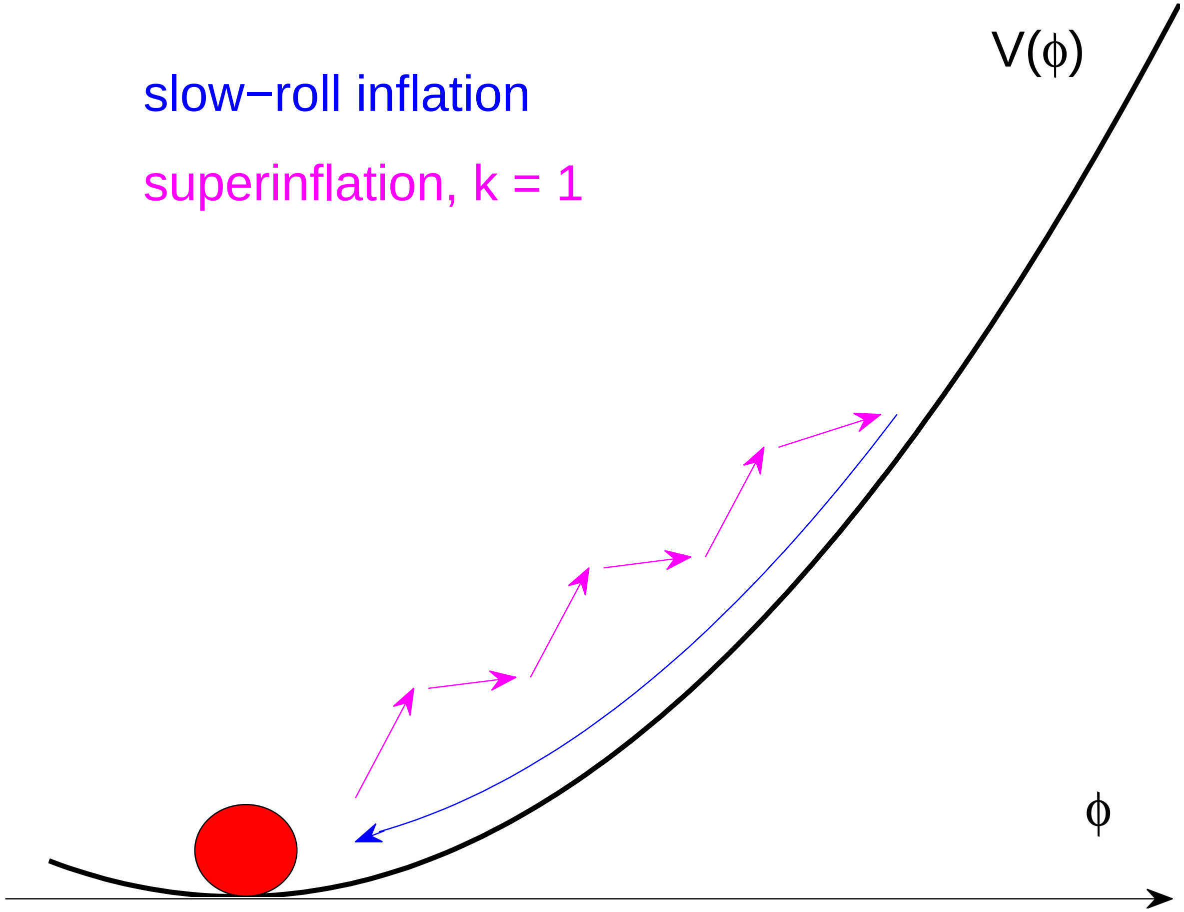
7. Bouncing Universe in $k = +1$



Field redshifts more rapidly than curvature term provided $\dot{\phi}^2 > V$ ($w > -1/3$).

As the field moves up the potential this condition becomes more difficult to satisfy and is eventually broken. Slow-roll inflation follows.

8. Bouncing Universe in $k = +1$, with self interacting potential



$$\phi_t^2 \propto \frac{1}{\dot{\phi}_{\rm init}} \frac{1}{q_{\rm init}^{3/2} \, a_*^3}$$

$\Rightarrow$ larger for lower j $\quad\Rightarrow\quad$ more e-folds.

9. The story so far...

1. Flat geometry

   - ϕ does not move high enough;
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   - ϕ does not move high enough;
   - ϕ_t independent of quantization parameter j.

2. Positively curved geometry

   - Allows oscillatory Universe;
   - For massless scalar field cycles are symmetric and consequently ever lasting;
   - Presence of a self interaction potential breaks symmetry and establishes initial conditions for inflation;
   - *Low* j results into *more* inflation.

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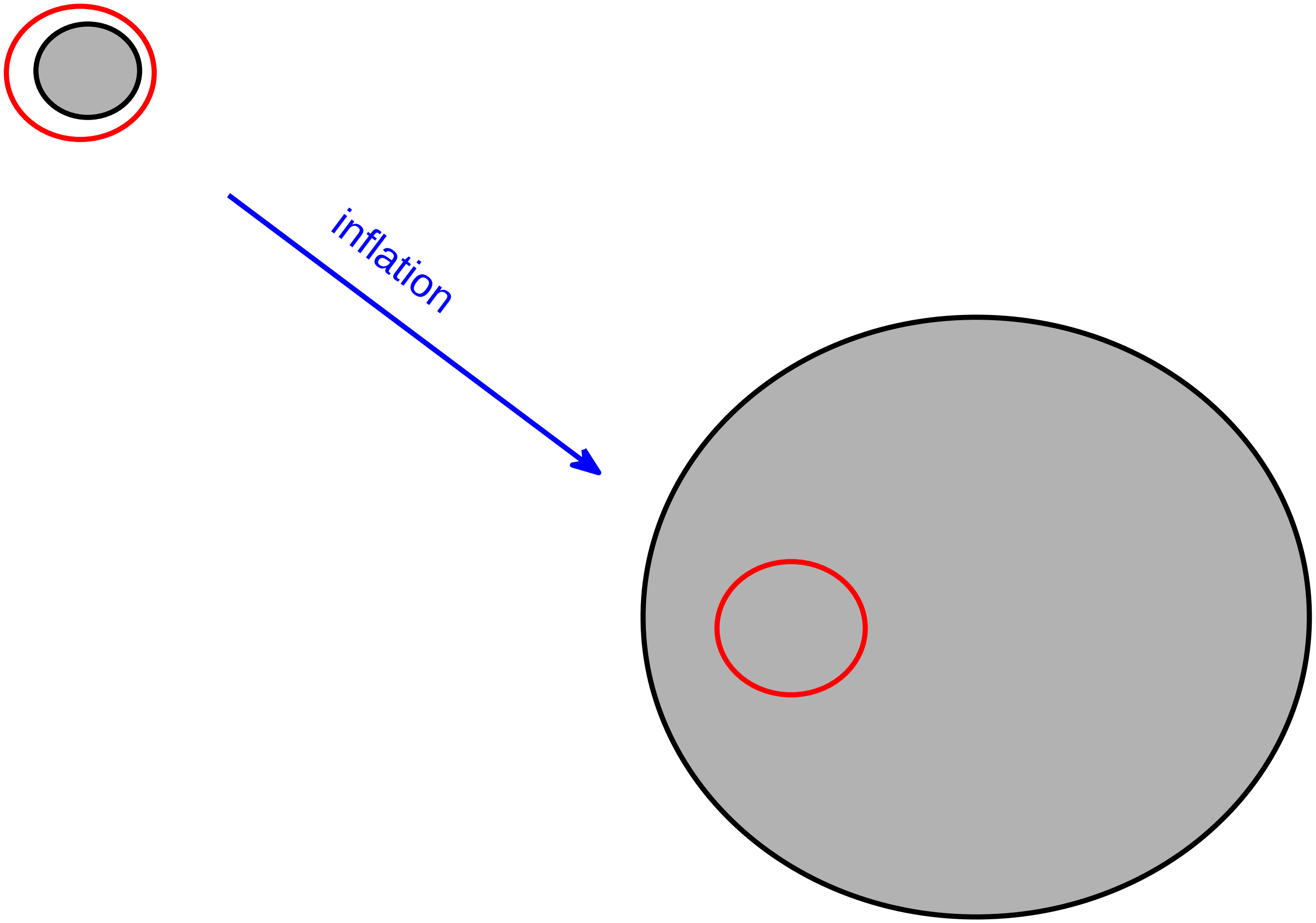
Can superinflation during the semi-classical phase replace standard slow-roll inflation?

- Does it solve the flatness and horizon problems?
- Does it give rise to a scale invariant spectrum of curvature perturbations?
- Is the spectrum of gravitational waves compactible with current bounds?

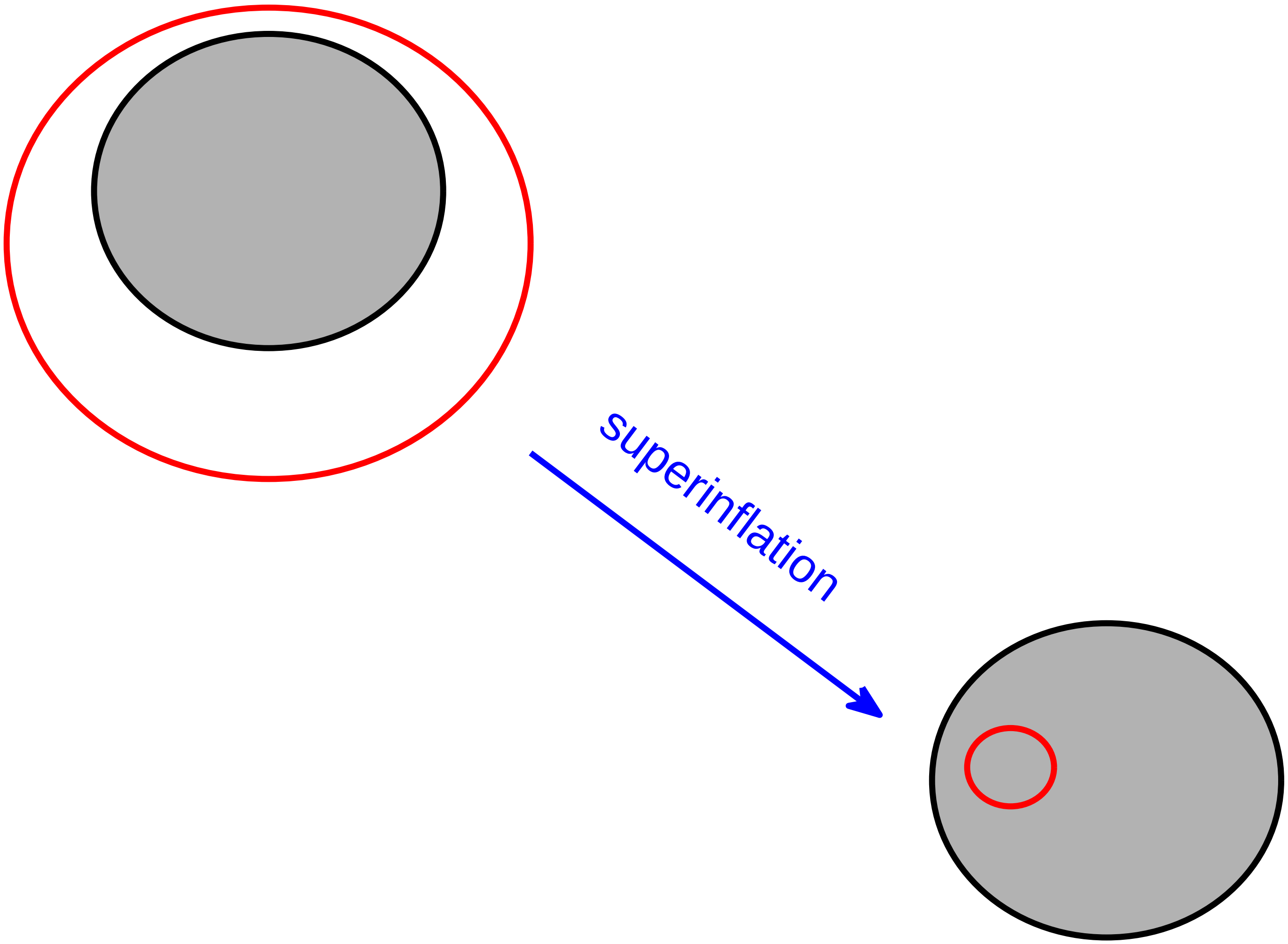
10. Inflation



e.g. Slow-roll inflation with scalar field(s).

Structure originates from quantum fluctuations of the field(s).

11. Superinflation



e.g. Ekpyrotic/cyclic universe, phantom field.

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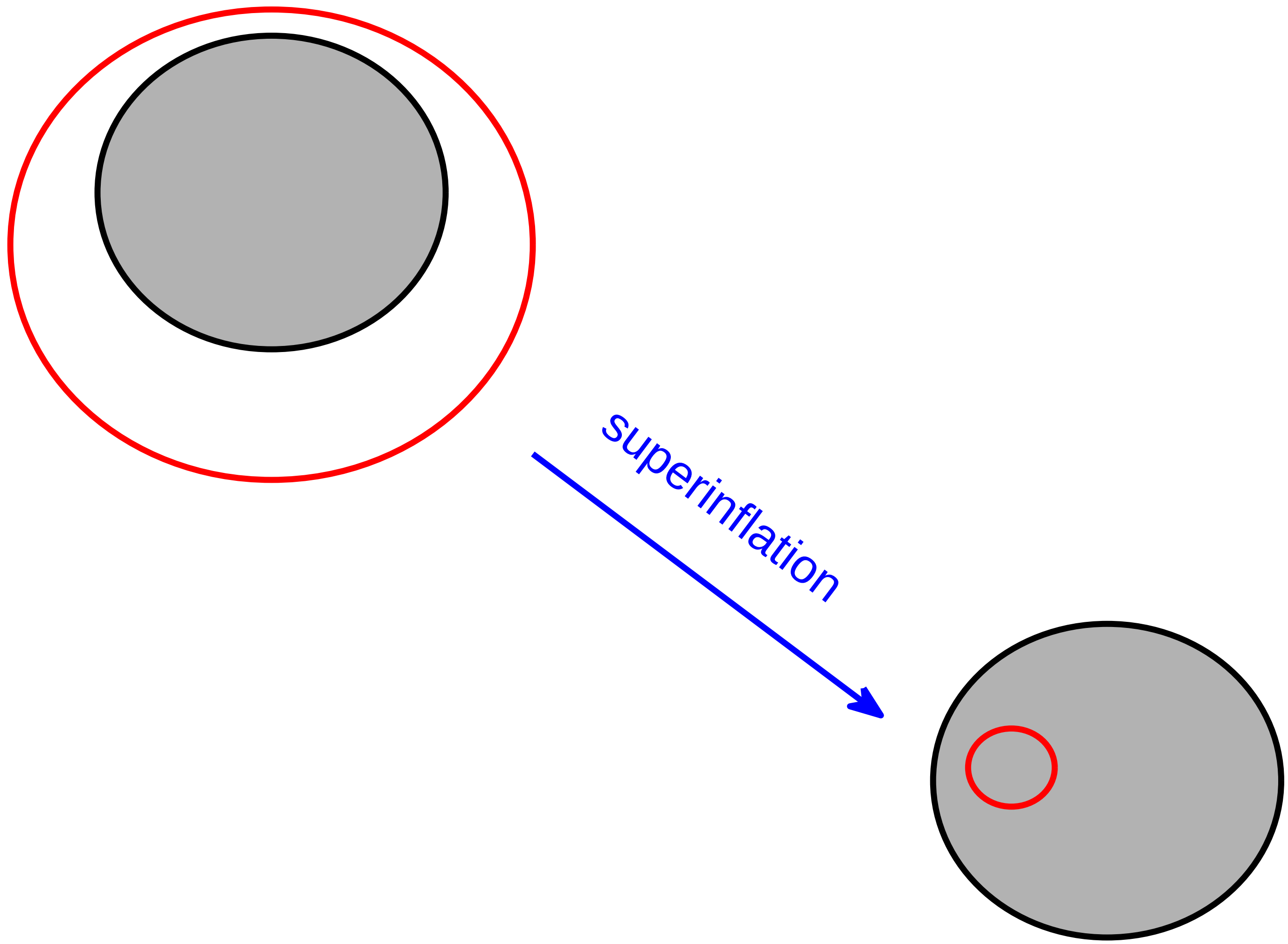
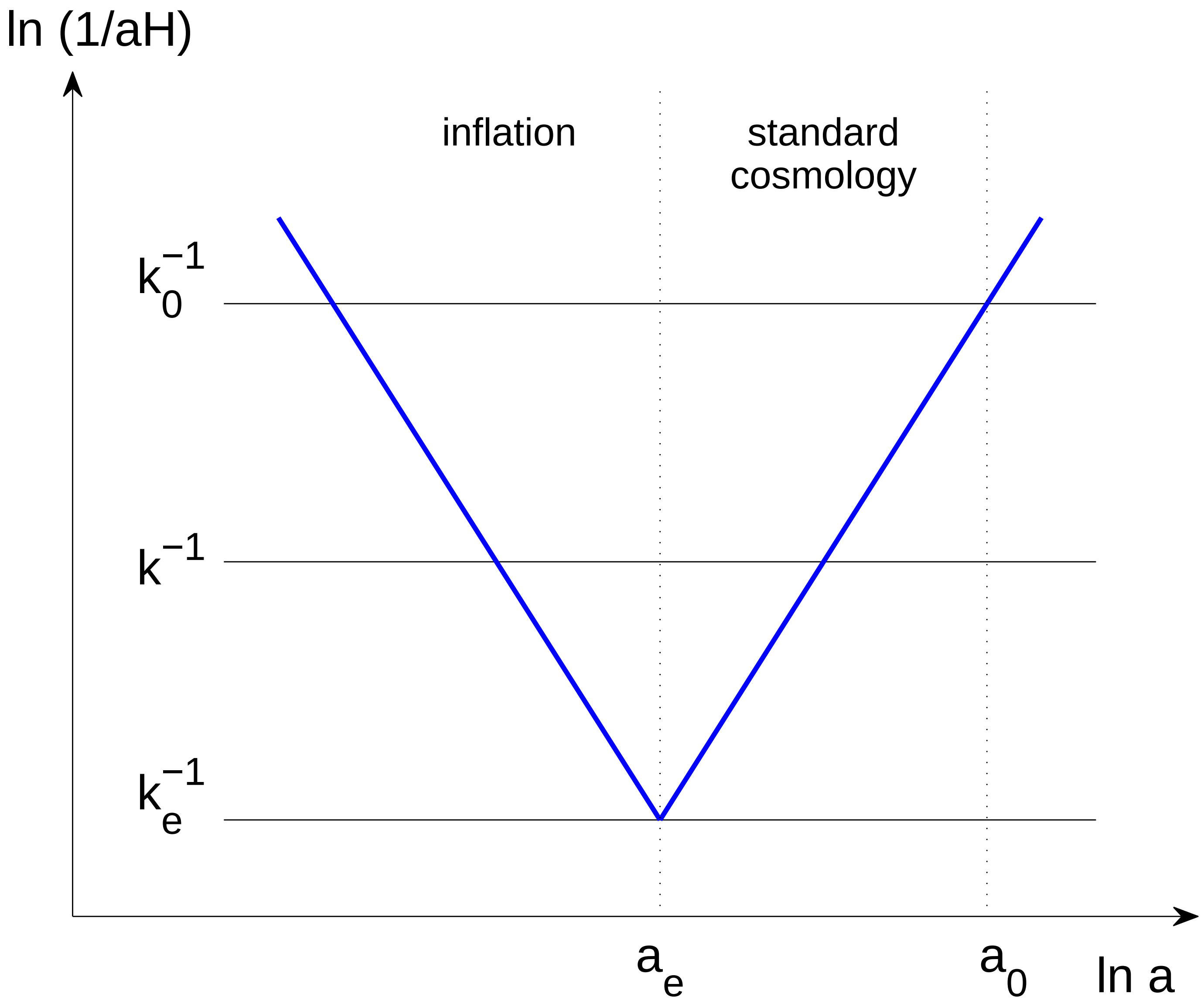
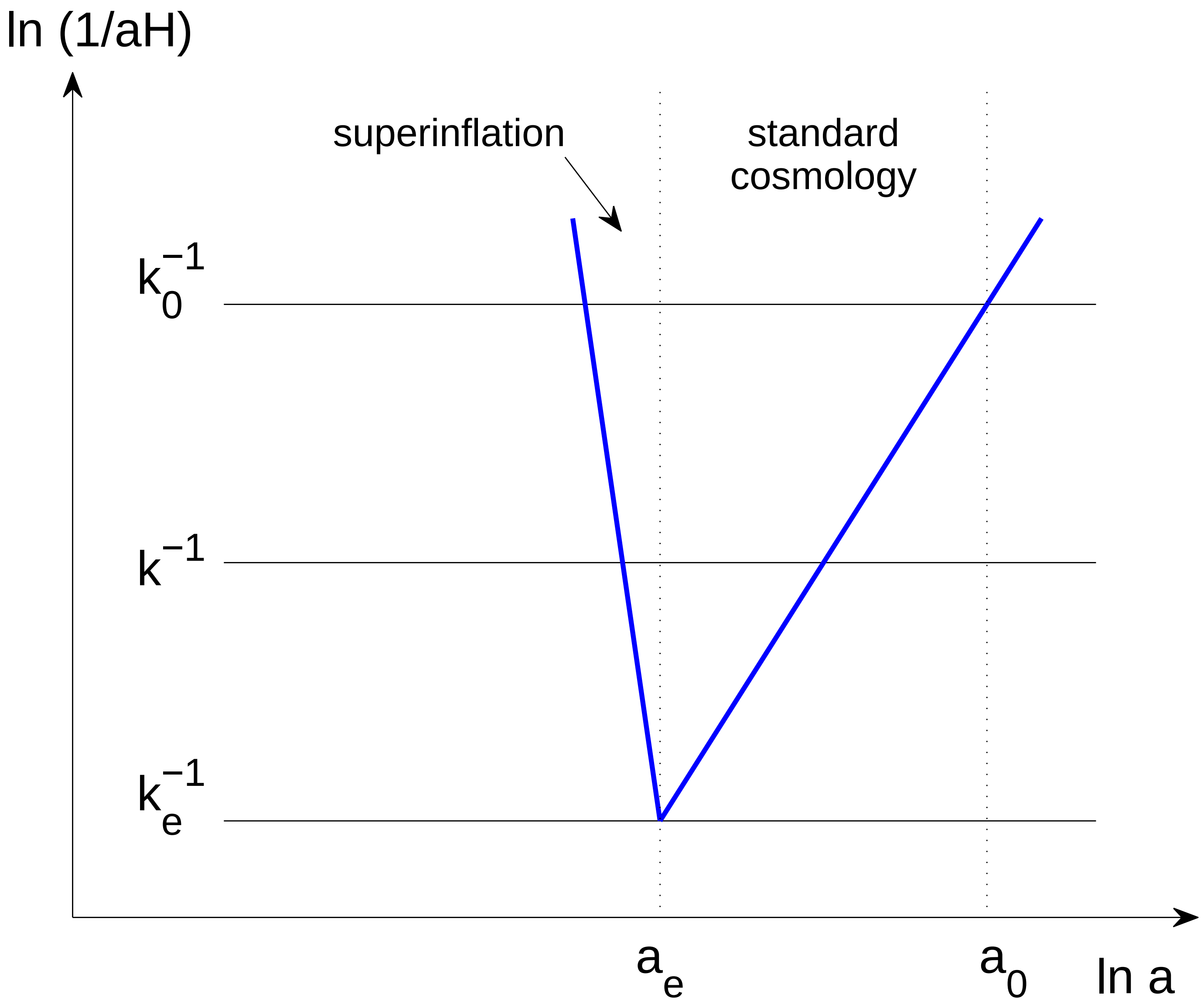


e.g. Ekpyrotic/cyclic universe, phantom field, LQC effects.

12. Inflation, and the horizon problem



13. Superinflation, and the horizon problem



14. Number of e-folds and the horizon problem

Requirement that the scale entering the horizon today exited N e-folds before the end of inflation:

$$\ln\left(\frac{a_{\rm end}H_{\rm end}}{a_N H_N}\right) = 68 - \frac{1}{2}\ln\left(\frac{M_{\rm Pl}}{H_{\rm end}}\right) - \frac{1}{3}\ln\left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right)^{1/4}$$

1. In standard inflation: $\ln\left(\frac{a_{\rm end}H_{\rm end}}{a_N H_N}\right) \approx \ln\left(\frac{a_{\rm end}}{a_N}\right) \equiv N \approx 60$

2. In LQC with $a = (-\tau)^p$ and $p \ll 1$

$$\ln\left(\frac{a_{\rm end}H_{\rm end}}{a_N H_N}\right) = \ln\frac{\tau_N}{\tau_{\rm end}} = \ln\left(\frac{a_N}{a_{\rm end}}\right)^{1/p} = -\frac{1}{p}N$$

$$N \approx -60\,p$$

Number of e-folds of super-inflation required to solve the horizon problem can be of only a few.

15. Scaling solution (inverse triad corrections)

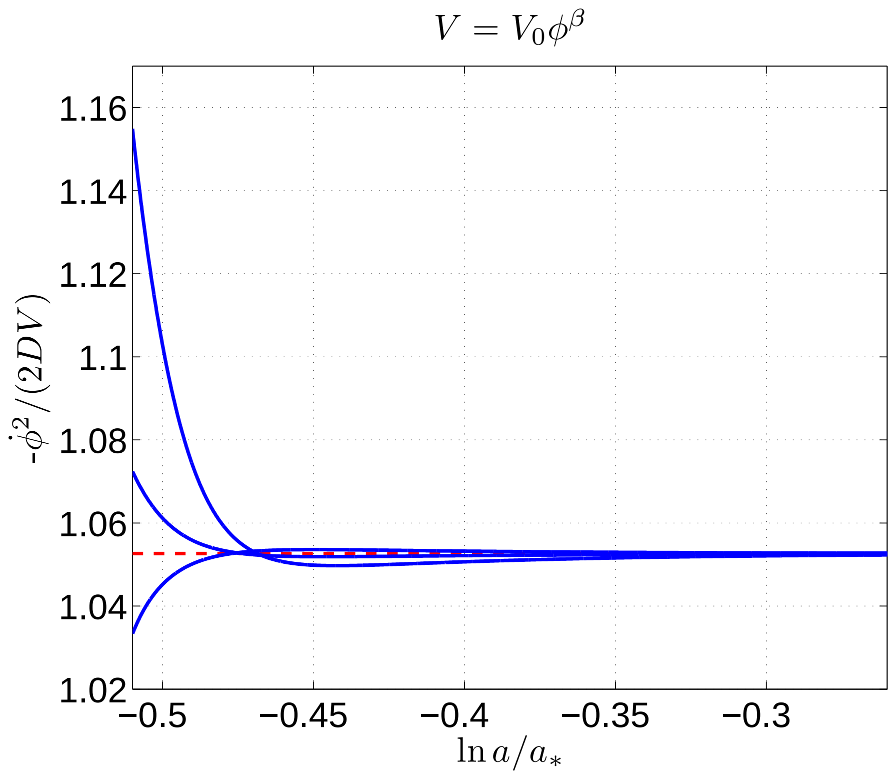
Scaling solution $\Leftrightarrow$ $\dot{\phi}^2/(2DV) \approx$ cnst.
Lidsey (2004)

$$a = (-\tau)^p$$

$$p = \frac{2\alpha}{2\bar{\epsilon} - (2+r)\alpha}$$

$$\bar{\epsilon} = \frac{1}{2}\frac{D}{S}\left(\frac{V_{,\phi}}{V}\right)^2$$

$$V = V_0\,\phi^\beta$$



$\beta = 4\bar{\epsilon}/(n-r)\alpha > 0$, $\alpha = 1 - n/6$, $D \propto a^n$, $S \propto a^r$.

Scaling solution is *stable* attractor for $\bar{\epsilon} > 3\alpha^2$ or $\beta > (n-6)/n \sim \mathcal{O}(1)$.

16. Perturbation equations

Define effective action that gives background equations of motion

$$S = \int d\tau \, d^3x \, a^4 \left(\frac{\phi'^2}{2Da^2} - \frac{\delta^{ij}}{a^2} \partial_i \phi \partial_j \phi - V \right)$$

Perturb field $\phi = \phi_b + \delta\phi$.

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Perturb field $\phi = \phi_b + \delta\phi$.

Define $u = a\delta\phi/\sqrt{D}$ and expand in plane waves:

$$\hat{u}(\tau, \mathrm{x}) = \int \frac{d^3\mathrm{k}}{(2\pi)^{3/2}} \left[\omega_k(\tau) \hat{a}_{\mathrm{k}} + \omega_k^*(\tau) \hat{a}^\dagger_{-\mathrm{k}} \right] e^{-i\mathrm{k}.\mathrm{x}}$$

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Obtain equation of motion

$$\omega_k'' + \left(D_* A^n (-\tau)^{np} k^2 + \frac{m_{\mathrm{eff}}^2 \tau^2}{\tau^2} \right) \omega_k = 0$$

where for the scaling solution

$$m_{\mathrm{eff}}^2\, \tau^2 = -2 + (3-2n)p + \frac{1}{2}(6 - 2n - n^2)p^2$$

17. General solution

General normalised solution is:

$$\omega_k(\tau) = \sqrt{\frac{\pi}{2|2+np|}}\sqrt{-\tau}\,H^{(1)}_{|\nu|}(x)$$

$$x \propto \frac{\sqrt{D}k}{aH}, \qquad \nu = -\frac{\sqrt{9-12p+8np-12p^2-4p^2n+2n^2p^2}}{2+np}$$

Define, by analogy with standard inflation, effective horizon $\frac{\sqrt{D}}{aH}$ or effective wavenumber $\sqrt{D}k$.

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On large scales ($x \ll 1$) $\quad \mathcal{P}_u \propto k^3|\omega_k|^2 \propto k^{3-2|\nu|}$

Near scale invariance for

$$p = -\frac{2}{\beta(n-r)+2(2+r)} = \frac{2\alpha}{2\bar{\epsilon}-(2+r)\alpha} \approx 0$$

Steep and *negative* potentials and *fast-roll* evolution

18. Holonomy corrections

Using holonomies as basic variables leads to a quadratic energy density contribution in the Friedmann equation

$$H^2 = \frac{1}{3}\rho\left(1 - \frac{\rho}{2\sigma}\right)$$

with $\rho < 2\sigma$. In this work we consider

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

The variation of the Hubble rate is

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\left(1 - \frac{\rho}{\sigma}\right)$$

Superinflation for $\sigma < \rho < 2\sigma$.

19. Scaling solution (quadratic corrections)

"Scaling solution" $\Leftrightarrow$ $\dot{\phi}^2/(2\sigma - V) \approx$ cnst.

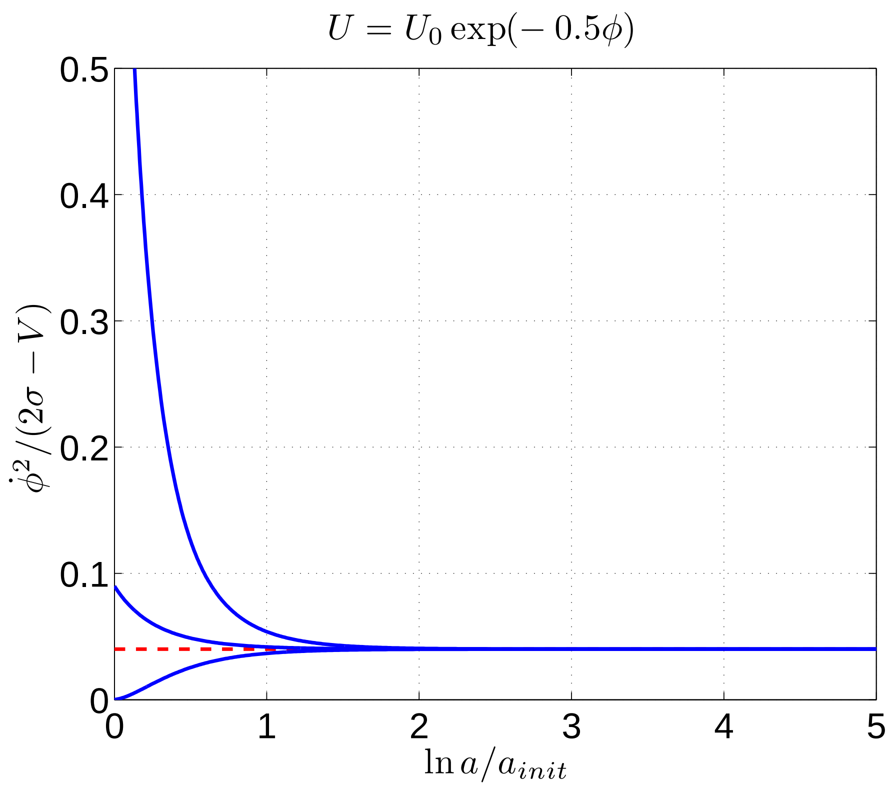
$$a = (-\tau)^p$$

$$p = -\frac{1}{\bar{\epsilon}+1}$$

$$\bar{\epsilon} = \frac{1}{2}\left(\frac{U_{,\phi}}{U}\right)^2$$

$$V = 2\sigma - U(\phi)$$

$$U = U_0\, e^{-\lambda\phi}$$



where $\lambda^2 = 2\bar{\epsilon}$.

Scaling solution is *stable* attractor for all λ or $\bar{\epsilon}$

20. Power spectrum of the perturbed field

Power spectrum is given by:

$$\mathcal{P}_u \propto k^3|\omega_k|^2 \propto k^{3-2|\nu|}$$

where $\nu = -\sqrt{1-4m_{\rm eff}^2\tau^2}/2$

For scaling solution $\qquad m_{\rm eff}^2\tau^2 = -2 + 3p(1+p)$

Near scale invariance $\Rightarrow \qquad p = -\frac{1}{\bar{\epsilon}+1} = -\frac{2}{2\lambda^2+2} \approx 0$

Steep and *positive* potentials and *fast-roll* evolution

21. Tensor spectrum – Inverse triad corrections

Bojowald and Hossain ('07)

$$h''_{\times,+} + 2\mathcal{H}\left[1 - \frac{1}{2}\frac{d\ln S}{d\ln a}\right] h'_{\times,+} - S^2\nabla^2 h_{\times,+} = 0$$

Quantize: $\hat{h} = \int (h_k a_{\mathrm{k}} + h_k^* a_{\mathrm{k}}^\dagger) e^{-i\mathbf{k}\cdot\mathbf{x}}$

$$h_k(\tau) = \frac{S^{1/2}}{\mathcal{H}^{1/2} a}\sqrt{\frac{-p\,\pi}{1+rp}}\, H_\nu^{(1)}(x)$$

$$\nu = \frac{1+p(r-2)}{2(1+pr)}, \qquad x = \frac{-pSk}{(1+pr)\mathcal{H}}$$

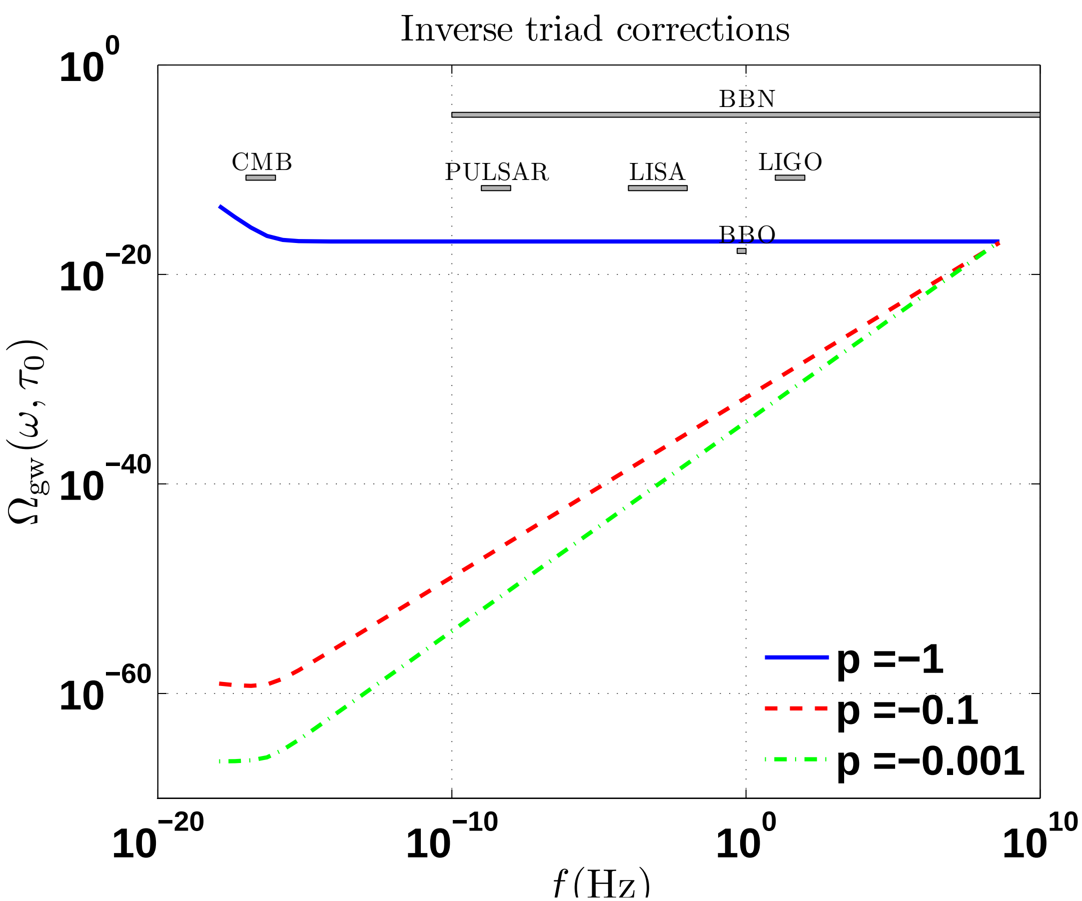
Primordial power spectrum: $P_h(\tau_{\mathrm{e}}, k) \propto k^{3-2\nu}$

For scaling solution $p \to 0$ or $\nu \to 1/2 \quad \Rightarrow \quad n_t \approx 2$.

22. Present abundance of gravitational waves

$$\mathcal{P}_h(\tau_0, k) \approx \left(\frac{k_0}{k}\right)^4 \left(1 + \frac{k}{k_{\rm eq}}\right)^2 \mathcal{P}_h(\tau_{\rm e}, k)$$

$$\Omega_{\rm gw} \approx \frac{1}{6}\left(\frac{k}{k_0}\right)^2 \mathcal{P}_h(\tau_0, k)$$



23. Tensor spectrum – Holonomy corrections

Bojowald and Hossain ('07)

$$h''_{\times,+} + 2\mathcal{H}h'_{\times,+} - \nabla^2 h_{\times,+} + T_Q h_{\times,+} = 2\Pi_Q$$

$$T_Q = \frac{a^2}{3}\frac{\rho^2}{2\sigma}\,, \qquad \Pi_Q = \frac{1}{2}\frac{\rho}{2\sigma}\left(\frac{a^2}{3}\rho - \phi'^2\right)$$

Quantize: $\hat{h} = \int (h_k a_{\mathrm{k}} + h_k^* a_{\mathrm{k}}^\dagger) e^{-i\mathbf{k}\cdot\mathbf{x}}$

$$h_k(\tau) = \frac{1}{\mathcal{H}^{1/2} a}\sqrt{-p\,\pi}\, H_\nu^{(1)}(-k\tau)$$

$$\nu = \frac{1}{2}\sqrt{1 + 4p + 12p^2}$$

Primordial power spectrum: $P_h(\tau_{\mathrm{e}}, k) \propto k^{3-2\nu}$

For scaling solution $p \to 0$ or $\nu \to 1/2 \qquad \Rightarrow \qquad n_t \approx 2.$

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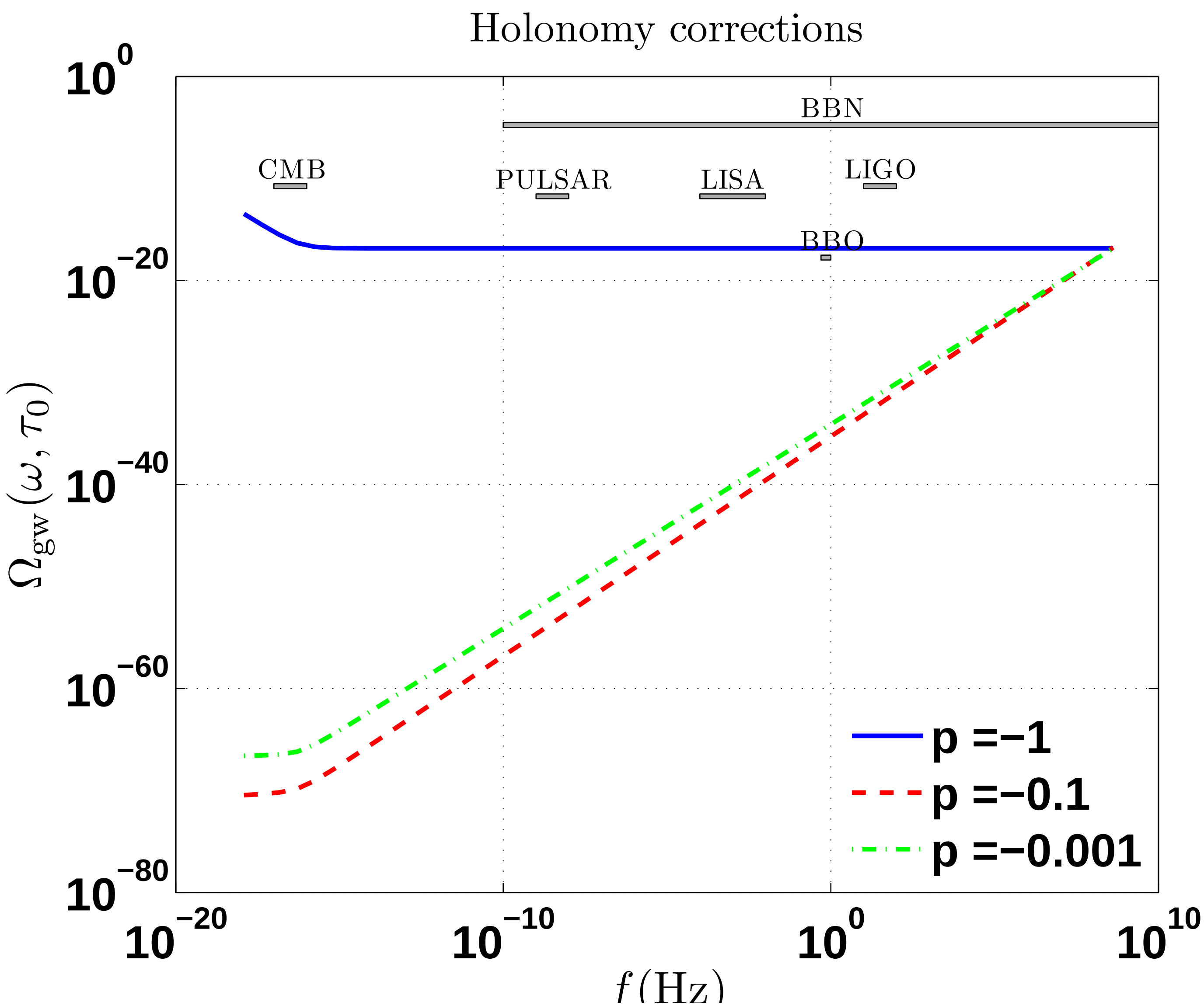

25. Summary and questions

1. Inverse triad corrections: Scale invariance for steep negative potentials, $V = V_0 \phi^\beta$;

2. Quadratic corrections: Scale invariance for steep positive potentials, $V = 2\sigma - U_0 \exp(-\lambda\phi)$;

3. Only a few e-folds necessary to solve the horizon problem;

4. Abundance of gravitational waves is highly suppressed with respect to standard inflation;

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3. Only a few e-folds necessary to solve the horizon problem;

4. Abundance of gravitational waves is highly suppressed with respect to standard inflation;

5. Are the flatness and monopole problems solved?

6. What is the power spectrum of the curvature perturbation?

7. Dynamics of multi-field superinflation? Assisted inflation? Non-gaussianities?

8. Processes of reheating?