On geometry, invariants and integrability in higher spin gauge theory

amplitudes in massive, noncompact, gauge field theories and Poisson sigma models in unfolded on-shell modules

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based on

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and fruitful conversations with M. Vasiliev

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Two parallel lines of research that may intersect!

- Is higher-spin gauge theory a limit of string theory? Or maybe it is the other way around? What constitutes the constructive principle underlying higher-spin gauge theory?
- Do higher-spin gauge theories exist as QFTs? Working hypothesis: "yes", based on the action principle for unfolded dynamics given by quasi-topological Poisson sigma models in ∞-dimensional target spaces – natural generalizations of the Cattaneo - Felder - Kontsevich model as well as topological conformal affine models.
- This constructive approach towards a fully nonlinear, off-shell formulation of interacting, higher-spin gauge fields – pioneered by Fradkin, Vasiliev and the Lebedev School – leads naturally to a framework for relativistic QFT based on infinite-dimensional, associative oscillator algebras and Howe-dual gauge algebras – these basic building blocks and algebraic structures have analogs in ordinary bosonic string theory in flat spacetime.

- The Poisson sigma models are natural generalizations of the one-dimensional Hamiltonian action to arbitrary dimensions providing an action formalism for unfolded dynamics:

(AKSZ,CF)

$$S = \int_B f^* [K + H] \ , \quad f : \ (B, \partial B) \ \rightarrow \ (T^* N, N) \ ,$$

where T^*N is a $\mathbb{Z}$-graded symplectic manifold with canonical form K. Gauge invariance requires that the "Hamiltonian" H has vanishing graded canonical Poisson bracket with itself:

(Voronov)

$$(H, H)_{P.B.} \ \equiv \ 0 \ .$$

- The resulting unfolded dynamics is a generally covariant formulation of classical field theory (including "rigid" field theories as a special case) based on the extension of Sullivan's free differential form algebra by infinite sets of zero-forms that contain all the local degrees of freedom on-shell.

Quasi-topological nature of unfolded dynamics

- The base manifold B can be of arbitrary even or odd dimension, and the target space N is an $\mathbb{N}$-graded manifold coordinatized by a (possibly duality extended) unfolded on-shell module – it has an infinite-dimensional submanifold of degree 0.
- The local degrees of freedom (observables) live on the boundary $M := \partial B$. The topological nature of the equations of motion is compensated by the presence of infinitely many zero-forms.
- The replacement of standard quadratic kinetic terms by infinite "towers" of topological first-order kinetic terms (in K) leads to novel properties in the ultra-violet as well as the infra-red that may have a bearing on what constitutes "finiteness" in QFT as well as the "cosmological constant problem".

Q-structure and Poisson structure

- The perturbatively defined Poisson sigma model

$$S = \int_B \Big[\underbrace{P_\alpha (dX^\alpha}_{K} + \underbrace{Q^\alpha(X)) + V(X) + \frac{1}{2} P_\alpha P_\beta \Pi^{\alpha\beta}(X) + \mathscr{O}(P^3)}_{H} \Big] ,$$

  is gauge-invariant iff $(H,H)_{P.B.} \equiv 0$.
- If H is quadratic in P this condition is equivalent to that

$$(\Pi,\Pi)_{S.B.} \equiv 0 , \quad \mathscr{L}_Q \Pi \equiv 0 , \quad \mathscr{L}_Q Q + \Pi V \equiv 0 , \quad \mathscr{L}_Q V \equiv 0 ,$$

  $\Rightarrow \Pi := \Pi^{\alpha\beta} \partial_\alpha \otimes \partial_\beta$ is a Poisson structure on N, and $Q := Q^\alpha \partial_\alpha$ is a compatible Q-structure deformed by a bulk observable V
- Locally on N, $Q = \Pi U$ and $V = \frac{1}{2} \Pi(U,U)$.
- If $V = 0$ a natural boundary condition is $P_\alpha|_M = 0$.
- The action can then be deformed by boundary terms $\oint_M \mathscr{O}(X)$ where $\mathscr{O}$ are observables obeying $\mathscr{L}_Q \mathscr{O} \equiv 0$.

Ultra-local initial value formulation

- In a coordinate chart $U \subset M$, the "initial data" for the unfolded equations of motion consists of boundary conditions on p-forms with $p > 0$ (transition functions) and the values at a single point for all non-Stückelberg zero-forms, referred to as the generalized Weyl tensors.
- The locally defined dynamical fields then arise as covariant "normal-coordinate" expansions, given in generic models by classical perturbative expansions, and in "integrable" models based on Maurer-Cartan-like equations, such as higher-spin gauge theories, by exact solutions.
- This "ultra-local" initial/boundary value formulation is not only meaningful for all types of field theories – it reproduces the results of the standard "local" formulation if such a formulation exists – but also crucial for theories with fundamental higher-derivative interactions such as higher-spin gauge theories.

Howe duality, geometry and observables

- Both higher-spin and string theory yield infinite-dimensional towers of "oscillators" transforming under infinite-dimensional symmetry algebras containing two maximal commuting subalgebras that defines "auxiliary" and "space-time" gauge symmetries.
- The oscillators arise within a "first-quantized" gauge field theory on an auxiliary bundle. The gauge-invariant auxiliary observables – the "vertex operators" – are the quantized representations of the space-time symmetry algebra forming an "algebraic ring" constituting the fiber of a "second-quantized" gauge field theory on a bundle over a "spacetime" (Dixmier's enveloping-algebra approach).
- The gauge-invariant space-time observables constitute the fiber for what one may think of as yet another "hyper" field theory (Deligne "recycling").

Unfolding, folding and recycling observables

$$\cdots \mathrm{Inv}\left[\mathrm{Inv}\left[F_n \hookrightarrow E_n \xrightarrow{\pi} M_n\right] \equiv F_{n+1} \hookrightarrow E_{n+1} \xrightarrow{\pi} M_{n+1} \ \ldots\right]\cdots$$

- The independent roles of the base manifolds M_n and M_{n+1} is manifest in higher-spin gauge theory and unfolded dynamics in general. Two key features are that the auxiliary field theory in E_n only couples to zero-forms on M_{n+1}, and the $M_{n+1} \supset M_D$, unfolded spacetime.
- This picture is blurred in standard string theory where spacetime is interpreted as a slice of $F_n \subset F_{n+1}$ given by the zero-modes of worldsheets.
- A natural termination of a seemingly perpetual recycling process could involve recurrences for the Poisson sigma models that are defined on homotopy classes of even-dimensional and odd-dimensional manifolds and observable-cycles.

Vasiliev's quasi-topological off-shell Poincaré gravity

- Suppose the observable algebra $F_{n+1} \supset \mathscr{A}$, the associative algebra of arbitrary polynomials on the phase space $\mathbb{R}^{2D}$.
- Assume E_{n+1} contains the subsection

$$F \ := \ dA + A * A \ \simeq \ 0 \ , \quad B \ := \ dB + A * B - B * A \ \simeq \ 0 \ ,$$

$$A \ \in \ \Omega^{[1]}(U) \otimes \mathscr{A} \ , \quad B \ \in \ \Omega^{[0]}(U) \otimes \mathscr{A} \ , \quad U \subseteq M_{n+1} \ .$$

- While the Maurer-Cartan form A is topological the "Riemann" zero-form B contains locally defined degrees of freedom.
- In the spontaneously broken "metric" phase

$$B(x,p) \ \stackrel{!}{=} \ p^2 + \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} p^s x^{s+n} \mathrm{Rie}(s+n,s) \ ,$$

  the off-shell normal-coordinate expansion of spin-s gauge fields on a space-time "slice" in F_{n+1}.
- On $M_D \subset M_{n+1}$ the spin-2 sector contains Borisov - Ogievetsky's nonlinear realization of off-shell Poincaré gravity based on $Diff_0(D)/O(D-1,1)$.

Local symmetries of (A, B) system

- As a Lie algebra, $\mathscr{A} \cong \mathfrak{poly}_0(D) = \mathfrak{poly}_+(D) \Subset \mathfrak{poly}_-(D)$, the smooth even and odd poly-vector fields on $\mathbb{R}^D$.
- Various structures break this symmetry to subalgebras:
  - Linear momenta $p_a \Rightarrow \mathfrak{diff}_0(D) = \mathfrak{diff}_+(D) \Subset \mathfrak{diff}_-(D)$, the smooth even and odd vector fields on $\mathbb{R}^D$.
  - Linear coordinates and momenta $(x^a, p_a) \Rightarrow$ linear affine $\mathfrak{igl}(D)$
  - Metric-conformal structure $\eta_{ab} \Rightarrow$ various conformal $\mathfrak{so}(D-t+1, t+1) \supset \mathfrak{iso}(D-t, t)$ and the Howe-dual

$$\mathfrak{sp}(2) \ : \ T^+ := p^2 \ , \ T^0 := \frac{1}{2}\{x, p\} \ , \ T^- := x^2 \ .$$

  - $\ker ad_{T^+} \Rightarrow$ the off-shell higher-spin algebra

$$h(\mathfrak{iso}(D-1,1)) \ := \ \mathscr{U}[\mathfrak{iso}(D-t,t)] \cap \mathscr{A} \ := \ \frac{\mathscr{U}[\mathfrak{iso}(D-t,t)]}{\mathscr{I}[\mathscr{A}]}$$

    where $\mathscr{I}[\mathscr{A}]$ is the vanishing ideal in $\mathscr{A}$. This algebra contains a subalgebra isomorphic to $\mathfrak{diff}_0(D)$ (preserving linear x^a).

Phases, observables and geometry

- Different phases of a dynamical system are characterized by their observable contents – the nature of observables may change drastically in passing from one phase to another.
- In gauge theories observables must be gauge invariant on-shell $\Rightarrow$ they can acquire well-defined expectation values that break the gauge symmetry classically.
- noncompact gauging $\Rightarrow$ observables are conserved quantities:
    - nonlocal functionals that remain invariant under small deformations of the submanifold (homotopy cylinder) of the base-manifold where they are evaluated – *e.g.* a decorated Wilson loop, a cycle on which a conserved form is evaluated, or a single point where closed zero-form charges are evaluated.
- Thus, in noncompact gauge theories the geometry of the base manifold is deeply reflected in the observable content – it may differ drastically (even locally) between different phases.

- "unbroken" phase: observable decorated Wilson lines

$$W_C(A,B) \;=\; Tr_{\mathscr{A}} P \left[\Big(\prod_{p\in C}^{*} B(p) \Big) \star \exp_\star \oint_C A \right] \;,$$

  on a universal $\mathfrak{poly}_0(D)$-bundle that measure transition functions for A and expectation values for B.

- "soldered" (or "softly broken") phase: observable de Rham cohomology ring $H^*_{DR}(M)$ on $\mathfrak{poly}_+(D)$-bundle with base $M \overset{\text{loc}}{\simeq} \mathfrak{poly}_-(D)$; subring from generalized Pontryagin classes

$$\mathscr{C} \;=\; STr_{\mathscr{A}} P_\star(E, B_+, B_-) \;, \quad d\mathscr{C} \;\simeq\; 0 \;, \quad \mathscr{C} \;\not\simeq\; d\mathscr{C}' \;,$$

  where $STr_{\mathscr{A}}(f\star g) = (-1)^{\epsilon(f)} STr_{\mathscr{A}}(g \star f)$ and

$$\epsilon(B_\pm) \;=\; \pm B_\pm \;, \quad A \;=\; E+\Omega \;, \quad \epsilon(E) \;=\; -E \;, \quad \epsilon(\Omega) \;=\; \Omega \;.$$

$$D_\Omega E \;\simeq\; 0 \;, \quad D_\Omega B_\pm + [E, B_\mp]_\star \;\simeq\; 0 \;, \quad d\Omega + \Omega\star\Omega + E\star E \;\simeq\; 0 \;.$$

Broken metric phase of the (A, B) system

- Consistent truncation (Weyl order)

$$A = p_a A^a(x) \ , \quad B = p_a p_b B^{ab}(x) + B(x)$$

- $\langle \det B^{ab} \rangle \neq 0$ & choice of signature $\eta^{ab} \Rightarrow B = T^+ + B'$

$$B' \cap \mathrm{Im}\ ad_{T^+} \ : \ \text{Stückelberg fields} \ ,$$

$$A \cap \mathrm{coker}\ ad\,T^+ \ : \ \text{broken gauge fields} \ .$$

- Gauge fix $\Rightarrow \ker\ ad_{T^-} \overset{!}{\ni} B' = \sum_{s,n=0}^{\infty} p^s x^{s+n} \mathrm{Rie}(s+n, s)$

$$A \simeq \underbrace{e^a P_a + \omega^{ab} M_{ab}}_{\in \ker\ ad_{T^+}} + L \underbrace{\left[e^a \frac{\partial}{\partial p^a} B' \right]}_{\in \ker\ ad_{T^-}} , \quad L := \left(ad_{T^0}|_{\mathrm{coker}\ ad_{T^0}} \right)^{-1}$$

$$T^a := \nabla e^a \simeq 0 \, , \quad R^{ab} := d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \simeq e^c \wedge e^d \mathrm{Rie}^a{}_{c,}{}^b{}_d \ .$$

Off-shell and on-shell invariants for Poincaré gravity

- The spin-2 truncation of the (A, B)-system has two types of soldered off-shell observables:
  - Topological invariants: $\left\{STr_{\mathscr{A}} P_\star(E, B'_-, B'_+)\right\} \cap H^*_{DR}(M)$ are off-shell closed, globally defined, local functions. By a general theorem they are topological. Since they are parity odd they can be identified as the Pontryagin class; indeed

$$STr \underbrace{E \star \cdots \star E}_{4n \text{ factors}} \propto p_{4n}(e, R(2,2)) + d(\text{globally defined } (4n-1)\text{-form})$$

  - Dynamical ϵ-invariants: non-topological, off-shell closed and globally defined D-forms given by

$$\mathscr{C}^{[D]}(E, B) = \epsilon_{a[D]} e^{a_1} \cdots e^{a_D} L^{[0]}(B') ,$$

    for arbitrary Lorentz-invariant functions $L^{[0]}$.
- Given a Lagrangian D-form one may ask for further on-shell conserved p-forms with $p < D$.

Where are all the local observables in classical gravity?

- In ordinary classical D-dimensional gravity we seek perturbative statements about weakly curved systems involving "local" degrees of freedom – for example a "dilute gas" of well-separated Schwarzschild solutions that can remain classically stable for a very long time.
- By analyzing the gravitational radiation in such a multi-body system it should be possible to disentangle at least some information about the positions and masses of separate bodies.
- So there should exist "classical amplitudes" in degree 0, accessible to local observers, doing the job of removing gauge artifacts in a curvature expansion.
- However, not many local observables in degree $p < D$ are known:
  - $\exists$ one exact Noether $(D-1)$-form for each globally defined vector field. Some yield quasi-observables (AdM mass and Wald entropy) given certain boundary conditions.
  - $\exists$ on-shell conserved $(D-p-1)$-forms from "free" p-forms

Enter higher-spin symmetry and cosmological constant

- In higher-spin gauge theory the observable/geometry issue becomes acute since the nonabelian gauge symmetry blurs all standard classical notions of geometry and the fields transform together with (nonpolynomial) higher-derivative constructs.
- The unfolded approach presents two countermeasures:
  - an "integrable" bundle based on associative oscillator algebras not much different from the (A, B) system
  - an additional massive parameter in the form of a "bare" cosmological constant Λ
- These facilitate the explicit construction of infinite towers of on-shell conserved p-forms with $p \geqslant 0$ in infinite-dimensional higher-spin geometries containing D-dimensional spacetime as a submanifold.
- The conserved forms are given by nonpolynomial derivative expansions that are computable by "localizing" supertraces.

Remainder of presentation

- Zoom in on Vasiliev's equations
- Perturbative structure of zero-form "amplitudes"
- Comment on multi-ray system
- Related (open) issues:
  - Multi-body systems?
  - Do (independent) zero-form charges exist for (A)dS gravities? or
  - Do there exist (nonlinear) consistent lower-spin truncations of higher-spin gauge theories?

Vasiliev's 4D on-shell $(\widehat{A}, \widehat{B}, \widehat{C})$-system

- On-shell anti-de Sitter analog of 4D (A, B) system:
  - strip traces off Riemann by expanding in $\mathfrak{sl}(2;\mathbb{C})$-doublet oscillators $Y^{\underline{\alpha}} = (y^\alpha, \bar{y}^{\dot{\alpha}})$
  - $\Rightarrow$ on-shell higher-spin algebra $\mathfrak{hs}(4) \supset \mathfrak{sp}(4) \cong \mathfrak{so}(2,3)$ and $T^\pm \approx T^0 \approx 0$
  - $\Rightarrow$ B does not source A (topological (A, B)-system)
- quasi-topological deformation $\Leftarrow$ doubled master fields in

$$\widehat{\mathscr{A}} = \Big[\underbrace{\mathscr{A}(Y) \otimes \mathscr{A}(Z)}_{\text{two commuting copies}}\Big] \otimes_{\text{twist}} \underbrace{\mathscr{A}(K, \bar{K})}_{\text{outer Kleinians}} ,$$

subject to integrable differential constraints in Z

$$\widehat{D}\widehat{C} \approx 0 , \quad \widehat{C} \star \widehat{B} \approx \widehat{B} \star \widehat{C} , \quad \widehat{C} := C^{\underline{\alpha}} \widehat{S}_{\underline{\alpha}}(Y, Z; K, \bar{K}) ,$$

$$\widehat{C} \star \widehat{C} \approx iC^2\Big(1 + \widehat{B} \star \underbrace{(\kappa K + \bar{\kappa}\bar{K})}_{\text{central}}\Big) , \quad \underbrace{\kappa := (-1)^N_\star , \quad \bar{\kappa} := (-1)^{\bar{N}}_\star}_{\text{inner Kleinians}}$$

where $C^{\underline{\alpha}}$ is anti-commuting and $N = a^+ \cdot a^-$ is the number operator for $a^\pm = \frac{1}{2}(y \pm z)$ *idem* $\bar{N}$.

Traces and localized supertraces

- Defining $Y^{\pm} = \frac{1}{2}(Y \pm Z)$, two traces on $\widehat{\mathscr{A}}$ are:

$$\widehat{T}r_{+}\left[\widehat{f}\right] := \int_{\mathbb{R}^8} d^4Y d^4Z \underbrace{f^w(Y,Z;K,\bar{K})}_{\text{symbol in } w\text{-order}}|_{K=\bar{K}=0} \,,$$

$$\widehat{T}r_{-}\left[\widehat{f}\right] := \widehat{T}r_{+}\left[(-1)_{\star}^{\widehat{N}} \star \widehat{f}\right] \,, \quad \widehat{N} := Y^{+} \cdot Y^{-} = -\frac{1}{2} Y \cdot Z \,,$$

obeying $\widehat{T}r_{-}\left[\widehat{f} \star \widehat{g}\right] = (-1)^{\widehat{\epsilon}(\widehat{f})} \widehat{T}r_{-}\left[\widehat{g} \star \widehat{f}\right]$.

- If $\int_{\mathbb{R}^8} d^4Y d^4Z \; f^w(Y,Z) < \infty$ then it is independent of w.
- For example, if a trace is finite in Weyl order then it is also finite (and given by the same value) in normal order.
- The inner Kleinians have Weyl and normal-ordered forms

$$(-1)_{\star}^{\widehat{N}} = \underbrace{\delta^4(Y)\,\delta^4(Z)}_{\text{Localized!}} = \underbrace{:\; e^{YZ} \;:}_{\text{Analytical!}} \,.$$

- Models respecting spin-statistics arise upon imposing

$$\pi\bar{\pi}(\widehat{A},\widehat{B},\widehat{C}) := K\bar{K}\star(\widehat{A},\widehat{B},\widehat{C})\star K\bar{K} = (\widehat{A},\widehat{B},\widehat{C}) \;\Rightarrow\; \widehat{\epsilon}(\widehat{S}_{\underline{\alpha}}) \;=\; -\widehat{S}_{\underline{\alpha}}\,.$$

- The bosonic model has (at least) two phases:
  - observable decorated Wilson lines (including zero form charges) on $(\widehat{A},\widehat{B},\widehat{C}) \hookrightarrow \widehat{E}_{\rm bos} \xrightarrow{\pi} M$ with Ehresmann connection $\widehat{A}$.
  - observable p-form charges in $H^*_{DR}(M)$ on soldered bundle $(\widehat{\Omega},\widehat{E},\widehat{B}_\pm,\widehat{C}_\pm) \hookrightarrow \widehat{E}_{\rm bos} \xrightarrow{\pi} M$ with Ehresmann connection $\widehat{\Omega} = \frac{1}{2}(1+\pi)\widehat{A}$, generalized vielbein $\widehat{E} = \frac{1}{2}(1-\pi)\widehat{A}$ and $M \overset{\rm loc}{\simeq} \frac{1}{2}(1-\pi)\widehat{\mathscr{A}_{\rm bos}}$.
- The local degrees of freedom on M are contained in the locally defined zero-forms $\widehat{B}$ and $\widehat{S}_{\underline{\alpha}}$.
- $\langle\widehat{B}\rangle = 0$ and $\langle\widehat{S}_{\underline{\alpha}}\rangle = z_\alpha \Rightarrow \widehat{S}_{\underline{\alpha}}$ is topological and

$$\widehat{B} = \underbrace{\widehat{\Phi}_0(Y,Z)}_{\text{topological}} + \underbrace{\widehat{\Phi}_+(Y,Z)}_{\text{quasi-topological}}\star K + \underbrace{\widehat{\Phi}_-(Y,Z)}_{\text{quasi-topological}}\star\bar{K} + \underbrace{\widehat{\Phi}_1(Y,Z)}_{\text{topological}}\star K\bar{K}\,.$$

- Zero-form charges are given by the supertraces

$$\widehat{\mathscr{C}}^{[0]} \;=\; \widehat{T}r_{-}\left[P_{\star}(\widehat{S}_{\underline{\alpha}},\widehat{B})\right]\;,\; P_{*} \text{ an arbitrary } \star\text{-polynomial.}$$

- $\widehat{\epsilon}(\widehat{S}_{\underline{\alpha}})=-1$ and $[\widehat{S}_{\underline{\alpha}},\widehat{B}]_{\star}\approx 0 \Rightarrow$ independent supertraces are

$$\widehat{\mathscr{C}}^{[0]}_{(n)} \;:=\; \widehat{T}r_{-}\left[(\widehat{B})^{\star n}\right]\;,\qquad n=1,2,\dots\;.$$

- If $\langle\widehat{S}_{\underline{\alpha}}\rangle\neq z_{\alpha}$ there are local degrees of freedom on M associated to $\widehat{S}_{\underline{\alpha}}$, and they are captured by cyclic traces which depend on global properties of Z-space.

Perturbative B-expansion and homotopy integrals

- $\langle \widehat{S}_\alpha \rangle = z_\alpha \Rightarrow$ Weyl-curvature expansion

$$\widehat{A} \approx A + \widehat{A}_{(1)}(Y,Z|A;B) + \widehat{A}_{(2)}(Y,Z|A;B,B) + \cdots \ .$$

$$\widehat{B} \approx B + \widehat{B}_{(2)}(Y,Z|B,B) + \widehat{B}_{(3)}(Y,Z|B,B,B) + \cdots \ ,$$

$$\widehat{C} \approx \widehat{C}_{(1)}(Y,Z|B) + \widehat{C}_{(2)}(Y,Z|A;B,B) + \cdots \ .$$

- The perturbative expansion in w-order is of the schematic form:

$$\frac{\partial}{\partial Z} \widehat{f}^{w}_{(n)} \approx \sum_{p+q=n} \widehat{f}^{w}_{(p)} \star \widehat{f}^{w}_{(q)} =: \widehat{j}^{w}_{(n)}(Y,Z)$$

$$\Rightarrow \widehat{f}^{w}_{(n)} \approx \oint_{C^w_n} \frac{dt_n}{2\pi i} \widehat{j}^{w}_{(n)}(Y, t_n Z)\gamma(t_n) \ , \quad \gamma(t) = \log \frac{t}{1-t} \ ,$$

provided $\widehat{j}^{w}_{(n)}(Y,tZ)$ is analytic in the whole t-plane and C^w_n encloses the branch cut in γ along $[0,1]$.

Perturbative associativity and homotopy contours

- The nth order is well-defined if $\{C^w_k\}_{k<n}$ admit smooth deformations such that
  - $\widehat{f}^w_{(p)} \star \widehat{f}^w_{(q)}$ are finite for $p, q < n$ since then $\widehat{j}^w_{(n)}(Y, Z)$ is real-analytic in Z if the initial data (A, B) is real-analytic in Y.
  - $\widehat{f}^w_{(p)} \star \left(\widehat{f}^w_{(q)} \star \widehat{f}^w_{(r)}\right) = \left(\widehat{f}^w_{(p)} \star \widehat{f}^w_{(q)}\right) \star \widehat{f}^w_{(r)}$ for $p, q, r < n$ since then the nth order Z-space differential equation is integrable.
- The first condition is weaker than the second condition: finiteness may hold for topologically distinct sets $\{C^w_k\}_{k<n}$ (separated by poles).
- The second condition requires absence of poles inbetween $\{C^w_k\}_{k<n}$: $\exists$ closed curve $\Lambda^w_{\rm reg} \subset \mathbb{C}$ encircling $[0, 1]$ such that $C^w_k = \Lambda^w_{\rm reg}$ for $k < n \Rightarrow$ finiteness (no "nested" singularities)
- This may deform $\{C^w_k\}_{k<n}$ far away from $[0, 1]$; even so, Z-space integrability still holds: when the derivative lands on a $\star$-factor in $\partial_Z(\widehat{f}^w_{(p)} \star \widehat{f}^w_{(q)})$, the "highest-order" homotopy integral in that factor can be deformed back from $\Lambda^w_{\rm reg}$ to $[0, 1]$ by the induction hypothesis.

Perturbative associativity and unfolded integrability

- Perturbative associativity $\Rightarrow \widehat{F} \approx 0$ and $\widehat{D}\widehat{B} \approx 0 \ \forall Z$ if

$$\widehat{F}|_{Z=0} \;=\; F + \sum_{n=1}^{\infty} J^{[2]}_{(n)}(A, A; B, \dots, B) \;\approx\; 0 \;,$$

$$\widehat{D}\widehat{B}|_{Z=0} \;=\; DB + \sum_{n=2}^{\infty} P^{[1]}_{(n)}(A; B, \dots, B) \;\approx\; 0 \;,$$

constituting full perturbative unfolded equations of motion that are integrable on a universal base manifold M.

- The perturbative associativity also assures conservation on M of the zero-form charges

$$\widehat{\mathscr{C}}^{[0]}_{(n)} \;=\; \sum_{p_1+\cdots p_n=n}^{\infty} \underbrace{\widehat{T}r_{-}\left[\widehat{B}_{(p_1)} \star \cdots \star \widehat{B}_{(p_n)}\right]}_{=:\ \widehat{\mathscr{C}}_{(n|n+m)}\ ,\ \ m\ :=\ p_1+\cdots+p_m-m} \;.$$

Regularization of space-time non-locality

- Pert. associativity holds if (A,B) are arbitrary polynomials.
- e^a invertible on $M_4 \subset M \Rightarrow$ standard equations of motion for higher-spin gauge fields; trial calculations indicate that

$$\underbrace{\{\nabla^2\phi\}+\{\Lambda\phi\}}_{\text{Killing unitarizable}} + \underbrace{\sum_{n=0}^{\infty} {\frac{\alpha_n}{\Lambda}}^{n-1} \{\nabla^{2n}\phi^2\}}_{\substack{\text{Born-Infeld tail; strongly coupled for}\\ \text{Killing normalizable fields}}} + \mathscr{O}(\phi^3) \approx 0 \ .$$

- Normal-ordered plane-wave decomposition of initial data

$$B \ = \ T(\Pi) \ : \ e^{M\cdot Y} \ : \ , \quad M \ := \ (\mu_\alpha, \bar\mu_{\dot\alpha}) \ , \quad \Pi \ := \ \frac{\partial}{\partial M} \ ,$$

  where T is a "polarization tensor", *e.g.*
  - Killing normalizable fields ("singular" B)
  - Unitarizable run-away/singular modes ("pseudo-singular" B)
  - Non-unitarizable "instantons" ("regular" B)
- Regularize $\Leftrightarrow$ deform closed homotopy contours $\{C_k^{::}\} \to \Lambda_{\rm reg}^{::}$.

Computation of zero-form charges

- The mth perturbative correction $\widehat{\mathscr{C}}_{(n|n+m)}$ is given by

$$\left[\prod_{k=1}^{n+m} \widehat{T}_k(\Pi_k, \widetilde{\Pi}_k)\right] \left[\prod_{i=1}^{2m} \oint_{C_i^{::}} \frac{\gamma(t_i)dt_i}{2\pi i}\right] \sum_{m_1+m_2=m} \widehat{T}r_-\left[P_{m_1,m_2}\right] ,$$

where P_{m_1,m_2} are $\star$-polynomials of the schematic form

$$P_{m_1,m_2}\left(\{t_i\}| \underbrace{\{: \; e^{M_k\cdot Y+\widetilde{M}_k Z} \; :\}}_{n+m \text{ entries}}; \underbrace{\kappa,\ldots,\kappa}_{m_1 \text{ entries}}, \underbrace{\bar{\kappa},\ldots,\bar{\kappa}}_{m_2 \text{ entries}}\right)\Big] .$$

- "Squeezed" plane-waves $g(\tau,\rho,\widetilde{\rho}) := \; : e^{\tau y\cdot z+\rho\cdot y+\widetilde{\rho}\cdot z} :$ obey

$$g(\tau,0,0)\star g(\tau',0,0) \;=\; g(\tau+\tau'-2\tau\tau',0,0) ,$$

and thus $P_{m_1,m_2} = g(\{t_i,\mu_k,\widetilde{\mu}_k\})\bar{g}(\{t_i,\bar{\mu}_k,\bar{\widetilde{\mu}}_k\})e^{\sigma_0(\{t_i,\mu_k,\widetilde{\mu}_k\})}$

$$g(\{t_i,\mu_k,\widetilde{\mu}_k\}) \;=\; g\left(\tau(\{t_i\}),\rho(\{t_i,\mu_k,\widetilde{\mu}_k\}),\widetilde{\rho}(\{t_i,\mu_k,\widetilde{\mu}_k\})\right) .$$

- "monomial domination": τ depend polynomially on $\{t_i\}$ such that $\tau \sim \prod_{i=1}^{2m} t_i$ as $|t_i|\to\infty$ *idem* ρ, $\widetilde{\rho}$ and σ_0.

- Normal and Weyl order are points in the ∞-dimensional affine space of "universal" and $\mathfrak{sp}(4)$-invariant orderings.
- Going smoothly from between them may deform

$$\{C_i^{::}\} \xrightarrow{::\,\to\,\text{Weyl}} \Lambda^{\text{loc}} \quad \text{far away from } [0,1].$$

- The supertrace localizes in the Weyl order, where

$$\widehat{T}r_-\left[g(\tau,\rho,\widetilde{\rho})\right] \;=\; \frac{1}{(1-\tau)^2}\exp\frac{\rho\cdot\widetilde{\rho}}{1-\tau}\;.$$

- The Jacobian pre-factor is potentially "anomalous" – can destroy associativity unless $\{C_i^{::}\}$ deform into some Λ^{loc}.
- Monomial domination $\Rightarrow$ if Λ^{loc} is sufficiently large then

$$\frac{1}{1-\tau} \sim \prod_{i=1}^{2m}\frac{1}{t_i}$$

which is manifestly free from nested singularities.

Amplitude formula

- A zero-form charge is thus given by its residues at $\{t_i = \infty\}$ leading to the amplitude formula $(t_i x_i = 1)$

$$\widehat{\mathscr{C}}^{[0]}_{(n)} = \sum_{m=0}^{\infty} \left[\prod_{k=1}^{n+m} \widehat{T}_k(\Pi_k, \widetilde{\Pi}_k) \right] \left[\prod_{i=1}^{2m} \oint_0 \frac{\log(1-x_i) dx_i}{2\pi i} \right]$$

$$\sum_{m_1+m_2=m} \frac{1}{(1-\tau)^2 \prod_i x_i^2} \exp\left(\underbrace{\frac{\rho \cdot \widetilde{\rho}}{1-\tau} + \sigma_0}_{\text{they both } \sim \ \prod_i \frac{1}{x_i}} \right) .$$

Applicability

- The applicability of the amplitude formula depends on the polarization tensor $\widehat{T}(\Pi)$:
    - Instantons (regular initial data): perturbative associativity holds for generic closed homotopy contours; the amplitude formula applies after the $\Lambda^{\rm loc}$-deformation that assures perturbative associativity of localization.
    - Killing normalizable fields (singular initial data): trial calculations based on Laplace transforms of $\widehat{T}$ indicate that perturbative associativity requires large $\Lambda^{::}_{\rm reg}$; the amplitude formula applies again since $\Lambda^{\rm loc} \sim \Lambda^{::}_{\rm reg}$ (no conflict!).
    - Run-away/singular fields (pseudo-singular initial data): trial calculations indicate $\widehat{j}^{::}_{(n)}(Y, tZ) \sim \log(1-t) \Rightarrow$ perturbative associativity requires open contours $\Lambda^{::}_{\rm reg} = [0,1] \Rightarrow$ conflict with localization and the zero-form charges appear to be formally divergent.

Many-ray and many-body systems

- SEE BLACKBOARD

Outlook into "Ring Theory"

- generalized Poisson sigma models in fiber and total space
- unbroken and soldered phases
- ∞-dimensional graded oscillators and formal geometry
- (conformal) affine extensions of fiber algebra
- noncompact associative algebras and bi-modules (noncompact topological open strings, Langlands, Dixmier,...)
- the "mixed-symmetry problem" and nonassociativity
- Question: Contact with standard QFT/string theory? or is unfolded QFT well-defined "Quantum Engineering"?

- THANK YOU !

Higher-spin versus Kac-Moody approaches

- In ordinary string theory the auxiliary gauging is better understood than the gauging in spacetime except, or course, for the massless sector.
- The situation is somewhat reversed in higher-spin gauge theory where the auxiliary gauging is introduced somewhat by hand.
- Similarly, in the Kac-Moody approach to supergravity it is natural to extend the program from the adjoint representation to more general representations in the enveloping algebra modulo some suitable ideal reflecting an auxiliary gauging – *c.f.* generalized Sugawara constructions.
- This state of affairs motivates a combined approach enhanced with conformal affine constructions perhaps along the lines of the integrable, topological, noncompact Wess-Zumino-Witten models containing multi-singleton composites studied by J. Engquist, L. Tamassia and me.

Local degrees of freedom: observables and phases

- The locally defined unfolded configuration space is given by the space of gauge orbits in the space of the zero-forms. After removing Stückelberg fields (the image of the massive shift symmetries) this space is given by the Weyl zero-forms modulo the nonabelian gauge transformations.
- Unfolded field theories with noncompact gauge algebras, such as gravity, supergravity and higher-spin gauge theory, have two characteristic phases: an unbroken phase in which all gauge symmetries, including local translations, are locally defined, and a soldered phase with observable de Rham cohomology in positive degrees which requires the local translations to be softly broken.
- By their very definition observables, if they exist, are generalized conserved charges. The unbroken observables are expected to be slightly more nonlocal than soldered observables, which are more standard boundary vertex operators for the Poisson sigma model.

Unbroken phase, Lax pairs and decorated Wilson loops

- If all gauge symmetries are locally defined then the observables are presumably given by highly nonlocal higher-dimensional generalizations of Lax pairs in two-dimensional integrable models.
- In higher-spin gauge theory the unbroken observables are given explicitly by decorated Wilson loops.
- The unfolded formulation of the nonlinear realization of off-shell Poincaré gravity suggests that similar constructs exist also in gravitational systems.

Soldered phase: equivariant de Rham cohomology

- Observable p-forms requires that some of the gauge transformations – the local translations – act as diffeomorphisms, while the remaining unbroken gauge symmetries – the generalized Lorentz transformations – act as locally defined rotations.
- The observables are then given by integrals over closed cycles of globally defined and on-shell closed composite forms, referred to as conserved forms, that are algebraic functions of the zero-forms and the broken gauge fields, referred to as the generalized vielbein.

Nonlocality and translations versus transvections

- If the noncompact gauge algebra is reductive, such as in Poincaré supergravities, or if the conserved forms are restricted to be given by polynomial derivative expansions, then there are only finitely many conserved charges (barring notions such as dual Weyl zero-forms).
- If, on the other hand, the noncompact gauge algebra is semisimple, *e.g.* if there is a nonvanishing cosmological constant, and if the conserved forms are allowed to have nonpolynomial derivative expansions, then there are additional conserved forms in the free (weak-curvature) limit that may give rise to conserved forms in the full interacting theory.
- The aforementioned statement also holds for $p = 0$: the simplest example is the bilinear forms on the self-dual, infinite-dimensional, gauge-algebra module containing the free Weyl zero-forms. For a free many-body system the bilinear zero-form charges are given by two-particle amplitudes plus self-interactions.

Unfolded QFT

Instead of starting from the standard action postulates, one may think of a perturbative QFT as follows:

- Symmetry: a noncompact algebra $\mathfrak{g}$ with a maximal compact subalgebra $\mathfrak{h}$ and a simple subalgebra $\mathfrak{m} \subset \mathfrak{g}$
- Spectrum: an ∞-dimensional $\mathfrak{g}$-module $\mathfrak{M}$ that decomposes into irreps with discrete eigenvalues of $C_2[\mathfrak{g}]$ such that $\mathfrak{D} := \mathfrak{M}|_{\mathfrak{h}}$ is unitarizable and $\mathfrak{T} := \mathfrak{M}|_{\mathfrak{m}}$, referred to as the twisted-adjoint module, consists of $\mathfrak{m}$-types $|\Theta_\alpha\rangle$ (tensors)
- Harmonic expansion: the equivariant maps (changes of basis) $\mathscr{S} : \mathfrak{D} \to \mathfrak{T}$ and $\mathscr{S}^* : \mathfrak{T} \to \mathfrak{D}$ encode completeness relations/boundary conditions that may require $\mathfrak{D}$ to consist of more than just lowest-energy representations

The Weyl zero-form module $\mathfrak{R}^{[0]}$

- Let $L: U \to G/H$ where $U \overset{\text{loc}}{\simeq} \mathfrak{g}/\mathfrak{h}$ is a coordinate chart; define $L^{-1}dL =: E + \Omega \ \in \ \mathfrak{g}/\mathfrak{m} \oplus \mathfrak{m}$; set $|\Phi\rangle := L^{-1}\mathscr{S}|\lambda\rangle$ with $|\lambda\rangle \in \mathfrak{D}$; The resulting generalized Bargmann-Wigner equations read

$$(\nabla + E)|\Phi\rangle \ \approx \ 0 \ , \quad \nabla \ := \ d + \Omega \ ,$$

$$|\Phi\rangle := \Phi^\alpha |\Theta_\alpha\rangle \ \in \ \widetilde{\mathfrak{R}}^{[0]} \ := \ \Omega^{[0]}(U) \otimes \mathfrak{T} \ ,$$

  where Φ^α are referred to as the generalized Weyl tensors.
- The Weyl zero-form modules
  - are Harish-Chandra modules generated by covariant differentiation of the "smallest" Weyl tensors, referred to as primary Weyl tensors
  - contain all the local degrees of freedom of the field theory
  - are infinitely indecomposable if $\mathfrak{g} = \mathfrak{iso}(D-1,1)$ and $M^2 = 0$ (the "strictly massless" case)
  - are self-dual if $|\Lambda| + |M^2| > 0$

- Integrating Bianchi identities yields potentials and Stückelberg fields in various $\mathbb{N}$ degrees:

(BMV,ASV,Skvortsov,BIS)

$$\left[\nabla+\mu+E+\sum_{k=1}^{\infty}\Sigma^{[k+1]}(E,\ldots,E)\right]|X\rangle \;\approx\; 0\;,$$

$$|X\rangle \;:=\; X^{\alpha}|\Theta_{\alpha}\rangle \;\in\; \mathfrak{R}:=\biguplus_{p\in\mathbb{N}}\Omega^{[p]}(U)\otimes\mathfrak{T}(p)\;,$$

where

  - μ are massive integration constants
  - $\Sigma^{[1+k]}$ describes k-fold integrations whereby p-forms "source" $(p+k)$-forms, which are dual to Chevalley-Eilenberg cocycles "gluing" $\mathfrak{R}^{[p]}$ to $\mathfrak{R}^{[p+k]}$.

- The on-shell modules are not unique – *e.g.* trace extensions in positive degrees. (*c.f.* Campoleoni,Francia,Mourad,Sagnotti)
- The "minimal" massive modules ($|\Lambda|+|M^2|>0$) are radial reductions of the minimal massless Skvortsov modules in flat spacetime. (BIS)

- For maximally symmetric spacetimes with cosmological constant Λ, each $\mathfrak{m}$-type in $\mathfrak{R}$ has multiplicity 1. This implies realizations in terms of oscillators and cell-operators:
  - Fock space $\mathscr{F}_\pm$ from bosonic (+) or fermionic (-) oscillators

$$[\alpha^a_\xi, \bar\alpha^\eta_b]_\mp \ = \ \delta^a_\beta \delta^\eta_\xi \ ,$$

where a labels an $\mathfrak{m}$-vector if $\Lambda = 0$ and a $\mathfrak{g}$-vector if $\Lambda \neq 0$, and $\xi = 1, 2, \ldots, \nu_\pm \to \infty$ labels rows (+) or columns (-).
  - Three-graded Howe-dual algebras

$$\widetilde{\mathfrak{g}} \ = \ \widetilde{\mathfrak{g}}^{(-1)} \uplus \widetilde{\mathfrak{g}}^{(0)} \uplus \widetilde{\mathfrak{g}}^{(+1)} \ = \ \begin{cases} \mathfrak{sp}(2\nu_+) & \text{bosons} \\ \mathfrak{so}(2\nu_-) & \text{fermions} \end{cases}$$

$T^{\xi\eta} \ = \ \bar\alpha^\xi \cdot \bar\alpha^\eta$, $N^\alpha_\eta \ = \ \frac{1}{2}[\bar\alpha^\xi, \alpha_\eta]_\pm$ and $T_{\xi\eta} \ = \ \alpha_\xi \cdot \alpha_\eta$.
  - Gauging $\widetilde{\mathfrak{g}}^{(+1)} \ = \ \{T_{\xi\eta}\}_{\forall \xi,\eta} \ \cup \ \{N^\xi_\eta\}_{\xi<\eta}$ yields the Schur module

$$\mathfrak{R} \ = \ \mathscr{S} \otimes \Omega(U) \ , \quad \mathscr{S} \ \cong \ \mathscr{S}_\pm \ = \ (\ker \widetilde{\mathfrak{g}}^{(+1)}) \ \cap \ \mathscr{F}_\pm \ ,$$

and $(d + \Omega + \Sigma)|X\rangle \approx 0$ with Σ realized explicitly by means of Olvert-Metsaev cell-operators that commute "weakly" to $\widetilde{\mathfrak{g}}^{(+1)}$.

- For $V = 0$, $|B| = 2$ and $|X^\alpha| = 0$, the Poisson sigma model describes the formal $\star$-product quantization of a standard zero-graded Poisson manifold coordinatized by $\mathfrak{R}^{[0]}$. The local degrees of freedom of the CFK model are the constant on-shell values for $X^\alpha|_M$.
- The gauged CFK model arises upon exchanging some coordinates and momenta, *viz.*

$$(X^\alpha; P_\alpha) \;=\; (\phi^i, \xi^r; \eta_i, \lambda_r) \;\leftrightarrow\; (\widetilde{X}^\alpha, \widetilde{P}_\alpha) \;=\; (\phi^i, \lambda_r; \eta_i, \xi^r) \;,$$

  where ξ_r now function as Lagrange multipliers implementing the curvature constraints $F_r := d\lambda_r + f_r^{st}\lambda_s\lambda_t \approx 0$ – the Hamiltonian constraints on boundary observables generated by Hamiltonian functions U^r obeying $\Pi(U^r, U^s) = f_t^{rs} U^t$ are spontaneously broken by integration constants.
- Since $\xi^r|_M = 0$ and the one-forms do not propagate any local degrees of freedom, it follows that the local degrees of freedom of the gauged model are the on-shell values for $\phi^i|_M$