Scattering amplitudes/Wilson loops duality and their symmetries

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Based on work in collaboration with

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Outline

- General properties of scattering amplitudes
- Dual conformal invariance – hidden symmetry of planar amplitudes
- Scattering amplitude/Wilson loop duality in $\mathcal{N} = 4$ SYM
- Dual superconformal symmetry of amplitudes

General properties of amplitudes in gauge theories

Tree amplitudes:

- ✔ Are well defined in $D = 4$ dimensions (free from UV and IR divergences)
- ✔ Respect the classical (Lagrangian) symmetries of the gauge theory
- ✔ Gluon tree amplitudes are the same in all gauge theories

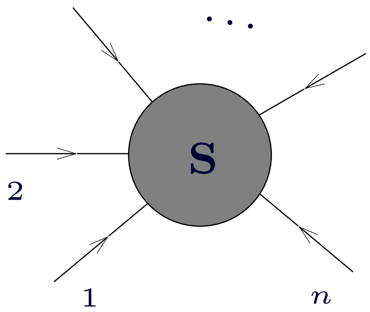
All-loop amplitudes:

- ✔ Loop corrections are not universal (gauge theory dependent)
- ✔ Free from UV divergences
- ✔ Suffer from IR divergences $\rightarrow$ are not well defined in $D = 4$ dimensions
- ✔ Some of the classical symmetries (dilatations, conformal boosts,...) are broken

Two main questions in this talk:

- ✔ Do tree amplitudes have a hidden, dynamical symmetry?
- ✔ What happens to this symmetry at loop level?

Gluon scattering amplitudes in $\mathcal{N} = 4$ SYM

$\mathcal{A}_n =$ 

- ✗ Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$: momentum $(p_i^\mu)^2 = 0$, helicity $h = \pm 1$, color a
- ✗ IR divergences $\mapsto$ require IR regularization ($D = 4 - 2\epsilon_{\rm IR}$, $\epsilon_{\rm IR} < 0$)
- ✗ Closely related to QCD gluon amplitudes

✔ Color-ordered planar partial amplitudes

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1} T^{a_2} \dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1, p_2, \dots, p_n) + \text{[Bose symmetry]}$$

- ✗ Classified according to their helicity content $h_i = \pm 1$
- ✗ $\mathcal{N} = 4$ supersymmetry relations:

$$A^{++\dots+} = A^{-+\dots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A_n^{--+\dots+}\,, \qquad A^{(\mathrm{next-to-MHV})} = A_n^{---+\dots+}\,, \qquad \dots$$

- ✗ *Weak/strong coupling corrections to all MHV amplitudes in $\mathcal{N} = 4$ SYM are described by a single function of the 't Hooft coupling and the kinematical invariants!* [Parke,Taylor]

$$A_n^{\mathrm{MHV}} = \delta^{(4)}(p_1 + \dots + p_n) A_n^{(\mathrm{tree})}(p_i, h_i) M_n^{\mathrm{MHV}}(\{s_{ij}\}; \lambda)$$

From MHV amplitudes to MHV superamplitude in $\mathcal{N}=4$ SYM

- On-shell helicity states in $\mathcal{N}=4$ SYM:

$$G^{\pm} \text{ (gluons } h=\pm 1), \qquad \Gamma_A,\, \bar{\Gamma}^A \text{ (gluinos } h=\pm\tfrac{1}{2}), \qquad S_{AB} \text{ (scalars } h=0)$$

- Can be combined into a single on-shell superstate [Mandelstam],[Brink et al]

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)$$

- Combine all MHV amplitudes into a single MHV superamplitude [Nair]

$$\mathcal{A}_n^{\mathrm{MHV}} = (\eta_1)^4(\eta_2)^4 \times A\left(G_1^- G_2^- G_3^+ \ldots G_n^+\right) + (\eta_1)^4(\eta_2)^2(\eta_3)^2 \times A\left(G_1^- \bar{S}_2\, S_3 \ldots G_n^+\right) + \ldots$$

- Spinor-helicity formalism: [Xu,Zhang,Chang'87]
  - commuting spinors: λ^α (helicity -1/2), $\tilde{\lambda}^{\dot{\alpha}}$ (helicity 1/2)
  - on-shell momenta:

$$p_i^2 = 0 \quad \Leftrightarrow \quad p_i^{\alpha\dot{\alpha}} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha\, \tilde{\lambda}_i^{\dot{\alpha}}$$

Tree MHV superamplitude

- ✔ All MHV amplitudes are combined into a single superamplitude (spinor contractions $\langle ij\rangle = \lambda_i^\alpha \epsilon_{\alpha\beta}\lambda_j^\beta$) [Nair'88]

$$\mathcal{A}_n^{\rm MHV}(p_1,\eta_1;\ldots;p_n,\eta_n) = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

- ✔ On-shell $\mathcal{N}=4$ supersymmetry:

$$q_\alpha^A = \sum_i \lambda_{i,\alpha}\eta_i^A\,, \qquad \bar{q}_{A\,\dot\alpha} = \sum_i \tilde\lambda_{i,\dot\alpha}\frac{\partial}{\partial\eta_i^A} \qquad \Longrightarrow \qquad q_\alpha^A\,\mathcal{A}_n^{\rm MHV} = \bar{q}_{A\,\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

- ✔ (Super)conformal invariance [Witten'03]

$$k_{\alpha\dot\alpha} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha\partial\tilde\lambda_i^{\dot\alpha}} \qquad \Longrightarrow \qquad k_{\alpha\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

  Much less trivial to verify for NMHV amplitudes

- ✔ The MHV superamplitude has another, dual superconformal symmetry

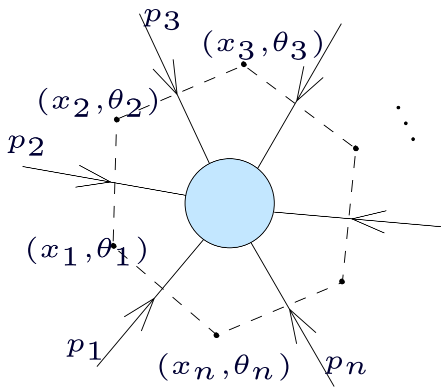
  [Drummond,Henn,Korchemsky,ES'08]

  acts on the dual coordinates x_i^μ and their superpartners $\theta_{i\,\alpha}^A$

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu\,, \qquad \lambda_i^\alpha\,\eta_i = \theta_i^\alpha - \theta_{i+1}^\alpha$$

Dual $\mathcal{N}=4$ superconformal symmetry I

✔ Chiral dual superspace $(x_{\alpha\dot\alpha}\,,\,\theta^A_\alpha\,,\,\lambda_\alpha)$:



✗ $p = \sum_{i=1}^n p_i = 0 \;\longrightarrow\; p_i \equiv \lambda_i \tilde\lambda_i = x_i - x_{i+1}\,, \quad x_{n+1} = x_1$

✗ $q = \sum_{i=1}^n \lambda_i \eta_i = 0 \;\longrightarrow\; \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i - \theta_{i+1})^A_\alpha\,, \quad \theta_{n+1} = \theta_1$

✔ MHV superamplitude in dual superspace: Impose cyclicity through delta functions

$$\mathcal{A}_n^{\rm MHV} = i(2\pi)^4 \frac{\delta^{(4)}\,(x_1 - x_{n+1})\;\delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

✔ $\mathcal{N}=4$ supersymmetry in dual chiral superspace:

$$Q_{A\,\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\theta_i^{A\,\alpha}}\,, \qquad \bar Q^A_{\dot\alpha} = \sum_{i=1}^n \theta_i^{A\,\alpha}\frac{\partial}{\partial x_i^{\dot\alpha\alpha}}\,, \qquad P_{\alpha\dot\alpha} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\dot\alpha\alpha}}$$

✔ Dual supersymmetry of the amplitude

$$Q_{A\,\alpha}\mathcal{A}_n^{\rm MHV} = \bar Q^A_{\dot\alpha}\mathcal{A}_n^{\rm MHV} = P_{\alpha\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

Dual $\mathcal{N} = 4$ superconformal symmetry II

- ✔ Super-Poincaré $+$ inversion $=$ conformal supersymmetry:
  - ✗ Inversion in dual superspace

$$I[\lambda_i^{\alpha}] = (x_i^{-1})^{\dot\alpha\beta}\lambda_{i\beta}\,, \qquad I[\theta_i^{\alpha\,A}] = (x_i^{-1})^{\dot\alpha\beta}\theta^{A}_{\beta\,i}$$

  - ✗ Neighboring contractions are dual conformally covariant

$$I[\langle i\, i+1\rangle] = (x_i^2)^{-1}\langle i\, i+1\rangle$$

  - ✗ Only in $\mathcal{N} = 4$

$$I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^{8}\,\delta^{(4)}(x_1 - x_{n+1})$$
$$I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8}\,\delta^{(8)}(\theta_1 - \theta_{n+1})$$

- ✔ The tree-level MHV superamplitude is covariant under dual conformal inversions

$$I\left[\mathcal{A}_n^{\rm MHV}\right] = (x_1^2 x_2^2 \ldots x_n^2) \times \mathcal{A}_n^{\rm MHV}$$

- ✔ ***Dual superconformal covariance is a property of all tree-level superamplitudes (NMHV, N^{2}MHV,...) in $\mathcal{N} = 4$ SYM theory***

[DHKS], [Brandhuber,Heslop,Travaglini], [Drummond,Henn]

Does (dual) (super)conformal symmetry survive loop corrections?

All-loop planar (super)amplitudes factorize into an IR divergent and a finite part

$$\mathcal{A}_n^{(\text{all-loop})} = \text{Div}(1/\epsilon_{\text{IR}})\ [\text{Fin} + O(\epsilon_{\text{IR}})]$$

✔ IR divergences (poles in ϵ_{IR}) exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],...

$$\text{Div}(1/\epsilon_{\text{IR}}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty}\lambda^l\left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}}\right)\sum_{i=1}^{n}(-s_{i,i+1})^{l\epsilon_{\text{IR}}}\right\}$$

✔ *IR divergences* are in one-to-one correspondence with *UV divergences* of Wilson loops

[Ivanov,Korchemsky,Radyushkin'86]

$$\Gamma_{\text{cusp}}(\lambda) = \textstyle\sum_l \lambda^l \Gamma_{\text{cusp}}^{(l)} = \text{\textit{cusp anomalous dimension of Wilson loops}}$$

$$G(\lambda) = \textstyle\sum_l \lambda^l G^{(l)} = \text{\textit{collinear anomalous dimension}}$$

✔ *IR divergences break conformal + dual conformal symmetry*

$$k_{\alpha\dot\alpha}\mathcal{A}_n^{(\text{all-loop})} \neq 0 \quad \Longrightarrow \quad \text{(conformal anomaly)}$$

$$K_{\alpha\dot\alpha}\mathcal{A}_n^{(\text{all-loop})} \neq 0 \quad \Longrightarrow \quad \text{(dual conformal anomaly)}$$

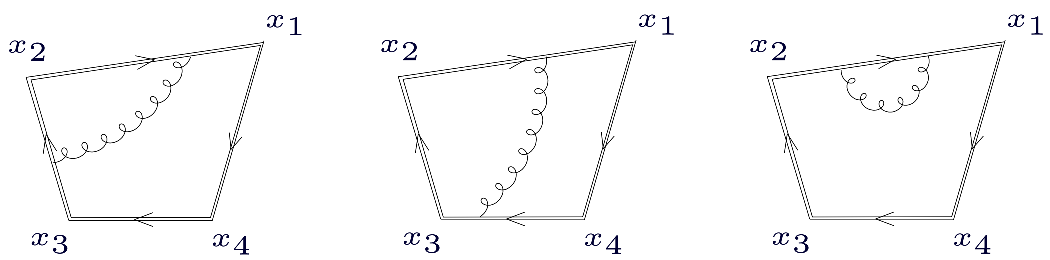
✔ Dual conformal anomaly can be determined from Wilson loop/scattering amplitude duality

[Alday,Maldacena'07], [Drummond,Henn,Korchemsky,ES'07]

MHV amplitudes/Wilson loop duality I

Simplest example:

- ✔ $n = 4$ light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$)

$\ln W(C_4) =$ 

$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm UV}^2} \left[(-x_{13}^2 \mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2 \mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \right\} + O(g^4)$$

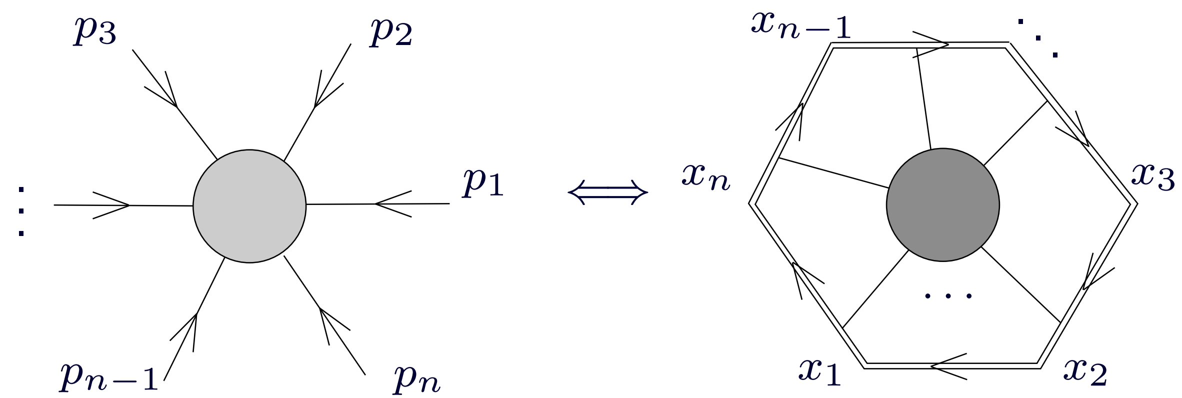
- ✔ Compare with $n = 4$ gluon amplitude

$$\ln \mathcal{A}_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm IR}^2} \left[(-s/\mu_{\rm IR}^2)^{-\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{-\epsilon_{\rm IR}} \right] + \frac{1}{2} \ln^2 \left(\frac{s}{t} \right) + \text{const} \right\} + O(g^4)$$

- ☞ Identify the light-like segments with the on-shell gluon momenta $x_{i,i+1} = p_i$
- ☞ finite $\sim \ln^2(s/t)$ corrections coincide to one loop (constant terms are different)
- ☞ UV div. of the light-like Wilson loop versus IR div. of the gluon amplitude

$\mu^2 := 1/\mu_{\rm IR}^2\,, \qquad \epsilon_{\rm UV} := -\epsilon_{\rm IR} \qquad \Leftarrow \qquad \Big\{$ *The two objects are defined for different* $D = 4 - 2\epsilon$ / *There is a mismatch of* $1/\epsilon$ *poles to higher loops*

MHV scattering amplitudes/Wilson loop duality II



MHV amplitudes are dual to light-like Wilson loops

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} \sim \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like polygon}$$

- ✔ At strong coupling, agrees with the BDS ansatz for $n = 4$ [Alday,Maldacena]
- ✔ At weak coupling, the duality was verified against the BDS ansatz for:
  - ✗ $n = 4$ (rectangle) to two loops [Drummond,Henn,Korchemsky,ES]
  - ✗ $n \geq 5$ to one loop [Brandhuber,Heslop,Travaglini]
  - ✗ $n = 5,\ 6$ to two loops [Drummond,Henn,Korchemsky,ES]
  - ✗ results on WL for $n \geq 7$ to two loops [Anastasiou,Brandhuber,Heslop,Khoze,Spence,Travaglini]

Wilson loops match the BDS ansatz for $n = 4, 5$ *but not for* $n \geq 6$

Dual conformal anomaly

Dual conformal symmetry of the amplitudes $\Leftrightarrow$ Conformal symmetry of Wilson loops

Dual conformal anomaly $\Leftrightarrow$ Conformal anomaly of Wilson loops

- ✔ Why do Wilson loops have a conformal anomaly in $\mathcal{N}=4$ SYM?

  - ✗ Were the Wilson loop well defined (= finite) in $D=4$ dimensions, it would be conformally invariant

$$W(C_n)=W(C'_n)$$

  - ✗ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C'_n) \times \text{[cusp anomaly]}$$

- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$\ln W(C_n) = F_n^{(WL)} + \text{[UV divergences]} + O(\epsilon)$$

Under special conformal transformations (boosts), to all orders, [Drummond,Henn,Korchemsky,ES]

$$K^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\text{cusp}}(\lambda) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of light-like Wilson loops

The consequences of the conformal Ward identity for the finite part of the Wilson loop W_n

- ✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)
  $\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(\lambda)\ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(\lambda)\sum_{i=1}^{5}\ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right)\ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

  Exactly the BDS ansatz for the 4- and 5-point MHV amplitudes!

- ✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x^2_{13}x^2_{46}}{x^2_{14}x^2_{36}}, \qquad u_2 = \frac{x^2_{24}x^2_{15}}{x^2_{25}x^2_{14}}, \qquad u_3 = \frac{x^2_{35}x^2_{26}}{x^2_{36}x^2_{25}}$$

  General solution of the Ward identity contains *an arbitrary function* of the conformal cross-ratios.

- ✔ Crucial test: go to six points at two loops where the answer is not determined by conformal symmetry [Drummond,Henn,Korchemsky,ES] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

$$F_6^{(\text{WL})} = F_6^{(\text{MHV})} \neq F_6^{(\text{BDS})}$$

  The Wilson loop/scattering amplitude duality holds at $n = 6$ to two loops!

Dual conformal symmetry beyond MHV I

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} + \mathcal{A}_n^{\rm NMHV} + \mathcal{A}_n^{\rm N^2MHV} + \ldots + \mathcal{A}_n^{\overline{\rm MHV}}$$

- ✔ The tree superamplitude $\mathcal{A}_n^{(\rm tree)}$ is covariant under dual superconformal transformations
- ✔ At loop level, this symmetry becomes anomalous due to IR divergences
- ✔ The dual superconformal symmetry is restored in the ratio of superamplitudes $\mathcal{A}_n$ and $\mathcal{A}_n^{\rm MHV}$

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} \times \Big[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\Big]$$

The ratio

$$R_n = 1 + R_n^{\rm NMHV} + R_n^{\rm N^2MHV} + \ldots$$

is *IR finite* and, most importantly, it is *(super)conformally invariant* ! [Drummond,Henn,Korchemsky,ES]

- ✔ Wilson loop/superamplitude duality involves a new ingredient

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A)/W_n(x_i) = \mathcal{A}_n^{\rm MHV\,(tree)} \times \big[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\big]$$

The Wilson loop $W_n(x_i)$ takes care of the anomalous contribution

The 'ratio function' R_n is a dual (super)conformal invariant

What is the operator definition of the dual superconformal invariant R_n ?

Dual conformal symmetry beyond MHV II

- ✔ One-loop NMHV superamplitudes
  - ✗ n–gluon one-loop NMHV known [Bern,Dixon,Kosower'04]
  - ✗ New result: [Drummond,Henn,Korchemsky,ES'08]

    One-loop NMHV superamplitude $\Leftrightarrow$ dual (super)conformal invariant

$$R_n^{\rm NMHV} = \sum_{p,q,r=1}^{n} M_{pqr}(x_{ij}) \frac{\langle q-1\, q\rangle \langle r-1\, r\rangle\, \delta^{(4)}(\langle p|x_{pq}x_{qr}|\theta_{rp}\rangle + \langle p|x_{pr}x_{rq}|\theta_{qp}\rangle)}{x_{qr}^2 \langle p|x_{pr}x_{r\,q-1}|q-1\rangle \langle p|x_{pr}x_{r\,q}|q\rangle \langle p|x_{pq}x_{q\,r-1}|r-1\rangle \langle p|x_{pq}x_{q\,r}|r\rangle}$$

  - ✗ The helicity structure is invariant under both dual and ordinary CSUSY
  - ✗ At tree level all $M_{pqr} = 1$, why?
  - ✗ At loop level all $M_{pqr}(x_{ij}) = 1 + g^2 M_{pqr}^{(\rm one-loop)} + ?\, O(g^4)$ are dual conformal invariants made of finite combinations of one-loop scalar box integrals. But they are not superconformal, why?

- ✔ All N^{p}MHV tree-level superamplitudes are obtained from BCFW recursion [Drummond,Henn'08]
  - ✗ Each term in them is both dual and ordinary superconformal [Korchemsky,ES]
  - ✗ What fixes the relative coefficients? Unitarity?

Conclusions and recent developments

- ✔ MHV amplitudes in $\mathcal{N}=4$ SYM theory
  - ✗ has dual conformal symmetry both at weak and at strong coupling
  - ✗ dual to light-like Wilson loops
- ✔ This symmetry is a part of dual superconformal symmetry of all planar superamplitudes in $\mathcal{N}=4$ SYM
  - ✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)
  - ✗ Imposes non-trivial constraints on the loop corrections
- ✔ Dual superconformal symmetry is now explained through the AdS/CFT correspondence by a combined bosonic [Kallosh,Tseytlin] and fermionic T duality symmetry [Berkovits,Maldacena], [Beisert,Ricci,Tseytlin,Wolf]
- ✔ The closure of ordinary and dual superconformal symmetry has a Yangian structure - integrability? [Drummond,Henn,Plefka'09]
- ✔ What is the generalization of the Wilson loop/amplitude duality beyond MHV?