Double Field Theory, String Field Theory and T-Duality

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What is string theory?

- Supergravity limit - misses stringy features
- Winding modes, T-duality, cocycles, algebraic structure not Lie algebra,...
- On torus $\{x^a\}$ extra dual coordinates $\{\tilde{x}_a\}$
- String field theory: interactions, T-duality
- Double field theory on doubled torus

$$g_{ab}(x^a, \tilde{x}_a), \; b_{ab}(x^a, \tilde{x}_a), \; \phi(x^a, \tilde{x}_a)$$

Earlier versions: Siegel, Tseytlin

Strings on a Torus

$$\mathbb{R}^{n-1,1} \times T^d$$

- Coordinates $x^i = (y^\mu, x^a)$ $x^a \sim x^a + 2\pi$
- Momentum $p_i = (k_\mu, p_a)$
- Winding w^a $(p_a, w^a) \in \mathbb{Z}^{2d}$
- Fourier transform $(k_\mu, p_a, w^a) \to (y^\mu, x^a, \tilde{x}_a)$
- Doubled Torus $\mathbb{R}^{n-1,1} \times T^{2d}$ $\tilde{x}_a \sim \tilde{x}_a + 2\pi$
- String Field Theory gives infinite set of fields $\phi(y^\mu, x^a, \tilde{x}_a)$

$$n + d = D = 26 \ or \ 10$$

T-Duality

- Interchanges momentum and winding
- Equivalence of string theories on dual backgrounds with very different geometries
- String field theory symmetry, provided fields depend on both $x, \tilde{x}$ Kugo, Zwiebach
- For fields $\psi(y)$ not $\psi(y, x, \tilde{x})$ Buscher
- Aim: generalise to fields $\psi(y, x, \tilde{x})$

Generalised T-duality Dabholkar & CMH

Free field equn, M mass in D dimensions

$$M^2 \equiv -(k^2+p^2+w^2) = \frac{2}{\alpha'}(N+\bar{N}-2)$$

Constraint

$$L_0 - \bar{L}_0 = N - \bar{N} - p_a w^a = 0$$

Massless states $N = \tilde{N} = 1 \qquad M^2 = 0 \qquad p_a w^a = 0$

Constrained fields $\phi(y, x, \tilde{x})$

$$\Delta\phi = 0 \qquad\qquad \Delta \equiv -\frac{2}{\alpha'}\frac{\partial}{\partial x^a}\frac{\partial}{\partial \tilde{x}_a}$$

$$h_{ij}(y^\mu, x^a, \tilde{x}_a),\ b_{ij}(y^\mu, x^a, \tilde{x}_a),\ d\,(y^\mu, x^a, \tilde{x}_a)$$

$$h_{ij} \to \{h_{\mu\nu}, h_{\mu a}, h_{ab}\}$$

Torus Backgrounds

$$\alpha' = 1$$

$$G_{ij} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix} \qquad E_{ij} \equiv G_{ij} + B_{ij}$$

$$x^i = \{y^\mu, x^a\} \qquad \tilde{x}_i = \{\tilde{y}_\mu, \tilde{x}_a\} = \{0, \tilde{x}_a\}$$

Left and Right Derivatives

$$D_i = \frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}, \quad \bar{D}_i = \frac{\partial}{\partial x^i} + E_{ki}\frac{\partial}{\partial \tilde{x}_k}$$

$$\Delta = \frac{1}{2}(D^2 - \bar{D}^2) = -2\frac{\partial}{\partial \tilde{x}_i}\frac{\partial}{\partial x^i}$$

$$\Box = \frac{1}{2}(D^2 + \bar{D}^2) \qquad D^2 = G^{ij} D_i D_j$$

Kinetic Operator

$$\Box = \frac{1}{2}(D^2 + \bar{D}^2) = \partial^t \mathcal{H}(E) \partial$$

$$\partial = \begin{pmatrix} \frac{\partial}{\partial \tilde{x}_i} \\ \frac{\partial}{\partial x^j} \end{pmatrix}$$

$$E_{ij} \equiv G_{ij} + B_{ij}$$

$$D \times D$$

Generalised Metric

$$2D \times 2D$$

$$\mathcal{H}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

Closed String Field Theory

Matter CFT + Ghost CFT: General State

$$|\Psi\rangle = \sum_I \int dk \sum_{p,w} \psi_I(k,p,w)\, \mathcal{V}^I |k,p,w\rangle$$

or in position space

$$|\Psi\rangle = \sum_I \int [dy dx d\tilde{x}]\, \psi_I(y,x,\tilde{x})\, \mathcal{V}^I |y,x,\tilde{x}\rangle$$

Vertex operators, ghost number 2 $\mathcal{V}^I$

Infinite set of fields $\psi_I(y,x,\tilde{x})$

SFT gives action for component fields

Closed String Field Theory Zwiebach

$$S = \frac{1}{2}\langle\Psi|c_0^- \, Q|\Psi\rangle + \frac{1}{3!}\{\Psi,\Psi,\Psi\} + \frac{1}{4!}\{\Psi,\Psi,\Psi,\Psi\} + \cdots$$

Symmetry $\delta\Psi = Q\Lambda + [\Lambda,\Psi] + \ldots$

String fields ghost number 2, parameters $|\Lambda\rangle$ ghost number 1 are constrained:

$$(L_0 - \bar{L}_0)|\Psi\rangle = 0, \quad (b_0 - \bar{b}_0)|\Psi\rangle = 0,$$
$$(L_0 - \bar{L}_0)|\Lambda\rangle = 0, \quad (b_0 - \bar{b}_0)|\Lambda\rangle = 0,$$

String Products $[A,B],\ [A,B,C],\ [A,B,C,D],\ldots$

$$\{\Psi,\Psi,\ldots,\Psi\} = \langle\Psi|c_0^-[\Psi,\ldots,\Psi]\rangle$$

$$[\Psi_1, \Psi_2] \equiv \int \frac{d\theta}{2\pi} e^{i\theta(L_0 - \bar{L}_0)} b_0^- [\Psi_1, \Psi_2]'$$

[A,B]' inserts the states A,B in 3-punctured sphere that defines the vertex

$$[A, B] = (-)^{AB} [B, A]$$

Graded, like a super-Lie bracket

$$\begin{aligned} & [A, [B, C]] \pm [B, [C, A]] \pm [C, [A, B]] \\ = \quad & Q[A, B, C] \pm [QA, B, C] \pm [A, QB, C] \pm [A, B, QC] \end{aligned}$$

Failure of graded Jacobi = failure of Q to be a derivation

Homotopy Lie Alegebra

Massless Fields

$$|\Psi\rangle = \int [dp]\ \Big(-\frac{1}{2}e_{ij}(p)\,\alpha^i_{-1}\bar{\alpha}^j_{-1}\,c_1\bar{c}_1$$

$$+\,d(p)\,(c_1c_{-1} - \bar{c}_1\bar{c}_{-1}) + \ldots\Big)|p\rangle$$

$$|\Lambda\rangle = \int [dp]\ \Big(i\lambda_i(p)\,\alpha^i_{-1}c_1 - i\bar{\lambda}_i(p)\,\bar{\alpha}^i_{-1}\bar{c}_1 + \mu(p)\,c_0^+\Big)|p\rangle$$

- Use in action, gauge transformations
- Fix μ symmetry, eliminate auxiliary fields
- Gives action and symmetries for $e_{ij} = h_{ij} + b_{ij},\ d$
- Background $E_{ij} = G_{ij} + B_{ij}$

Quadratic Action

$$S^{(2)} = \int [dxd\tilde{x}] \Big[\frac{1}{2} e_{ij} \Box e^{ij} + \frac{1}{4} (\bar{D}^j e_{ij})^2 + \frac{1}{4} (D^i e_{ij})^2$$
$$-2\, d\, D^i \bar{D}^j e_{ij} \; - \; 4\, d\, \Box\, d \Big]$$

Invariant under

$$\delta e_{ij} = \; \bar{D}_j \lambda_i + D_i \bar{\lambda}_j \, ,$$

$$\delta d \; = -\frac{1}{4} D \cdot \lambda - \frac{1}{4} \bar{D} \cdot \bar{\lambda}$$

using constraint $\quad \Delta \lambda = \Delta \bar{\lambda} = 0$

Discrete Symmetry

$$e_{ij} \to e_{ji} \, , \; D_i \to \bar{D}_i \, , \; \bar{D}_i \to D_i \, , \; d \to d$$

Comparison with Conventional Actions

Take $\quad B_{ij} = 0 \qquad \tilde{\partial}_i \equiv G_{ik} \dfrac{\partial}{\partial \tilde{x}_k}$

$$D_i = \partial_i - \tilde{\partial}_i \,, \quad \bar{D}_i = \partial_i + \tilde{\partial}_i$$

$$\square = \partial^2 + \tilde{\partial}^2 \quad \Delta = -2\,\partial_i \tilde{\partial}^i$$

$$e_{ij} = h_{ij} + b_{ij}$$

Usual quadratic action $\quad \displaystyle\int dx \; L[\,h, b, d; \partial\,]$

$$L[\,h, b, d; \partial\,] = \; \frac{1}{4} h_{ij} \partial^2 h_{ij} + \frac{1}{2} (\partial^j h_{ij})^2 - 2\, d\, \partial^i \partial^j \, h_{ij}$$

$$-4\, d\, \partial^2\, d + \frac{1}{4} b_{ij} \partial^2 b_{ij} + \frac{1}{2} (\partial^j b_{ij})^2$$

Double Field Theory Action

$$S^{(2)} = \int [dx d\tilde{x}] \; \Big[\; L[\,h, b, d; \partial\,] + L[\,h, b, -d; \tilde{\partial}\,]$$

$$+ (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\, d\, \partial^i \tilde{\partial}^j b_{ij}$$

Action + dual action + strange mixing terms

$$\begin{aligned} \delta h_{ij} &= \quad \partial_i \epsilon_j + \partial_j \epsilon_i \; + \; \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \,, \\ \delta b_{ij} &= -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i)\,, \\ \delta d \; &= -\,\partial \cdot \epsilon + \tilde{\partial} \cdot \tilde{\epsilon}\,. \end{aligned}$$

Diffeos and B-field transformations mixed

Dilaton

$$\delta h_{ij} = \quad \partial_i \epsilon_j + \partial_j \epsilon_i + \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \,,$$

$$\delta d = - \partial \cdot \epsilon + \tilde{\partial} \cdot \tilde{\epsilon} \,.$$

$\phi = d + \frac{1}{4} \eta^{ij} h_{ij}$ invariant under ϵ transformation

In non-linear theory d is a density, dilaton scalar is ϕ

$$e^{-2d} = e^{-2\phi} \sqrt{-g}$$

$\tilde{\phi} = d - \frac{1}{4} \eta^{ij} h_{ij}$ invariant under $\tilde{\epsilon}$ transformation

Dual dilaton. Under T-duality d is invariant, $\phi \to \tilde{\phi}$

Cubic Terms in Action

$$\int [dx d\tilde{x}] \Big[\; 4\, e_{ij} (D^i \bar{D}^j d) d + 4\, d^2 \,\Box\, d$$

$$+ \frac{1}{4}\, e_{ij} \Big(\, (D^i e_{kl})(\bar{D}^j e^{kl}) - (D^i e_{kl})\, (\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl}) \Big)$$

$$+ \frac{1}{2} d \, \Big(2 e^{ij} (\bar{D}_j \bar{D}^k e_{ik} + D_i D^k e_{kj}) + \frac{1}{2} (D_k e_{ij})^2 + \frac{1}{2} (\bar{D}_k e_{ij})^2$$

$$+ \; (D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 \Big) \; \Big]$$

$$\delta_\lambda e_{ij} = \bar{D}_j \lambda_i + \frac{1}{2} \Big[(D_i \lambda^k) e_{kj} - (D^k \lambda_i) e_{kj} + \lambda_k D^k e_{ij} \Big]$$

$$\delta_\lambda d = -\frac{1}{4} D \cdot \lambda + \frac{1}{2} (\lambda \cdot D)\, d$$

action invariant
to this order

Linearised Symmetries: diffeos on doubled space?

$$
\begin{aligned}
\delta h_{ij} &= \quad \partial_i \epsilon_j + \partial_j \epsilon_i \; + \; \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \,, \\
\delta b_{ij} &= -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i) \,, \\
\delta d \; &= - \, \partial \cdot \epsilon + \tilde{\partial} \cdot \tilde{\epsilon} \,.
\end{aligned}
$$

Non-linear terms & algebra $\Rightarrow$ NOT doubled diffeos

Diffeos after field redefs

$$e^{\pm}_{ij} \equiv e_{ij} \pm \frac{1}{2} e_i{}^k e_{kj} + O(e^3)$$

For fields independent of $\tilde{x}$, δe^{+}_{ij} gives ϵ diffeos

For fields independent of x, δe^{-}_{ij} gives $\tilde{\epsilon}$ diffeos

No field redef can give both kinds of diffeo

T-Duality Transformations of Background

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d;\mathbb{Z})$$

T-duality

$$E' = (aE+b)(cE+d)^{-1}$$

$$X \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

transforms as a vector

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}$$

T-Duality is a Symmetry of the Action

Fields $e_{ij}(x,\tilde{x}),\ d(x,\tilde{x})$

Background E_{ij}

$$E' = (aE+b)(cE+d)^{-1}$$

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}$$

Action invariant if:

$$e_{ij}(X) = M_i{}^k\, \bar{M}_j{}^l\, e'_{kl}(X') \qquad M \equiv d^t - E\, c^t$$

$$d(X) = d'(X') \qquad \bar{M} \equiv d^t + E^t c^t$$

Conjecture for full non-linear transformations:

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}$$

$$\mathcal{E} = E + e$$

$$d'(X') = d(X)$$

Linearising in e_{ij} gives previous result

Projectors and Cocycles

Naive product of constrained fields does not satisfy constraint

$$L_0^- \Psi_1 = 0, L_0^- \Psi_2 = 0 \quad \text{but} \quad L_0^- (\Psi_1 \Psi_2) \neq 0$$

$$\Delta A = 0, \Delta B = 0 \qquad \text{but} \qquad \Delta(AB) \neq 0$$

String product has explicit projection

$$[\Psi_1, \Psi_2] \equiv \int \frac{d\theta}{2\pi} e^{i\theta(L_0 - \bar{L}_0)} b_0^- [\Psi_1, \Psi_2]'$$

Double field theory requires projections, novel forms

SFT has non-local cocycles in vertices, DFT should too

Cocycles and projectors not needed in cubic action

Double Field Theory

- New limit of strings, captures some of the magic of string theory
- Constructed cubic action, quartic will have new stringy features
- T-duality, cocycles, homotopy Lie, constraints
- Simpler than SFT, can address stringy issues in simpler setting
- Generalised Geometry doubles Tangent space, DFT doubles coordinates. Geometry?