Black Hole Partition Functions and Duality

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Summary of Talk

The OSV conjecture (2004) states that the microstates of any $N = 2, D = 4$ BPS black hole are captured by the topological string:

$$Z_{BH} = |Z_{top}|^2 \tag{1}$$

However:

- Duality invariance/covariance not manifest.
- Black hole degeneracies are sometimes captured by genus 2 Siegel modular forms. Complicated objects, do not lead to (1).

Need to change the OSV relation into

$$Z_{BH} \propto |Z_{top}|^2$$

to make it compatible with duality invariance.
Will encounter non-holomorphic deformation of special geometry.
Relation LEEA $\leftrightarrow$ topological string amplitudes more subtle than previously envisioned.

OSV Proposal

Ooguri + Strominger + Vafa, hep-th/0405146

Let's consider

- BPS black holes in four-dimensional $N = 2$ supergravity theories,
- dyonic, with electric/magnetic charges (q, p), single-center

Define a mixed black hole partition function Z_{BH} in terms of black hole microstate degeneracies $d(p, q)$ (a suitable index),

$$Z_{BH}(p, \phi) = \sum_q d(p, q)\, \mathrm{e}^{\pi q\, \phi} \quad \xrightarrow{\text{ILPT}} \quad d(p, q) = \int d\phi\, Z_{BH}(p, \phi)\, \mathrm{e}^{-\pi\, q\, \phi}$$

Here, ϕ are the electrostatic potentials. OSV proposal:

$$Z_{BH} \equiv \mathrm{e}^{4\pi\, \mathrm{Im}\, F_{top}} \ ,$$

with F_{top} topological free energy. If true,

$$d(p, q) = \int d\phi\, \mathrm{e}^{\pi [4\, \mathrm{Im}\, F_{top} - q\, \phi]} \ ,$$

universal formula in terms of topological string data.

Duality Symmetries

Weak topological string coupling g_{top}:

$$F_{top}(g_{top}, z^A) = \sum_{g=0}^{\infty} g_{top}^{2g-2}\, F_g(z^A) \quad , \quad \text{holomorphic}$$

IIA: z^A Kähler class moduli of Calabi-Yau threefold.

F_g's enter in the Wilsonian action as follows:

- metric on Kähler class moduli space is computed from F_0
- higher F_g's ($g \geq 1$) are coupling functions for higher-curvature terms proportional to the square of the Weyl tensor.

$N = 2$ theory may have duality symmetries. Duality invariance requires the F_g's ($g \geq 1$) to acquire non-holomorphic corrections:

- needed in the LEEA to make symmetries of the theory manifest;
- encoded in the holomorphic anomaly equations of the topological string.

This Talk

- Work at weak topological string coupling g_{top}.
- Describe a method to include non-holomorphic corrections into the OSV proposal, necessary for duality covariance. Suggests consistent non-holomorphic deformation of special geometry. Departure from topological string.
- Use saddle-point arguments to infer measure factor in OSV integral.
- Confront with proposal for microstate degeneracy in a specific $N = 2$ model, the S-T-U model.

  `A. Sen + C. Vafa, hep-th/9508064 ,   J. David, arXiv:0711.1971`

  Agreement!

With `Justin David, Bernard de Wit and Swapna Mahapatra, arXiv:0810.1233`
`Bernard de Wit and Swapna Mahapatra, arXiv:0808.2627`
`Bernard de Wit, Jürg Käppeli and Thomas Mohaupt, hep-th/0601108.`

BPS black holes in $D = 4, N = 2$: extremal, supported by (VM) complex scalar fields Y^I $(I = 0, \ldots, n)$, charges (p^I, q_I).

Calabi-Yau compactifications: Wilsonian Lagrangian contains higher-curvature interactions $\propto$ Weyl^2 $\rightarrow$ encoded in holomorphic homogeneous function $F(Y, \Upsilon)$. Here Υ is the Weyl background.

Υ-expansion $F(Y, \Upsilon) = \sum_{g=0}^{\infty} (Y^0)^{2-2g}\, \Upsilon^g\, F_g \longrightarrow$ topol. string

Attractor mechanism: `Ferrara, Kallosh, Strominger`

$$\text{at horizon} \qquad Y^I \rightarrow Y^I_{\rm Hor}(p, q) \quad , \quad \Upsilon \rightarrow -64$$

Attractor equations (in the presence of Weyl^2):

$$\begin{aligned} Y^I - \bar{Y}^{\bar{I}} &= i\, p^I \quad , \quad \text{magnetic} \quad , \quad F_I = \partial F(Y, \Upsilon)/\partial Y^I \\ F_I - \bar{F}_{\bar{I}} &= i\, q_I \quad , \quad \text{electric} \quad , \quad F_\Upsilon = \partial F(Y, \Upsilon)/\partial \Upsilon \end{aligned}$$

Electro/magnetostatic potentials: $Y^I + \bar{Y}^{\bar{I}} \quad , \quad F_I + \bar{F}_{\bar{I}}$

Variational Principle

Attractor equations can be obtained from a variational principle, based on a BPS entropy function Σ:

$$\Sigma(Y,\bar{Y},p,q) = \mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - q_I(Y^I + \bar{Y}^{\bar{I}}) + p^I(F_I + \bar{F}_{\bar{I}}) \ ,$$

where $\mathcal{F}$ is the free energy

$$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = -i\left(\bar{Y}^{\bar{I}} F_I - Y^I \bar{F}_{\bar{I}}\right) - 2i\left(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_{\bar{\Upsilon}}\right)$$

Stationary points: set $\Upsilon = -64$

$$\begin{aligned} \delta\Sigma &= i(Y^I - \bar{Y}^{\bar{I}} - ip^I)\,\delta(F_I + \bar{F}_{\bar{I}}) - i(F_I - \bar{F}_{\bar{I}} - iq_I)\,\delta(Y^I + \bar{Y}^{\bar{I}}) \\ \delta\Sigma &= 0 \longleftrightarrow \text{attractor equations} \end{aligned}$$

At attractor point, get macroscopic (Wald's) entropy:

$$\pi\Sigma|_{\text{attractor}} = \mathcal{S}_{\text{macro}}(p,q)$$

So far, Wilsonian.

Duality Transformations: Entanglement with Υ!

Charges (p^I, q_I) undergo duality transformations. If these constitute symmetries of LEEA, macroscopic entropy is invariant under them.

These leave Υ invariant, but act as symplectic $Sp(2n+2, Z)$ transformations on the vector $(Y^I, F_I(Y, \Upsilon))$ in the attractor equations. Entanglement with the Weyl background!

Departure from topological string approach.

Precise form of $N = 2$ LEEA not known $\longrightarrow$ cannot rely on an action principle to incorporate non-holomorphic corrections needed for duality invariance. Instead, demand:

- attractor equations retain their form;
- they follow from a variational principle based on free energy $\mathcal{F}$,

$$\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = -i \left(\bar{Y}^{\bar{I}} F_I - Y^I \bar{F}_{\bar{I}} \right) - 2i \left(\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_{\bar{\Upsilon}} \right) ,$$

but now based on a general function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, not necessarily holomorphic.

Non-Holomorphic Corrections

Variation of $\mathcal{F}$: with the attractor value $\Upsilon = -64$

$$\begin{aligned} \delta\mathcal{F} &= i(Y^I - \bar{Y}^{\bar{I}})\,\delta(F_I + \bar{F}_{\bar{I}}) - i(F_I - \bar{F}_{\bar{I}})\,\delta(Y^I + \bar{Y}^{\bar{I}}) \\ &\quad - \left[i\left(2\Upsilon\,\delta F_\Upsilon + Y^I\,\delta F_I - F_I \delta Y^I\right) + c.c. \right] \end{aligned}$$

Vanishing of second line: at least two solutions, namely

- F holomorphic, $F(\lambda Y, \lambda^2 \Upsilon^2) = \lambda^2 F(Y, \Upsilon)$, usual Wilsonian case
- $F = 2i\,\Omega$, Ω real, $\Omega(\lambda Y, \lambda \bar{Y}, \lambda^2 \Upsilon, \lambda^2 \bar{\Upsilon}) = \lambda^2\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$

  Without loss of generality:

$$F = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$$

When Ω is harmonic (i.e. $\Omega =$ holo + anti-holo), get back usual Wilsonian case.

Consistent Deformation of Special Geometry?

Second option seems to be a consistent non-holomorphic deformation of special geometry, e.g.:

- under symplectic (duality) transformations, $(Y^I, F_I) \to (\tilde{Y}^I, \tilde{F}_I)$; can show that

$$\tilde{F}_I = \frac{\partial \tilde{F}}{\partial \tilde{Y}^I} \;;$$

  $(\mathcal{L}, F)$ and $(\tilde{\mathcal{L}}, \tilde{F})$ in same equivalence class;
- in addition, can show that

$$F_\Upsilon \longrightarrow F_\Upsilon \;,$$

  i.e. F_Υ transforms as a scalar.

  It follows that $\mathcal{F}$ and Σ are invariant under duality transformations that define a symmetry.

Duality Invariant OSV Integral

Consider the following duality invariant integral, expressed in terms of entropy function Σ, with the attractor value $\Upsilon = -64$,

$$\int d(Y^I + \bar{Y}^{\bar{I}})\, d(F_I + \bar{F}_{\bar{I}})\, \mathrm{e}^{\pi\Sigma(Y,\bar{Y},p,q)} = \int dY\, d\bar{Y}\, \Delta^-(Y,\bar{Y})\, \mathrm{e}^{\pi\Sigma(Y,\bar{Y},p,q)}$$

Duality covariance requires measure factor

$$\Delta^- = \left|\det\left[\mathrm{Im}\left[F_{JK} - F_{J\bar{K}}\right]\right]\right|$$

Evaluate integral in saddle-point approximation about attractor point:

$$\mathrm{e}^{\pi\Sigma|_{\mathrm{attractor}}} = \mathrm{e}^{\mathcal{S}_{\mathrm{macro}}(p,q)} \qquad \text{for large charges}$$

Duality invariant. Expect saddle-point approximation to hold for dyonic black holes.

Suggests to identify the above with $d(p,q)$, as in OSV.

Prediction for Mixed Partition Function

On the other hand, when only integrating over $Y^I - \bar{Y}^{\bar{I}}$ in saddle-point approximation so that $Y^I = (\phi^I + ip^I)/2$, get modified OSV-type integral,

$$d(p,q) = \int d\phi \, \sqrt{\Delta^-(p,\phi)} \, \mathrm{e}^{\pi \left[\mathcal{F}_E(p,\phi) - q_I \, \phi^I\right]} \, ,$$

where

$$\mathcal{F}_E = 4 \left[\mathrm{Im} F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})\right] |_{Y^I = (\phi^I + ip^I)/2 \quad , \quad \Upsilon = -64}$$

Inverting yields prediction for $N = 2$ mixed black hole partition function,

$$Z_{BH}(p,\phi) = \sum_q d(p,q) \, \mathrm{e}^{\pi q_I \, \phi^I} = \sqrt{\Delta^-} \; \mathrm{e}^{\pi \mathcal{F}_E} = \sqrt{\Delta^-} \; \mathrm{e}^{4\pi \Omega^{\mathrm{nonholo}}} \; \mathrm{e}^{\pi \mathcal{F}_E^{\mathrm{holo}}}$$

Test requires knowledge of microscopic degeneracies.

$N = 2$ S-T-U Model

Specific $N = 2$ model with exact duality symmetry, $\Gamma(2) \in SL(2, Z)$:

S-T-U-model, A. Sen + C. Vafa, hep-th/9508064

Demanding (Y^I, F_I) to transform accordingly,

- S-T-Υ-mixing. Under S-duality,

$$T^a \longrightarrow T^a + \frac{ic}{\Delta_S\,(Y^0)^2}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b} \quad , \quad \Delta_S = icS + d$$

- yields Ω with non-holomorphic corrections that is related, but not identical, to the solution of the non-holomorphic anomaly equation of TS. Due to entanglement with Weyl background.

Dyonic microstate degeneracy in terms of Siegel modular form of weight zero. J. David, arXiv:0711.1971

$$Z_{BH}(p,\phi) = \sum_q d(p,q)\, \mathrm{e}^{\pi q_I\, \phi^I} = \sqrt{\Delta^-}\; \mathrm{e}^{4\pi\Omega^{\text{nonholo}}}\; \mathrm{e}^{\pi\mathcal{F}_E^{\text{holo}}}$$

Agreement (up to certain level of accuracy)!

Conclusions

- We made a proposal for a measure factor in the OSV-integral that is compatible with duality covariance in $N = 2$ models.
- In doing so, we proposed a method for incorporating non-holomorphic terms needed for duality invariance. We found indications that this is a consistent non-holomorphic deformation of special geometry.

  LEEA encoded in a non-holomorphic function F not known. Precise relationship between LEEA (1PI graphs) and topological string (connected graphs) remains to be worked out.
- Our proposal yields results in agreement with direct calculations of Z_{BH} in $N = 4$ models, and also in the $N = 2$ S-T-U-model.

Thanks!