Black Hole Partition Functions and Duality

Gabriel Lopes Cardoso

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The **OSV conjecture** (2004) states that the microstates of any $N = 2, D = 4$ BPS black hole are captured by the topological string:

$$Z_{BH} = |Z_{top}|^2 \quad (1)$$

However:

- **Duality invariance/covariance** not manifest.
- Black hole degeneracies are sometimes captured by genus 2 Siegel modular forms. Complicated objects, do not lead to (1).

Need to change the OSV relation into

$$Z_{BH} \propto |Z_{top}|^2$$

to make it compatible with duality invariance.

Will encounter **non-holomorphic deformation** of special geometry. Relation LEEA $\leftrightarrow$ topological string amplitudes more subtle than previously envisioned.
Let’s consider
- BPS black holes in four-dimensional $N = 2$ supergravity theories,
- dyonic, with electric/magnetic charges $(q, p)$, single-center

Define a mixed black hole partition function $Z_{BH}$ in terms of black hole microstate degeneracies $d(p, q)$ (a suitable index),

\[ Z_{BH}(p, \phi) = \sum_q d(p, q) e^{\pi q \phi} \quad \text{ILPT} \quad d(p, q) = \int d\phi Z_{BH}(p, \phi) e^{-\pi q \phi} \]

Here, $\phi$ are the electrostatic potentials. **OSV proposal:**

\[ Z_{BH} \equiv e^{4\pi \text{Im} F_{top}}, \]

with $F_{top}$ topological free energy. If true,

\[ d(p, q) = \int d\phi e^{\pi [4 \text{Im} F_{top} - q \phi]}, \]

universal formula in terms of topological string data.
Duality Symmetries

Weak topological string coupling $g_{\text{top}}$:

$$F_{\text{top}}(g_{\text{top}}, z^A) = \sum_{g=0}^{\infty} g_{\text{top}}^{2g-2} F_g(z^A), \quad \text{holomorphic}$$

IIA: $z^A$ Kähler class moduli of Calabi-Yau threefold.

$F_g$’s enter in the Wilsonian action as follows:

- metric on Kähler class moduli space is computed from $F_0$
- higher $F_g$’s $(g \geq 1)$ are coupling functions for higher-curvature terms proportional to the square of the Weyl tensor.

$N = 2$ theory may have duality symmetries. Duality invariance requires the $F_g$’s $(g \geq 1)$ to acquire non-holomorphic corrections:

- needed in the LEEA to make symmetries of the theory manifest;
- encoded in the holomorphic anomaly equations of the topological string.
Work at weak topological string coupling $g_{\text{top}}$.

Describe a method to include non-holomorphic corrections into the OSV proposal, necessary for duality covariance. Suggests consistent non-holomorphic deformation of special geometry. Departure from topological string.

Use saddle-point arguments to infer measure factor in OSV integral.

Confront with proposal for microstate degeneracy in a specific $N = 2$ model, the S-T-U model.


Agreement!

With Justin David, Bernard de Wit and Swapna Mahapatra, arXiv:0810.1233

Bernard de Wit and Swapna Mahapatra, arXiv:0808.2627

Bernard de Wit, Jürg Käppeli and ThomasMohaupt, hep-th/0601108.
BPS Black Holes

BPS black holes in $D = 4, N = 2$: extremal, supported by (VM) complex scalar fields $Y^I$ ($I = 0, \ldots, n$), charges $(p^I, q_I)$.

Calabi-Yau compactifications: Wilsonian Lagrangian contains higher-curvature interactions $\propto \text{Weyl}^2 \rightarrow$ encoded in holomorphic homogeneous function $F(Y, \Upsilon)$. Here $\Upsilon$ is the Weyl background. $\Upsilon$-expansion $F(Y, \Upsilon) = \sum_{g=0}^{\infty} (Y^0)^{2-2g} \Upsilon^g F_g \rightarrow$ topol. string

Attractor mechanism: Ferrara, Kallosh, Strominger

at horizon $Y^I \rightarrow Y^I_{\text{Hor}}(p, q)$, $\Upsilon \rightarrow -64$

Attractor equations (in the presence of Weyl$^2$):

\[
Y^I - \bar{Y}^\bar{I} = i p^I, \quad \text{magnetic}, \quad F_I = \partial F(Y, \Upsilon)/\partial Y^I
\]
\[
F_I - \bar{F}_{\bar{I}} = i q_I, \quad \text{electric}, \quad F_\Upsilon = \partial F(Y, \Upsilon)/\partial \Upsilon
\]

Electro/magnetostatic potentials:

$Y^I + \bar{Y}^\bar{I}$, $F_I + \bar{F}_{\bar{I}}$
Variational Principle

Attractor equations can be obtained from a variational principle, based on a BPS entropy function $\Sigma$:

$$\Sigma(Y, \bar{Y}, p, q) = \mathcal{F}(Y, \bar{Y}, \gamma, \bar{\gamma}) - q(\gamma^l + \bar{\gamma}^l) + p(F_l + \bar{F}_l),$$

where $\mathcal{F}$ is the free energy

$$\mathcal{F}(Y, \bar{Y}, \gamma, \bar{\gamma}) = -i \left( \bar{Y}^l F_l - Y^l \bar{F}_l \right) - 2i \left( \gamma F_\gamma - \bar{\gamma} \bar{F}_{\bar{\gamma}} \right)$$

Stationary points: set $\gamma = -64$

$$\delta \Sigma = i(Y^l - \bar{Y}^l - ip^l) \delta(F_l + \bar{F}_l) - i(F_l - \bar{F}_l - iq^l) \delta(Y^l + \bar{Y}^l)$$

$$\delta \Sigma = 0 \iff \text{attractor equations}$$

At attractor point, get macroscopic (Wald’s) entropy:

$$\pi \Sigma|_{\text{attractor}} = S_{\text{macro}}(p, q)$$

So far, Wilsonian.
Duality Transformations: Entanglement with $\Upsilon$!

Charges $(p^I, q_I)$ undergo duality transformations. If these constitute symmetries of LEEA, macroscopic entropy is invariant under them. These leave $\Upsilon$ invariant, but act as symplectic $Sp(2n + 2, \mathbb{Z})$ transformations on the vector $(\Upsilon^I, F_i(\Upsilon, \Upsilon))$ in the attractor equations. Entanglement with the Weyl background!

Departure from topological string approach.

Precise form of $N = 2$ LEEA not known $\rightarrow$ cannot rely on an action principle to incorporate non-holomorphic corrections needed for duality invariance. Instead, demand:

- attractor equations retain their form;
- they follow from a variational principle based on free energy $\mathcal{F}$,

$$\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = -i \left( \bar{Y}^I F_i - Y^I \bar{F}_i \right) - 2i \left( \Upsilon \bar{F}_\Upsilon - \bar{\Upsilon} \Upsilon \bar{F}_{\bar{\Upsilon}} \right),$$

but now based on a general function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, not necessarily holomorphic.
Non-Holomorphic Corrections

Variation of $\mathcal{F}$: with the attractor value $\gamma = -64$

$$\delta \mathcal{F} = i(Y^I - \bar{Y}^\bar{I}) \delta (F_I + \bar{F}_{\bar{I}}) - i(F_I - \bar{F}_{\bar{I}}) \delta (Y^I + \bar{Y}^\bar{I})$$

$$- \left[ i \left( 2\gamma \delta F_\gamma + Y^I \delta F_I - F_I \delta Y^I \right) \right] + \text{c.c.}$$

Vanishing of second line: at least two solutions, namely

- $F$ holomorphic, $F(\lambda Y, \lambda^2 \gamma^2) = \lambda^2 F(Y, \gamma)$, usual Wilsonian case
- $F = 2i \Omega$, $\Omega$ real, $\Omega(\lambda Y, \lambda \bar{Y}, \lambda^2 \gamma, \lambda^2 \bar{\gamma}) = \lambda^2 \Omega(Y, \bar{Y}, \gamma, \bar{\gamma})$

Without loss of generality:

$$F = F^{(0)}(Y) + 2i \Omega(Y, \bar{Y}, \gamma, \bar{\gamma})$$

When $\Omega$ is harmonic (i.e. $\Omega = \text{holo} + \text{anti-holo}$), get back usual Wilsonian case.
Consistent Deformation of Special Geometry?

Second option seems to be a consistent non-holomorphic deformation of special geometry, e.g.:

- under symplectic (duality) transformations, $(Y^I, F_I) \rightarrow (\tilde{Y}^I, \tilde{F}_I)$;
  
  can show that

  \[ \tilde{F}_I = \frac{\partial \tilde{F}}{\partial \tilde{Y}^I}; \]

  $(\mathcal{L}, F)$ and $(\mathcal{L}, \tilde{F})$ in same equivalence class;

- in addition, can show that

  \[ F_\gamma \rightarrow \tilde{F}_\gamma; \]

  i.e. $F_\gamma$ transforms as a scalar.

It follows that $\mathcal{F}$ and $\Sigma$ are invariant under duality transformations that define a symmetry.
Duality Invariant OSV Integral

Consider the following duality invariant integral, expressed in terms of entropy function $\Sigma$, with the attractor value $\Upsilon = -64$,

$$
\int d(Y^I + \bar{Y}^\bar{I}) d(F_I + \bar{F}_\bar{I}) e^{\pi \Sigma(Y, \bar{Y}, p, q)} = \int dY d\bar{Y} \Delta^-(Y, \bar{Y}) e^{\pi \Sigma(Y, \bar{Y}, p, q)}
$$

Duality covariance requires measure factor

$$
\Delta^- = | \text{det} \left[ \text{Im} \left[ F_{JK} - F_{J\bar{K}} \right] \right] |
$$

Evaluate integral in saddle-point approximation about attractor point:

$$
e^{\pi \Sigma|_{\text{attractor}}} = e^{S_{\text{macro}}(p, q)} \quad \text{for large charges}
$$

Duality invariant. Expect saddle-point approximation to hold for dyonic black holes.

Suggests to identify the above with $d(p, q)$, as in OSV.
Prediction for Mixed Partition Function

On the other hand, when only integrating over $Y^I - \bar{Y}^I$ in saddle-point approximation so that $Y^I = (\phi^I + ip^I)/2$, get modified OSV-type integral,

$$d(p, q) = \int d\phi \sqrt{\Delta - (p, \phi)} e^{\pi [F_E(p, \phi) - q I \phi^I]} ,$$

where

$$F_E = 4 \left[ \text{Im}F(Y, \bar{Y}, \gamma, \bar{\gamma}) - \Omega(Y, \bar{Y}, \gamma, \bar{\gamma}) \right] \mid_{Y^I = (\phi^I + ip^I)/2} , \ \gamma = -64$$

Inverting yields prediction for $N = 2$ mixed black hole partition function,

$$Z_{BH}(p, \phi) = \sum_q d(p, q) e^{\pi q I \phi^I} = \sqrt{\Delta -} e^{\pi F_E} = \sqrt{\Delta -} e^{4\pi \Omega^{\text{nonholo}}} e^{\pi F_E^{\text{holo}}}$$

Test requires knowledge of microscopic degeneracies.
Specific $N = 2$ model with exact duality symmetry, $\Gamma(2) \in SL(2, \mathbb{Z})$:

S-T-U-model, A. Sen + C. Vafa, hep-th/9508064

Demanding $(Y^I, F_I)$ to transform accordingly,

- S-T-$\Upsilon$-mixing. Under S-duality,

$$T^a \rightarrow T^a + \frac{ic}{\Delta_S (Y^0)^2} \eta^{ab} \frac{\partial \Omega}{\partial T^b} , \quad \Delta_S = icS + d$$

yields $\Omega$ with non-holomorphic corrections that is related, but not identical, to the solution of the non-holomorphic anomaly equation of $TS$. Due to entanglement with Weyl background.


$$Z_{BH}(p, \phi) = \sum_q d(p, q) e^{\pi q_l \phi^l} = \sqrt{\Delta - e^{4\pi \Omega_{\text{nonholo}}} e^{\pi \mathcal{F}_E^{\text{holo}}}}$$

Agreement (up to certain level of accuracy)!
Conclusions

- We made a proposal for a measure factor in the OSV-integral that is compatible with duality covariance in $N = 2$ models.
- In doing so, we proposed a method for incorporating non-holomorphic terms needed for duality invariance. We found indications that this is a consistent non-holomorphic deformation of special geometry.

LEEA encoded in a non-holomorphic function $F$ not known. Precise relationship between LEEA (1PI graphs) and topological string (connected graphs) remains to be worked out.

- Our proposal yields results in agreement with direct calculations of $Z_{BH}$ in $N = 4$ models, and also in the $N = 2$ S-T-U-model.

Thanks!