On unconstrained higher spins

of any symmetry

Dario Francia

Université Paris VII - APC

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Some reviews:

≻ "Higher-Spin Gauge Theories",
Proceedings of the First Solvay Workshop, Brussels on May 12-14, 2004, including:

→ X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev,
"Nonlinear higher spin theories in various dimensions," arXiv:hep-th/0503128;

→ N. Bouatta, G. Compere and A. Sagnotti,
"An introduction to free higher-spin fields," arXiv:hep-th/0409068;

≻ D. Sorokin,
"Introduction to the classical theory of higher spins," AIP Conf. Proc. **767**, 172 (2005) [arXiv:hep-th/0405069];

≻ D. F. and A. Sagnotti,
"Higher-spin geometry and string theory,"
J. Phys. Conf. Ser. **33** (2006) 57 [arXiv:hep-th/0601199].

≻ A. Fotopoulos and M. Tsulaia,
"Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation," arXiv:0805.1346 [hep-th].

≻ A. Sagnotti, D. Sorokin, P. Sundell, M. A. Vasiliev
Phys. Rept. *to appear*

Introduction I: string theory & higher-spins

Some basic features of ST:

➽ *Spectrum*: spectrum of vibrating string accomodates massless spin 1 and spin 2 particles (*"ST predicts Gravity"*) together with infinitely many *massive* states, with masses and spins related by (open strings) $m^2(J) \sim \frac{1}{\alpha'} J$

(*ST predicts massive higher-spins*)

➽ *UV finiteness*: tree level, high energy amplitude (here: elastic scattering of scalar particles exchanging *arbitrary-spin* intermediate particles; t-channel):

$$\mathcal{A}(s,t) \sim \sum_J g_J^2 \frac{(-s)^J}{t - m_J^2}$$

might be better behaved than any single (or finite number of) exchange.

Massive states as broken phase of massless, higher-spin phase ?

Introduction II: higher-spins & field theory

Symmetry group of space-time

$\Rightarrow$

fundamental particles (fields) labeled by two quantum numbers:

mass $m \geq 0$, and *spin* $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$

(more general labels in $D > 4$)

no indications about the existence of some "privileged" subset of values.

[*Majorana* 1932, *Dirac* 1936, *Fierz-Pauli*, *Wigner* 1939 . . .]

♋

<u>But</u>:

➽ no phenomenological input for (elementary) higher-spins,
(*high-spin "particles" do exist!*)

➽ no-go arguments against their interactions

[*Velo-Zwanziger*, *Coleman-Mandula*, *Aragone-Deser* . . .]

Why this "selection rule" ?

Introduction III: higher-spins & geometry

Central object in Maxwell, Yang-Mills (spin 1) and Einstein (spin 2) theories

is

$$\text{the curvature} : \begin{cases} A_\mu \; \rightarrow F_{\mu\,\nu}\,, \\ h_{\mu\nu} \rightarrow \mathcal{R}_{\mu\,\nu,\,\rho\,\sigma}\,. \end{cases}$$

it provides *dynamics* together with *geometrical meaning*.

What is the "geometry" (if any) underlying hsp gauge fields?

Plan

I. Higher spins in (Q)FT & ST

II. Unconstrained higher spins of any symmetry

III. Higher spins & Geometry

I. Higher spins in (Q)FT & ST

Free theory I: symmetric tensors

"Canonical" description of *free, symmetric* higher-spin gauge fields via

(Fang-) Fronsdal equations (1978):

➽ *Bosons* *(∼ spin 2 → $R_{\mu\nu} = 0$) :*

$$\mathcal{F}_{\mu_1 \ldots \mu_s} \equiv \Box \, \varphi_{\mu_1 \ldots \mu_s} - \partial_{\mu_1} \partial^{\alpha} \varphi_{\alpha \, \mu_2 \ldots \mu_s} + \ldots + \partial_{\mu_1} \partial_{\mu_2} \varphi^{\alpha}{}_{\alpha \, \mu_3 \ldots \mu_s} + \ldots = 0$$

- gauge invariant under $\delta \varphi = \partial \Lambda$ *iff* $\Lambda' (\equiv \Lambda^{\alpha}{}_{\alpha}) \equiv 0$;
- Lagrangian description *iff* $\varphi'' (\equiv \varphi^{\alpha\beta}{}_{\alpha\beta}) \equiv 0$.

➽ *Fermions* *(∼ spin $\frac{3}{2}$ → $\not\partial \, \psi_{\mu} - \gamma_{\mu} \not\psi = 0$) :*

$$\mathcal{S}_{\mu_1 \ldots \mu_s} \equiv i \, \{ \gamma^{\alpha} \partial_{\alpha} \, \psi_{\mu_1 \ldots \mu_s} - (\partial_{\mu_1} \gamma^{\alpha} \psi_{\alpha \, \mu_2 \ldots \mu_s} + \ldots) \} = 0$$

- gauge invariant under $\delta \psi = \partial \epsilon$ *iff* $\not\epsilon \equiv 0$;
- Lagrangian description *iff* $\not\psi' (\equiv \not\psi^{\alpha}{}_{\alpha}) \equiv 0$.

Free theory II: mixed-symmetry tensors

Generalisation to (spinor -) tensors of *any symmetry* type in

Labastida equations $(1986-1989)$:

➽ *Bosons* (2-families: $\varphi_{\mu_1\cdots\mu_s,\,\nu_1\cdots\nu_r} \equiv \varphi_{\,\mu_s,\,\nu_r}$):

$$\mathcal{F}_{\,\mu_s,\,\nu_r} \equiv \Box\varphi_{\,\mu_s,\,\nu_r} - \partial_\mu\,\partial^{\,\alpha}\,\varphi_{\,\alpha\mu_{s-1},\,\nu_r} - \partial_\nu\,\partial^{\,\alpha}\,\varphi_{\,\mu_s,\,\alpha\nu_{r-1}} + \partial^{\,2}{}_{\mu}\cdots + \partial^{\,2}{}_{\nu}\cdots + \partial_\mu\,\partial_\nu\cdots = 0$$

➾ gauge invariant under

$$\delta\,\varphi_{\,\mu_s,\,\nu_r} = \partial_\mu\,\Lambda^{(1)}{}_{\mu_{s-1},\,\nu_r} + \partial_\nu\,\Lambda^{(2)}{}_{\mu_s,\,\nu_{r-1}}$$

iff suitable combinations of *traces* of $\Lambda^{(1)}$ and $\Lambda^{(2)}$ vanish;

➾ Lagrangian description *iff* suitable combinations of *double traces* of $\varphi_{\,\mu_s,\,\nu_r}$ vanish.

➽ *Fermions* (2-families: $\psi^{a}{}_{\mu_1\cdots\mu_s,\,\nu_1\cdots\nu_r} \equiv \psi_{\,\mu_s,\,\nu_r}$):

$$\mathcal{S}_{\,\mu_s,\,\nu_r} \equiv i\,\{\gamma^{\,\alpha}\,\partial_\alpha\,\psi_{\,\mu_s,\,\nu_r} - \partial_\mu\,\gamma^{\,\alpha}\,\psi_{\,\alpha\,\mu_{s-1},\,\nu_r} - \partial_\nu\,\gamma^{\,\alpha}\,\psi_{\,\mu_s,\,\alpha\,\nu_{r-1}})\} = 0$$

➾ similar constraints, but *no Lagrangian description available* for the general case.

No-go results

Techniques allowing interacting theories for $s \leq 2$ tipically fail for $s \geq \frac{5}{2}$

Examples:

I. *Lagrangian* eom for *massive* fields of $s \geq 1$:

Velo-Zwanziger '69
Porrati -Rahman '08

$\rightarrow$ *non-causality*
$\rightarrow$ *loss of constraints*
$\rightarrow$ *failure to propagate*

II. S-matrix amplitudes - massless hsp particles $\leftrightarrow$ hsp symmetries:

Coleman-Mandula '67 - HLS '69
Benincasa-Cachazo '07

(under assumptions not always met in hsp theories)
not allowed symmetry generators carrying Lorentz indices other then those of the Poincaré group

III. Coupling with Gravity - propagation of waves on *Ricci-flat* bkg $\mathcal{R}_{\mu\nu} = 0$:

Aragone-Deser '79

$$\begin{cases} s = 3/2 : \ E_{\bar{\psi}_\mu} = \gamma^{\mu\nu\rho} D_\nu \psi_\rho & \rightarrow \delta E_{\bar{\psi}_\mu} = 0 \\ s = 5/2 : \ E_{\bar{\psi}_{\mu\nu}} = \not{D} \psi_{\mu\nu} - D_{(\mu} \not{\psi}_{\nu)} & \rightarrow \delta E_{\bar{\psi}_{\mu\nu}} \sim \text{“Riemann”} \end{cases}$$

Hsp & (Q)FT V: positive results

Idea: fluctuation of gravitational field over *non-flat* bkg

useful ?

- Riemann over (A)dS background [*Fradkin - Vasiliev* 1987]

$$R_{\mu\nu,\,\rho\sigma} = R^{(AdS)}_{\mu\nu,\,\rho\sigma} + \widehat{R}_{\mu\nu,\,\rho\sigma}\,, \qquad s.\ t \qquad \widehat{R}^2 \sim 0$$

$$\delta\, E_{\psi_{\mu\nu}} \sim (\widehat{R}_{\mu\nu,\,\rho\sigma} + \widehat{R}_{\mu\rho,\,\nu\sigma})\gamma^{\mu}\,\varepsilon^{\sigma} + \text{“}Ricci\ terms\text{''}\cdot\varepsilon\,,$$

$$E_{\psi_{\mu\nu}} = E^{(0)}_{\psi_{\mu\nu}} + \frac{i}{2\,\Lambda}(\widehat{R}_{\mu\nu,\,\rho\sigma} + \widehat{R}_{\mu\rho,\,\nu\sigma})\,\nabla\!\!\!\!/\,\psi^{\mu\sigma} \rightarrow \delta\, E_{\psi_{\mu\nu}} = 0\ !$$

The *cubic* vertex describing this non-minimal coupling is, schematically

$$V = \frac{i}{\Lambda}\int d^D x\sqrt{g}\{\bar{\psi}\widehat{R}\nabla\!\!\!\!/\,\psi + \bar{\psi}\,(\nabla\!\!\!\!/\,\widehat{R})\,\psi\}\,.$$

- Other *cubic* vertices for *self -interacting* or *mutually interacting* hsp:

- Bengtsson, Bengtsson, Brink (1983)
- Berends, Burgers, Van Dam (1984)
- Fradkin, Metsaev (1991), Metsaev (1993)
- Bekaert, Boulanger, Cnockaert, Leclercq, Sundell, Mourad (2006, 2008, 2009)
- Buchbinder, Fotopoulos, Irges, Petkou, Tsulaia (2006, 2007)

(First-order), cubic, hsp gauge theories do exist.

Vasiliev Theory

(*also Sezgin - Sundell*)

generalisation of the frame-like formulation of general relativity

- *Consistent, non-linear higher-spin eom are given, for symmetric tensors*,
- *Infinite-dimensional hsp algebra*, with generators T_s s.t. *the maxima sub-algebra closes up to spin* 2. For $s > 2$ (generators carry spin $s-1$), HS symmetry $\longrightarrow$ *infinite tower of HS gauge fields*.
- *Very little is known about the action*: basically only the cubic coupling;

♋

General features of interactions:

Need for infinitely many fields of increasing spin

Higher-derivative couplings

$\longleftrightarrow$

All reminiscent of String Theory

Hsp & Strings: tensionless SFT

➽ Consider the equations of motion for open String Field Theory

$$\mathcal{Q}\,|\Phi\rangle \;=\; 0\;,$$

where $\mathcal{Q}$ is the BRST charge, and evaluate the limit $\alpha' \to \infty$;
[*Bengtsson, Henneaux-Teitelboim, Lindström, Sundborg, D.F.-Sagnotti, Sagnotti-Tsulaia, Lindström-Zabzine, Bonelli, Savvidy, Buchbinder-Fotopoulos-Tsulaia-Petkou, . . .*]

➽ Actually, by restricting the attention to totally symmetric tensors it is possible to show that this equation splits into a series of *triplet* equations:

$$\begin{aligned} \Box\,\varphi \;&=\; \partial\,C\;, \\ C \;&=\; \partial\cdot\varphi \;-\; \partial\,D\;, \\ \Box\,D \;&=\; \partial\cdot C\;, \end{aligned}$$

together with the gauge transformations

$$\begin{aligned} \delta\,\varphi &= \partial\,\Lambda\,, \\ \delta\,C &= \Box\,\Lambda\,, \\ \delta\,D &= \partial\cdot\Lambda\,, \end{aligned}$$

where φ is a spin-s field, C a spin-$(s-1)$ field and D a spin-$(s-2)$ field, all

unconstrained.

[Extension of triplets to irreducible spin s $\rightarrow$ Buchbinder-Galajinski-Krykhtin 2007; frame-like analysis for reducible & irreducible cases $\rightarrow$ Sorokin-Vasiliev 2008]

The massless phase given by tensionless SFT involves

unconstrained fields

♋

- Calls for a generalisation of Fronsdal-Labastida theories,
- Moreover, absence of constraints is expected in a *geometric* description of higher-spin gauge fields (*here focus on symmetric tensors*):

linearised curvatures for higher spins:

[*de Wit-Freedman* '80]

$$\varphi_{\mu_1 \ldots \mu_s} \quad \longrightarrow \quad \mathcal{R}_{\mu_1 \ldots \mu_s;\, \nu_1 \ldots \nu_s} \sim \partial^{\,s}\, \varphi$$

s.t.

$$\delta\, \mathcal{R}_{\mu_1 \ldots \mu_s;\, \nu_1 \ldots \nu_s} \equiv 0$$

under $\quad \delta \varphi_{\mu_1 \ldots \mu_s} = \partial_{\mu_1} \Lambda_{\mu_2 \mu_3 \ldots \mu_s} + \partial_{\mu_2} \Lambda_{\mu_1 \mu_3 \ldots \mu_s} + \ldots$

for unconstrained gauge fields and gauge parameters

At least three indications suggest to reconsider the free theory :

① *No Lagrangians* for arbitrary mixed-symmetry fermions;

② *No constraints* from the tensionless limit of SFT;

③ *Constrained* theory $\not\leftrightarrow$ higher-spin *curvatures*.

♋

How to connect curvatures and dynamics?

➽ Not clear: $\mathcal{R}^{(s)}_{\mu_1\ldots\mu_s,\,\nu_1\ldots\nu_s}$ is a *higher-derivative* tensor, if $s \geq 3$;

➽ Let us concentrate on a slightly simpler, but related, issue:

are the constraints in the Fronsdal theory really necessary?

II. Unconstrained higher spins of any symmetry

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Unconstrained hsp I: local theory - symmetric tensors

Fronsdal	Unconstrained
$\mathcal{F}$ s. t. $\delta\mathcal{F} = 3\,\partial^3\Lambda'$	$\mathcal{A} \equiv \mathcal{F} - 3\,\partial^3\,\alpha \rightarrow \begin{cases}\delta\,\alpha = \Lambda', \\ \delta\,\mathcal{A} = 0\,.\end{cases}$
$\mathcal{F} = 0$	$\mathcal{A} = 0$
$\mathcal{L}_{\varphi''\equiv 0} = \frac{1}{2}\varphi\left(\mathcal{F} - \frac{1}{2}\,\eta\,\mathcal{F}'\right)$	$\mathcal{L} = ?$

Basic ingredient: *the Bianchi identity*:

$$\partial\cdot\mathcal{A} - \frac{1}{2}\,\partial\,\mathcal{A}' \equiv -\frac{3}{2}\,\partial^3\,(\varphi'' - \partial\cdot\alpha - \partial\,\alpha')$$

compare with gravity

$$\partial^{\alpha}\mathcal{R}_{\alpha\mu} - \frac{1}{2}\,\partial_{\mu}\,\mathcal{R} \equiv 0$$

Unconstrained hsp II: *local theory - symmetric tensors*

➽ Start with the trial Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\varphi\left(\mathcal{A} - \frac{1}{2}\eta\,\mathcal{A}'\right),$$

and compute its gauge variation: $\delta\,\varphi = \partial\,\Lambda \quad \rightarrow$

$$\delta\mathcal{L}_0 = \frac{3}{4}\binom{s}{3}\Lambda'\partial\cdot\mathcal{A}' + 3\binom{s}{4}\partial\cdot\partial\cdot\partial\cdot\Lambda\,(\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha')$$

Introduce a *Lagrange multiplier* β, s. t. $\delta\beta = \partial\cdot\partial\cdot\partial\cdot\,\Lambda$; then

$$\mathcal{L}\,(\varphi,\alpha,\beta) = \frac{1}{2}\,\varphi\,\left(\mathcal{A} - \frac{1}{2}\,\eta\,\mathcal{A}'\right) - \frac{3}{4}\,\binom{s}{3}\,\alpha\,\partial\cdot\mathcal{A}' - 3\,\binom{s}{4}\,\beta\,\mathcal{C}\,,$$

define a gauge-invariant, local, unconstrained Lagrangian for spin s.

[*D. F. - A. Sagnotti* 2005, 2006]

Generalisation to *(A)dS*: [*A. Sagnotti - M. Tsulaia* '03; *D. F. - J. Mourad - A. Sagnotti,* '07]

Unconstrained hsp III: local theory - mixed symmetry bosons

[A. Campoleoni - D. F. - J. Mourad - A. Sagnotti, 2008]

Here: Two-family fields $\varphi_{\mu_1 \dots \mu_{s_1} ; \nu_1 \dots \nu_{s_2}}$

Notation:
$$\begin{cases} \varphi_{\mu_1 \dots \mu_{s_1} ; \nu_1 \dots \nu_{s_2}} & \rightarrow \ \varphi \,, \\ \partial_{(\mu_1^i |} \varphi_{\dots ; | \mu_2^i \dots \mu_{s_i+1}^i) ; \dots} & \rightarrow \ \partial^i \varphi \,, \\ \partial^\lambda \varphi_{\dots ; \lambda \mu_2^i \dots \mu_{s_i}^i ; \dots} & \rightarrow \ \partial_i \varphi \,, \\ \varphi_{\dots ; \ {}^{\lambda}{}_{\mu_2^i \dots \mu_{s_i}^i} ; \dots ; \lambda \mu_2^j \dots \mu_{s_j}^j ; \dots} & \rightarrow \ T_{ij} \varphi \,. \end{cases}$$

upper indices $\leftrightarrow$ *added indices*

lower indices $\leftrightarrow$ *removed indices*

Families of *symmetric* indices $\longrightarrow$ *reducible* $gl\,(D)$ tensors

$\sim$

Basic *constrained* theory: [Labastida 1986, 1989]

$$\mathcal{F} = \Box \varphi - \partial^i \partial_i \varphi + \frac{1}{2} \partial^i \partial^j T_{ij} \varphi = 0 \,,$$

- gauge invariant under $\delta \varphi = \partial^i \Lambda_i$ *iff* $T_{(ij} \Lambda_{k)} \equiv 0$;
- Lagrangian description *iff* $T_{(ij} T_{kl)} \varphi = 0$.

$\rightarrow$ *not all traces vanish;*

$\rightarrow$ *the constraints are not independent.*

Unconstrained hsp IV: local theory - mixed symmetry bosons

Basic unconstrained kinetic tensor:

$$\mathcal{A} = \mathcal{F} - \frac{1}{2}\partial^i\partial^j\partial^k\alpha_{ijk},$$

But, due to linear dependence of constraints

$$\begin{cases}\alpha_{ijk} \equiv \alpha_{ijk}(\Phi) = \frac{1}{3}T_{(ij}\Phi_{k)}, \\ \delta\Phi_k = \Lambda_k .\end{cases}$$

$\sim$

To construct the Lagrangian $\rightarrow$ resort to *Bianchi identity*:

$$\partial_i\mathcal{A} - \frac{1}{2}\partial^j T_{ij}\mathcal{A} = -\frac{1}{4}\partial^j\partial^k\partial^l\mathcal{C}_{ijkl}$$

$$\mathcal{C}_{ijkl} = T_{(ij}T_{kl)}\varphi + \mathcal{C}_{ijkl}(\alpha)$$

As for symm case, take care of terms in $\propto \mathcal{C}_{ijkl}$ via a *Lagrange multiplier* β:

$$\mathcal{L} = \frac{1}{2}\langle\varphi, E_\varphi\rangle + \frac{1}{2}\langle\Phi_i, (E_\Phi)_i\rangle + \frac{1}{2}\langle\beta_{ijkl}, (E_\beta)_{ijkl}\rangle$$

where in particular the e.o.m. for φ, gauge fixing $\alpha_{ijk} = \frac{1}{3}T_{(ij}\Phi_{k)}$ to zero, is

$$E_\varphi = \mathcal{E}_\varphi + \frac{1}{2}\eta^{ij}\eta^{kl}\mathcal{B}_{ijkl} = 0,$$

$$\mathcal{E}_\varphi = \mathcal{F} - \frac{1}{2}\eta^{ij}T_{ij}\mathcal{F} + \frac{1}{36}\eta^{ij}\eta^{kl}\left(2T_{ij}T_{kl} - T_{i(k}T_{l)j}\right)\mathcal{F}.$$

[*A. Campoleoni - D. F. - J. Mourad - A. Sagnotti, 2009*]

The basic kinematical setting of Labastida [1987]

$$\begin{cases} \mathcal{S} \,=\, i\,\big(\,\partial\!\!\!/\,\psi \,-\, \partial^{\,i}\,\psi\!\!\!/_{\,i}\,\big) \,=0\,, \\ \delta\,\psi \,=\, \partial^{\,i}\,\epsilon_{\,i}\,, \\ T_{(\,ij}\,\psi\!\!\!/_{\,k\,)} \,=\, 0\,;\; \gamma_{(\,i}\,\epsilon_{\,j\,)} \,=\, 0\,, \end{cases}$$

can be easily turned to its *unconstrained* counterpart:

$$\begin{cases} \mathcal{W} \,=\, \mathcal{S} \,+\, i\,\partial^{\,i}\,\partial^{\,j}\,\xi_{\,ij} \,=\, 0\,, \\ \delta\,\psi \,=\, \partial^{\,i}\,\epsilon_{\,i}\,, \\ \xi_{\,ij}\,(\Psi) \,=\, \tfrac{1}{2}\,\gamma_{\,(\,i}\,\Psi_{\,j\,)}\,, \\ \delta\,\Psi_{\,i} \,=\, \epsilon_{\,i}\,, \end{cases}$$

BUT, in the constrained setting, *no Lagrangian available for fermions;*

- Using the Bianchi identity (here constrained theory, for simplicity)

$$\partial_{\,i}\,\mathcal{S} \,-\, \tfrac{1}{2}\,\partial\!\!\!/\,\gamma_{\,i}\,\mathcal{S} \,-\, \tfrac{1}{2}\,\partial^{\,j}\,T_{ij}\,\mathcal{S} \,-\, \tfrac{1}{6}\,\partial^{\,j}\,\gamma_{\,ij}\,\mathcal{S} \,=\, 0\,,$$

it is possible to find the *complete Lagrangian, for N-family fields*, in the form

$$\begin{cases} \mathcal{L} \,=\, \tfrac{1}{2}\,\langle\,\bar{\psi}\,,\sum_{p\,,\,q\,=\,0}^{N} k_{\,p\,,\,q}\;\eta^{\,p}\,\gamma^{\,q}\,(\,\gamma^{\,[q]}\,\mathcal{S}^{\,[p]}\,)\,\rangle \,+\, \text{h.c.}\,, \\ k_{\,p\,,\,q} \,=\, \frac{(-1)^{\,p+\frac{q(q+1)}{2}}}{p\,!\,q\,!\,(\,p+q+1\,)\,!}\;. \end{cases}$$

Generalised Weyl symmetries

Linearised Einstein equations in D-dimensions

$$\mathcal{E}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R} = 0$$

Reduction: $\mathcal{E}^{\alpha}{}_{\alpha} \sim (D-2)\,\mathcal{R}$;

Weyl shift: $\delta h_{\mu\nu} = \eta_{\mu\nu}\Omega \Rightarrow \delta\mathcal{E}_{\mu\nu} = (D-2)\,(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\Omega$;

Triviality: $\mathcal{E}_{\mu\nu}\,(D=2) \equiv 0$ ($\mathcal{L}$ is a total derivative) .

For mixed-symmetry fields more possibilities:

- Theories with (formal) shift symmetries and vanishing Einstein tensor

Example: $gl(D)$-irreducible bosonic two-column fields $\{p,\, q\}$ in $D \geq p+q$;

- Theories with (true) shift symmetries and non-vanishing Einstein tensor

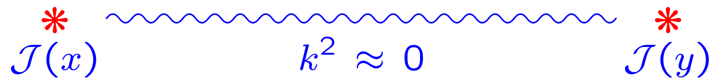
Example: $gl(D)$-irreducible fermionic $\{2,\, 1\}$ field in $D=4$, where the shift is

$$\delta\,\psi_{\mu\nu,\,\rho} = \gamma_{(\mu}\,\Omega^{(1)}{}_{\nu),\,\rho} + \gamma_\rho\,\Omega^{(2)}{}_{\mu\nu}$$

Massive theory & Current exchanges

❄ Massive Lagrangians from massless ones → K-K reduction from $D+1$ to D

❄ Response of the theory to the presence of an external source $\mathcal{J}$; *unitarity* : only transverse, on-shell polarisations mediate the interaction between distant sources:



tantamount to computing the *propagator*

➽ Straightforward in flat bkg;

$$s = 3: \quad \begin{cases} p^2\, \mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{J} - \frac{3}{D}\, \mathcal{J}' \cdot \mathcal{J}' & m = 0 \\ (p^2 - m^2)\, \mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{J} - \frac{3}{D+1}\, \mathcal{J}' \cdot \mathcal{J}' & m \neq 0 \end{cases}$$

(generalisation to hsp of the vDVZ discontinuity)

➽ Less direct to describe (partially) massive (A)dS fields$^{(*)}$;

$$s = 3: \quad \begin{cases} P_L^2\, \mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{J} - \frac{3}{D}\, \mathcal{J}' \cdot \mathcal{J}' & m = 0 \\ (P_L^2 - m^2)\, \mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{J} - 3\, \frac{m^2 L^2 + D + 1}{(D+1)(m^2 L^2 + D)}\, \mathcal{J}' \cdot \mathcal{J}' & m \neq 0 \end{cases}$$

(*no vDVZ discontinuity* for hsp on (A)dS)

$^{(*)} P_L^2 = \Box_L - 4\frac{D}{L^2}$

[*D.F. - J. Mourad - A. Sagnotti, '07, '08*]

Fronsdal constraints are, at least, not necessary

♋

- What is the *meaning* of the unconstrained theory?

- In particular, is there any relation with the *hsp geometry* described by de Wit and Freedman?.

III. Higher spins & Geometry

♋

linearised curvatures:

simplest gauge invariant tensors whose vanishing $\Rightarrow \varphi$ is pure gauge

- *Spin 1 [Maxwell]*: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ s. t. $\delta F_{\mu\nu} = 0$ under $\delta A_\mu = \partial_\mu \Lambda$;

  (but also $s = 3/2$)

- *Spin 2 [Einstein]*: $\delta h_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$;

  (but also $s = 5/2$)

$$\mathcal{R}^{(2)}{}_{\mu\mu,\,\rho\rho} = \partial_\mu^2 h_{\rho\rho} - \frac{1}{2}\partial_\mu \partial_\rho h_{\mu,\,\rho} + \partial_\rho^2 h_{\mu\mu}$$

- *Spin 3 [de Wit - Freedman]*: $\delta\varphi_{\alpha\beta\gamma} = \partial_\alpha \Lambda_{\beta\gamma} + \partial_\beta \Lambda_{\alpha\gamma} + \partial_\gamma \Lambda_{\beta\alpha}$ $\boxed{\Lambda^\alpha{}_\alpha \neq 0\,!}$

  (but also $s = 7/2$)

$$\mathcal{R}^{(3)}{}_{\mu\mu\mu,\,\rho\rho\rho} = \partial_\mu^3 \varphi_{\rho\rho\rho} - \frac{1}{3}\partial^2{}_\mu \partial_\rho \varphi_{\mu,\,\rho\rho} + \frac{1}{3}\partial_\mu \partial^2{}_\rho \varphi_{\mu\mu,\,\rho} - \partial_\rho^3 \varphi_{\mu\mu\mu}$$

and so on.

equations of motion ?

Hsp & Geometry II: generalised Ricci tensors

- $s = 3, 4$: saturate enough indices to restore the symmetries of φ:

$$\mathcal{R}_{\mu_1\mu_2\mu_3,\,\nu_1\nu_2\nu_3} \quad \to \quad \partial \cdot \mathcal{R}\,',$$
$$\mathcal{R}_{\mu_1\mu_2\mu_3\mu_4,\,\nu_1\nu_2\nu_3\nu_4} \quad \to \quad \mathcal{R}\,'',$$

- restore dimensions of P^2, introducing *inverse D'Alembertian*, generalising Maxwell and Einstein theories *via* a class of *candidate 'Ricci' tensors*:

$$(s=1) \qquad \partial \cdot \mathcal{R} = 0 \quad \to \quad \frac{1}{\Box^{n-1}} \partial \cdot \mathcal{R}^{[n-1]}{}_{\mu_1 \cdots \mu_{2n-1}} = 0 \qquad (s = 2n-1),$$
$$(s=2) \qquad \mathcal{R}\,' = 0 \quad \to \quad \frac{1}{\Box^{n-1}} \mathcal{R}^{[n]}{}_{\mu_1 \cdots \mu_{2n}} = 0 \qquad (s = 2n)$$

[*D.F. - A. Sagnotti, 2002*]

♋

- This possibility is *highly non-unique* → infinitely many -more singular- ones:

$$(s=3) \qquad \frac{1}{\Box} \partial \cdot \mathcal{R}\,' \quad \to \quad \mathcal{A}_{\varphi}(a) \equiv \frac{1}{\Box} \partial \cdot \mathcal{R}\,' + a \frac{\partial^2}{\Box^2} \partial \cdot \mathcal{R}\,'' = 0$$

Meaning?

Hsp & Geometry III: geometric Lagrangians

Spin s: the most general candidate "Ricci" tensor

$$\mathcal{A}_\varphi(a_1, \dots a_k, \dots)$$

is such that, *for almost all choices* of $a_1, \dots a_k, \dots$:

➻ (*CONSISTENCY*) the equation $\mathcal{A}_\varphi = 0$ implies the *compensator equation*

$$\mathcal{A}_\varphi(\{a_k\}) \equiv \mathcal{F} - 3\,\partial^3 \alpha_\varphi = 0\,,$$

with $\delta\,\alpha_\varphi = \Lambda'$ ➻ *Fronsdal form, after partial gauge-fixing*.

[the equation $\mathcal{F} = 3\partial^3\mathcal{H}$ first derived from curvatures by Damour-Deser 1987 (spin 3); then Dubois Violette-Henneaux 1999, 2001, Bekaert-Boulanger 2003, $\cdots$]

➻ (*LAGRANGIAN*) it is possible to define *identically divergenceless Einstein tensors* $\mathcal{E}_\varphi(a_1, \dots a_k, \dots)$ s.t.

$$\mathcal{L} = \frac{1}{2}\varphi\,\mathcal{E}_\varphi(\{a_k\}) \quad \longrightarrow \quad \mathcal{E}_\varphi(\{a_k\}) = 0 \quad \longrightarrow \quad \mathcal{A}_\varphi(\{a_k\}) = 0\,,$$

[*D.F. - J. Mourad - A. Sagnotti, 2007*]

Spin 2: massive theory as

quadratic deformation of the geometric theory:

➽ *Spin* 2 [*Fierz-Pauli*]

$$\mathcal{L}(m=0) = \frac{1}{2} h_{\mu\nu} \, (\mathcal{R}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\mathcal{R})$$

$$\mathcal{L}(m) = \frac{1}{2} h_{\mu\nu} \{ \underbrace{(\mathcal{R}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\mathcal{R})}_{\partial\cdot\mathcal{E}_{s=2}\equiv 0} - m^2 \underbrace{(h^{\mu\nu} - \eta^{\mu\nu} h^{\alpha}{}_{\alpha})}_{Fierz-Pauli\ mass\ term} \}$$

➽ *Spin* s: General idea: *higher traces* should appear in the mass term, s.t.

$$\mathcal{L} = \frac{1}{2}\varphi\{\mathcal{E}_{\varphi}(a_1, \ldots a_k, \ldots) - m^2 M_{\varphi}\} \quad \text{where} \quad \underbrace{M_{\varphi} = \sum \lambda_k \eta^k \varphi^{[k]},}_{generalised\ FP\ mass\ term}$$

➽ Fronsdal : $\partial \cdot \mathcal{E}_{Fronsdal} \neq 0 \;\Rightarrow$ need for *auxiliary fields*;

➽ Differently, for *all* geometric Einstein tensors $\mathcal{E}_{\varphi}$ we have $\partial \cdot \mathcal{E}_{\varphi} \equiv 0$!

➽ Indeed it is possible to define a consistent massive theory with

$$M_{\varphi} = \varphi - \eta\varphi' - \eta^2\varphi'' - \frac{1}{3}\eta^3\varphi''' - \cdots - \frac{1}{(2n-3)!!}\eta^n\varphi^{[n]} .$$

No auxiliary fields are needed [*D.F., '07*]

We found consistent formulations for unconstrained hsp

♋

on the other hand:

- Using curvatures $\longrightarrow$ *non-localities*;

- Minimal local Lagrangians $\longrightarrow$ *higher-derivatives*: $\sim \alpha \, \Box^{\,2} \alpha$

- BRST approach$^{(*)}$: to describe spin $s \longrightarrow \mathcal{O}(s)$ *auxiliary fields*

$\longrightarrow$

intrinsic 'singularity' of the unconstrained approach?

$^{(*)}$[*Pashnev - Tsulaia - Buchbinder et al.* 1997, ...]

Unconstrained hsp without higher derivatives

There is a simple, alternative interpretation of the minimal local Lagrangians:

- Consider the Fronsdal Lagrangian, together with a multiplier for ϕ'':

$$\mathcal{L} = \phi(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}') + \beta\phi''$$

$\mathcal{L}$ is gauge-invariant under $\delta\varphi = \partial\lambda,\ \delta\beta = \partial\cdot\partial\cdot\partial\cdot\lambda$, with $\lambda' = 0$

- Perform the *Stueckelberg substitution*

$$\phi \quad \rightarrow \quad \varphi - \partial\theta$$

obtaining an *unconstrained* Lagrangian, gauge invariant under

$$\delta\varphi = \partial\Lambda; \qquad\qquad \delta\theta = \Lambda$$

with an *unconstrained* parameter Λ.

- Only the trace of θ appears in $\mathcal{L}$ (after a redefinition of β)so that, defining $\theta' \equiv \alpha$ we recover the minimal Lagrangian

$$\mathcal{L}(\varphi, \alpha, \beta) = \frac{1}{2}\varphi\left(\mathcal{A} - \frac{1}{2}\eta\mathcal{A}'\right) - \frac{3}{4}\binom{s}{3}\alpha\,\partial\cdot\mathcal{A}' - 3\binom{s}{4}\beta\,\mathcal{C}$$

Unconstrained hsp without higher derivatives

Two basic observations:

- higher-derivative terms are simply due to the different dimensions of θ w.r.t. φ in $\phi \;\rightarrow\; \varphi - \partial\,\theta$;
- Under this substitution *any* function of ϕ would be (trivially) gauge-invariant.

This is too much!

What we want is to *extend* to the unconstrained level

a constrained gauge symmetry already present in the Lagrangian

♋

In this sense, maybe it is possible to improve the Stueckelberg idea.

Unconstrained hsp without higher derivatives

[*see also* *Buchbinder, Galajinsky, Krykhtin '07*]

- In $\delta\phi = \partial\Lambda$ separate *traceless* and *trace* parts of the parameter Λ:

$$\Lambda = \Lambda^t + \eta\,\Lambda^p\,,$$
$$\Lambda^p\,:\,\Lambda' = (\eta\,\Lambda^p)'$$

- introduce a new compensator θ_p, s.t. $\delta\,\theta_p = \partial\,\Lambda^p$ (so θ_p *is not* pure gauge)

- perform in $\mathcal{L}$ the substitution

$$\phi \;\rightarrow\; \varphi - \eta\,\theta_p$$

where $\varphi - \eta\,\theta_p$ *transforms as the 'old' Fronsdal field*.

- The corresponding "Ricci tensor" (and generalisations thereof)

$$\mathcal{A}_{\varphi,\theta} = \mathcal{F} - (D + 2s - 6)\,\partial^2\theta - \eta\,\mathcal{F}_\theta\,,$$

is the building-block of *unconstrained Lagrangians*, with a *minimal* content of auxiliary fields and *no higher-derivatives*

for bosons and fermions of any symmetry type

[*D. F. 2007; A. Campoleoni - D. F. - J. Mourad - A. Sagnotti; 2008; 2009*]

∼ *Perspectives* ∼

Still some open issues on the *free theory :*

- hsp supersymmetry multiplets;
- Noether currents;
- Dualities;
- Quantization
- ...

♋

whether or not allowing for a wider gauge symmetry might prove to be relevant, only a deeper insight into interactions will tell. Some possible directions:

To better understand what we already "know" :

∼ *Cubic interactions & eom:* Positive (preliminary) results for hsp interactions *are known*; but, very little is known about *explicit solutions*;

∼ *String Theory :* Closer look at *"massless" phase* of ST, to better understand what could make hsp interactions at all possible.

To go beyond

∼ *Quartic interactions :*

- For *spin* 1 (YM) and *spin* 2 (EH) *cubic vertex* implies *full Lagrangian*;
- for higher spins *nothing known about quartic couplings*; *but* "proper" hsp features from quartic coupling onwards: maybe worth the effort to try and overcome the "cubic" barrier.

Appendix: Hsp geometry & current exchanges, $m = 0$

Are all the geometrical Einstein tensors really equivalent?

- *Propagator* from Lagrangian equation with an external current:

$$\mathcal{E}_\varphi\,(a_1, \ldots a_k \ldots) = \mathcal{J} \quad \Rightarrow \quad \varphi = \mathcal{G}\,(a_1, \ldots a_k \ldots) \cdot \mathcal{J}$$

- *Current exchange* $\mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{G} \cdot \mathcal{J} \rightarrow$ consistency conditions on the *polarisations flowing*:

almost all geometric theories give the wrong result, but one.

The correct theory has a simple structure:

- The 'Ricci' tensor has the compensator form $\mathcal{A}_\varphi = \mathcal{F} - 3\,\partial^3\,\gamma_\varphi$;

- It satisfies the identities : $\begin{cases} \partial \cdot \mathcal{A}_\varphi - \frac{1}{2}\,\partial\,\mathcal{A}'_\varphi \equiv 0 \\ \mathcal{A}''_\varphi \equiv 0 \end{cases}$, and the Lagrangian is

$$\boxed{\mathcal{L} = \frac{1}{2}\,\varphi\,(\mathcal{A}_\varphi - \frac{1}{2}\,\eta\,\mathcal{A}'_\varphi + \eta^2\,\mathcal{B}_\varphi) - \varphi \cdot \mathcal{J}}$$

[*D.F. - J. Mourad - A. Sagnotti, 2007*]

Appendix: Hsp geometry & current exchanges, $m \neq 0$

- Consider the family of Lagrangians, for spin 4: [D.F. 2007, 2008]

$$\mathcal{L}(m) = \frac{1}{2}\varphi\{\mathcal{E}_\varphi(a_1, a_2) - m^2 M_\varphi\} - \varphi \cdot \mathcal{J},$$

  where $\mathcal{J}$ is a *conserved* current: $\partial \cdot \mathcal{J} = 0$.

- The divergence of the eom

$$\partial \cdot \{\mathcal{E}_\varphi(a_1, a_2) - m^2(\varphi - \eta\varphi' - \eta^2\varphi'')\} = \partial \cdot \mathcal{J} = 0,$$

  implies the *same consequences as in the absence of* $\mathcal{J}$.

- Actually, $\forall a_1, a_2$ the eom reduce to

$$\Box\varphi - \frac{\partial^2}{\Box}\varphi' - 3\frac{\partial^4}{\Box^2}\varphi'' - m^2(\varphi - \eta\varphi' - \eta^2\varphi'') = \mathcal{J},$$

- where a_1, a_2 disappeared; computing the product $\mathcal{J} \cdot \mathcal{J}$:

  (1) *only surviving contribution from the family of Einstein tensors is* $\Box\varphi$

  (2) *full structure of the propagator encoded in the coefficients of* M_φ

- Inverting the equation of motion we find the correct result

$$\mathcal{J} \cdot \varphi = \frac{1}{p^2 - m^2}\{\mathcal{J} \cdot \mathcal{J} - \frac{6}{D+3}\, J' \cdot J' + \frac{3}{(D+1)(D+3)}\, J'' \cdot J''\}$$

Appendix: Hsp geometry: uniqueness of mass deformation

The same mass term M_φ generates *infinitely many* consistent massive theories.

$\longrightarrow$

issue of uniqueness

I. ➽ Origin of the Fierz-Pauli mass-term, for $s = 2$: KK reduction ($\Box \to \Box - m^2$):

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\mathcal{R} \sim \Box\,(h - \eta\,h') + \ldots\,,$$

A similar mechanism for M_φ?

➽ For each Einstein tensor $\mathcal{E}_\varphi(a_1, \ldots, a_k)$ it is unambiguously defined the "pure massive" contribution of the reduction, neglecting singularities from $\frac{1}{\Box} \to \frac{1}{\Box - m^2}$:

$$\mathcal{E}_\varphi(a_1, \ldots, a_k) \sim \Box\,(\varphi + k_1\,\eta\,\varphi' + k_2\,\eta^2\,\varphi'' + \ldots) + \ldots\,,$$

where $k_i = k_i\,(a_1, \ldots, a_k)$.

➽ Is it possible to find a geometric theory whose "box" term encodes the coefficients of the generalised FP mass term M_φ?

Yes! Up to spin 11 *(at least) it is just the unique theory with the correct current exchange*.

II. ➽ Why the mass term works well with *all* geometric Einstein tensors? Not too strange, also true for spin 2: the non-local (wrong!) theory defined by the eom

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R} + \lambda\,(\eta - \frac{\partial^2}{\Box})\,\mathcal{R} - m^2\,(h - \eta\,h') = T_{\mu\nu}\,,$$

with $T_{\mu\nu}$ *conserved*, reduces to the Fierz system, and gives the correct current exchange!