Integrability in Supersymmetric Gauge Theories and Topological Strings

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Old story (1995):

$\mathcal{N} = 2$ supersymmetric Yang-Mills theory = Yang-Mills-Higgs system plus fermions:

- Higgs field falls into condensate $\langle \Phi \rangle \in \mathfrak{h}$, and breaks the gauge group up to maximal torus (in general position);

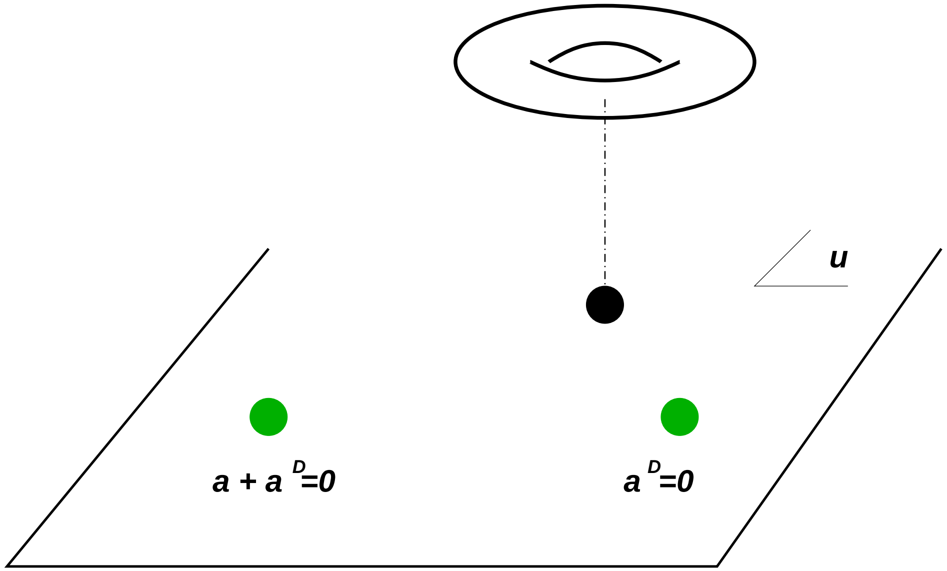
- supersymmetry ensures (partial) cancelation of perturbative corrections, and existence of light BPS states, with masses $\sim |q \cdot a + g \cdot a^D|$, (q, g) - set of electric and magnetic charges.

One may speak on *moduli space* of the theory: $u \sim \langle \mathrm{Tr}\Phi^2 \rangle$, or generally the set coefficients of

$$P(z) = \langle \det(z - \Phi) \rangle \tag{1}$$

Classical moduli space: singular point at the origin $u = 0$, where the gauge group restores, and nothing interesting ... but this is in domain of strong coupling, where quasiclassics does not work.

Quantum moduli space



Gauge group never restores, but there are singularities where BPS states become massless: e.g. the monopole at $a^D = 0$ and dyon at $a + a^D = 0$.

Seiberg-Witten theory: $\mathcal{N}=2$ supersymmetric Yang-Mills theory ($U(N_c)$ gauge group)

$$\mathcal{L}_0 = \frac{1}{g_0^2}\mathrm{Tr}\ \left(\mathsf{F}_{\mu\nu}^2 + |D_\mu\Phi|^2 + [\Phi,\bar{\Phi}]^2 + \dots\right) \tag{2}$$

so that $[\Phi,\bar{\Phi}] = 0 \Rightarrow \Phi = \mathrm{diag}(a_1,\dots,a_{N_c})$, and $D_\mu\Phi \Rightarrow [A_\mu,\Phi]^{ij} = A_\mu^{ij}(a_i - a_j)$, so that only $A_\mu^{ii} \equiv A_\mu^i$ remain massless.

SW theory gives a set of effective couplings $T_{ij}(a)$ in the low-energy $\mathcal{N}=2$ SUSY Abelian $U(1)^{\mathrm{rank}}$ gauge theory.

$$\mathcal{L}_{\mathrm{eff}} = \mathrm{Im}\ T_{ij}(a)\ F_{\mu\nu}^i F_{\mu\nu}^j + \dots \tag{3}$$

with $T_{ij} \underset{\text{weak coupling}}{\longrightarrow} \log\frac{a_i - a_j}{\Lambda} + O\left(\left(\frac{\Lambda}{a}\right)^{2N_c}\right)$.

$\mathcal{N}=2$ kinematics encodes nontrivial information in holomorphic prepotential $T_{ij}=\frac{\partial^2\mathcal{F}}{\partial a_i \partial a_j}$ (effective action is $\mathrm{Im}\int d^4\theta\mathcal{F}(\Phi)$).

The prepotential itself is determined by: Σ of genus=rank, with a meromorphic differential dS_{SW} such that

$$\delta dS_{SW} \simeq \text{holomorphic} \tag{4}$$

or by *an integrable system*.

Period variables $\{a_i = \oint_{A_i} dS_{SW}\}$ and $\mathcal{F}$ are introduced by

$$a_i^D = \oint_{B_i} dS_{SW} = \frac{\partial \mathcal{F}}{\partial a_i} \tag{5}$$

consistent by symmetricity of $\frac{\partial^2\mathcal{F}}{\partial a_i \partial a_j} = T_{ij}(a)$ period matrix of Σ (integrability from Riemann bilinear identities).

Famous example of Σ: let $P_{N_c}(z) = \langle \mathrm{det}(z - \Phi) \rangle$, then

$$w + \frac{\Lambda^{2N_c}}{w} = P_{N_c}(z) = \prod_{i=1}^{N_c} (z - v_i)$$
$$dS_{SW} \simeq z\frac{dw}{w} \qquad (6)$$

Integrable system is N_c-periodic Toda chain.

Simplest possible(?) example $N_c = 2$, $z \to$ momentum, $\log w \to$ coordinate, the curve Σ and dS_{SW} turn into the Hamiltonian and Jacobi form of physical pendulum or the 1d “sine-Gordon” ($\Lambda \to 0$: Liouville) system

$$w + \frac{\Lambda^4}{w} = z^2 - u$$

In fact the simplest possible example is $N_c = 1$ ($U(1)$ $\mathcal{N} = 2$ supersymmetric gauge theory?)

$$\Lambda \left(w + \frac{1}{w} \right) = z - v \tag{7}$$

giving rise to $\mathcal{F} = \frac{1}{2} a^2 t_1 + e^{t_1}$, with $\Lambda^2 = e^{t_1}$, $a = \oint z \frac{dw}{w} = v$.

Indeed, the Toda "chain" (dispersionless limit):

$$\frac{\partial^2 \mathcal{F}}{\partial t_1^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

Stringy solution $\mathcal{F} = \frac{1}{2} a^2 t_1 + e^{t_1}$: a system of particles $a^D = \frac{\partial \mathcal{F}}{\partial a} = a t_1$ with constant velocity = number = a.

Topological A-string on $\mathbb{P}^1$ with quantum cohomology OPE: $\varpi \cdot \varpi \simeq e^{t_1}\mathbf{1}$, primary operators $t_1 \leftrightarrow \varpi$, $a \leftrightarrow \mathbf{1}$: $\mathcal{F} \sim \langle \exp(a\mathbf{1} + t_1\varpi) \rangle$ is a truncated generation function.

Toda hierarchy - the descendants: $t_{k+1} \leftrightarrow \sigma_k(\varpi)$, $T_n \leftrightarrow \sigma_n(\mathbf{1})$, $(a \equiv -T_0)$ then

$$\mathcal{F} = \frac{a^2 t_1}{2} + e^{t_1} \Rightarrow \mathcal{F}(\mathbf{t}, a) \Rightarrow \mathcal{F}(\mathbf{t}, \mathbf{T}) \tag{8}$$

being still a solution to the Toda equation

$$\frac{\partial^2 \mathcal{F}}{\partial t_1^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

Solution is found via dual “Landau-Ginzburg” B-model (the $N_c = 1$ SW curve)

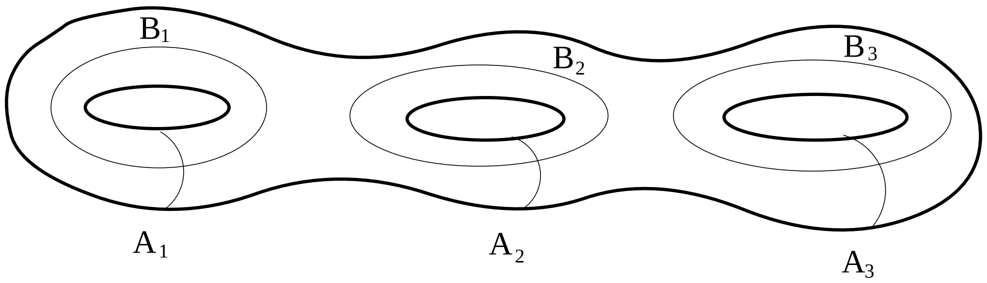
$$z = v + \Lambda\left(w + \frac{1}{w}\right) \tag{9}$$

by construction of a function with asymptotics,

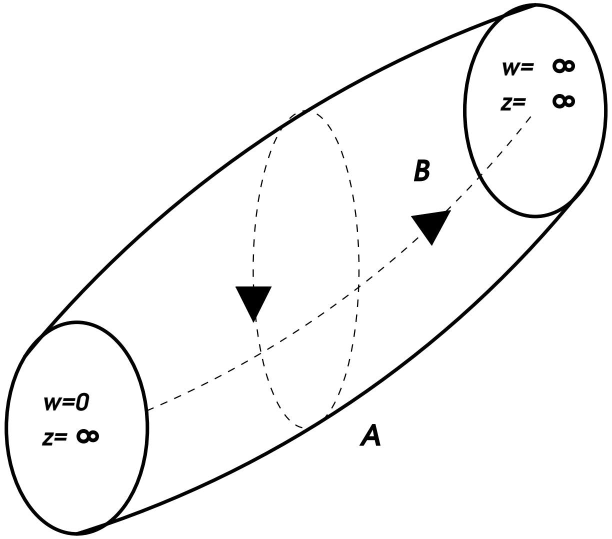
$$\begin{aligned} S(z) \underset{z\to\infty}{=} & \sum_{k>0} t_k z^k - 2\sum_{n>0} T_n z^n(\log z - c_n) + \\ & +2a\log z - \frac{\partial\mathcal{F}}{\partial a} - 2\sum_{k>0}\frac{1}{kz^k}\frac{\partial\mathcal{F}}{\partial t_k} \end{aligned} \tag{10}$$

($c_k = \sum_{i=1}^{k}\frac{1}{i}$), whose “tail” defines the gradients of prepotential (analogs of the dual periods), e.g.

$$\frac{\partial\mathcal{F}}{\partial a} \sim \int_B z\frac{dw}{w} \sim [S]_0$$



Smooth Riemann surface (of genus 3)
with fixed A- and B-cycles.



Cylinder $z = v + \Lambda\left(w + \frac{1}{w}\right)$
with degenerate B- cycle.

What is the sense of this oversimplified example?

Topological A-string: the prepotential counts asymptotics of the Hurwirz numbers, number of ramified covers by string world-sheets of the (target!) $\mathbb{P}^1$.

Gauge-string duality: sum over partitions $\equiv$ summing instantons in 4D $\mathcal{N} = 2$ SUSY gauge theory (Nekrasov partition function).

$U(1)$ gauge theory: non-commutative instantons, Toda hierarchy - the deformation of the UV prepotential

$$F_{UV,0} = \tfrac{1}{2}\tau\Phi^2 \to F_{UV} = \sum_{k>0} \frac{t_k}{k+1}\Phi^k$$

with $\tau = t_1 \sim \log \Lambda$.

Partition function in deformed gauge theory (at $T_n = \delta_{n,1}$)

$$Z(a,\mathbf{t};\hbar) = \sum_{\mathbf{k}} \frac{\mathbf{m}_{\mathbf{k}}^2}{(-\hbar^2)^{|\mathbf{k}|}} \; e^{\frac{1}{\hbar^2}\sum_{k>0}\frac{t_k}{k+1}\mathrm{ch}_{k+1}(a,\mathbf{k},\hbar)} \sim \tag{11}$$
$$\sim \exp\left(\frac{1}{\hbar^2}\mathcal{F}(a,\mathbf{t}) + \ldots\right)$$

is some over set of partitions $\mathbf{k} = k_1 \geq k_2 \geq \ldots$ with the Plancherel measure

$$\mathbf{m}_{\mathbf{k}} = \prod_{i<j} \frac{k_i - k_j + j - i}{j-i} = \frac{\prod_{1\leq i<j\leq \ell_{\mathbf{k}}}(k_i - k_j + j - i)}{\prod_{i=1}^{\ell_{\mathbf{k}}}(\ell_{\mathbf{k}} + k_i - i)!} \tag{12}$$

and particular (Chern) polynomials

$$\mathrm{ch}_0(a,\mathbf{k}) = 1, \quad \mathrm{ch}_1(a,\mathbf{k}) = a, \quad \mathrm{ch}_2(a,\mathbf{k}) = a^2 + 2\hbar^2|\mathbf{k}|$$
$$\mathrm{ch}_3(a,\mathbf{k}) = a^3 + 6\hbar^2 a|\mathbf{k}| + 3\hbar^3 \sum_i k_i(k_i + 1 - 2i) \tag{13}$$
$$\ldots$$

or

$$\left(e^{\frac{\hbar u}{2}} - e^{-\frac{\hbar u}{2}}\right) \sum_{i=1}^{\infty} e^{u(a+\hbar(\frac{1}{2}-i+k_i))} = \sum_{l=0}^{\infty} \frac{u^l}{l!} \, \mathrm{ch}_l(a, \mathbf{k}, \hbar) \qquad (14)$$

coming from the Chern classes of the universal bundle over the instanton moduli space.

The $\mathbf{T}$-dependence $Z(a, \mathbf{t}) \to Z(a, \mathbf{t}, \mathbf{T})$ is restored from the Virasoro constraints

$$L_n(\mathbf{t}, \mathbf{T}; \partial_{\mathbf{t}}, \partial_{\mathbf{T}}; \partial^2_{\mathbf{t}}) Z(a, \mathbf{t}, \mathbf{T}; \hbar) = 0, \quad n \geq -1 \qquad (15)$$

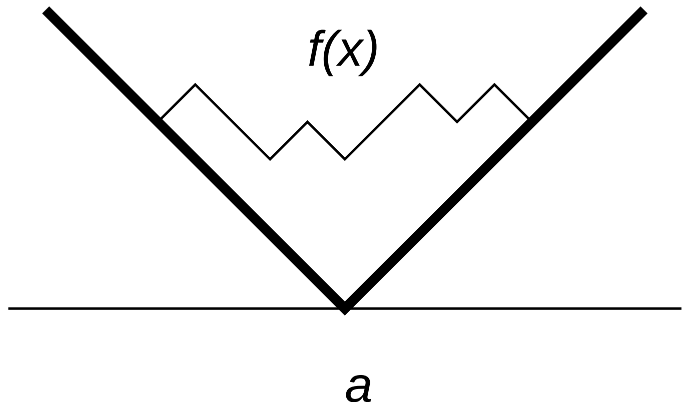
Non Abelian theory: $U(N_c)$ gauge group, nontrivial SW theory. Partition function more complicated, but quasiclassics always given by solution to *the same* functional problem:

$$\mathcal{F} = \int dx f''(x) F_{UV}(x) - \frac{1}{2}\int_{x>\tilde{x}} dx d\tilde{x} f''(x) f''(\tilde{x}) F(x-\tilde{x}) + \\ + \sum_{i=1}^{N_c} a_i^D \left(a_i - \frac{1}{2}\int dx\ x f''(x) \right) \tag{16}$$

with $F_{UV}(x) = \sum_{k>0} t_k \frac{x^{k+1}}{k+1}$, and

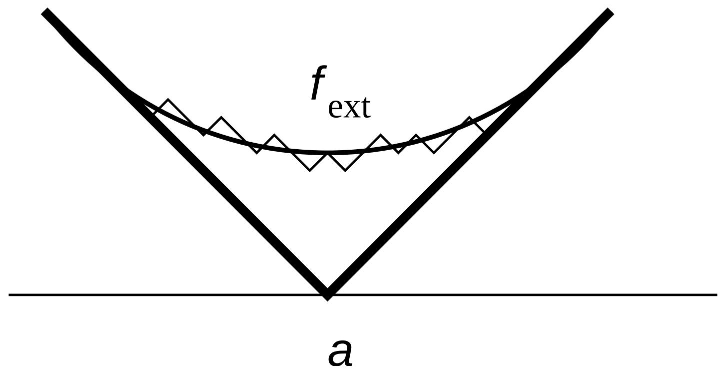
$$\log \mathbf{m_k^2} \to F(x) \propto x^2 \left(\log x - \frac{3}{2} \right)$$

when integrated with (double derivative of the) shape function

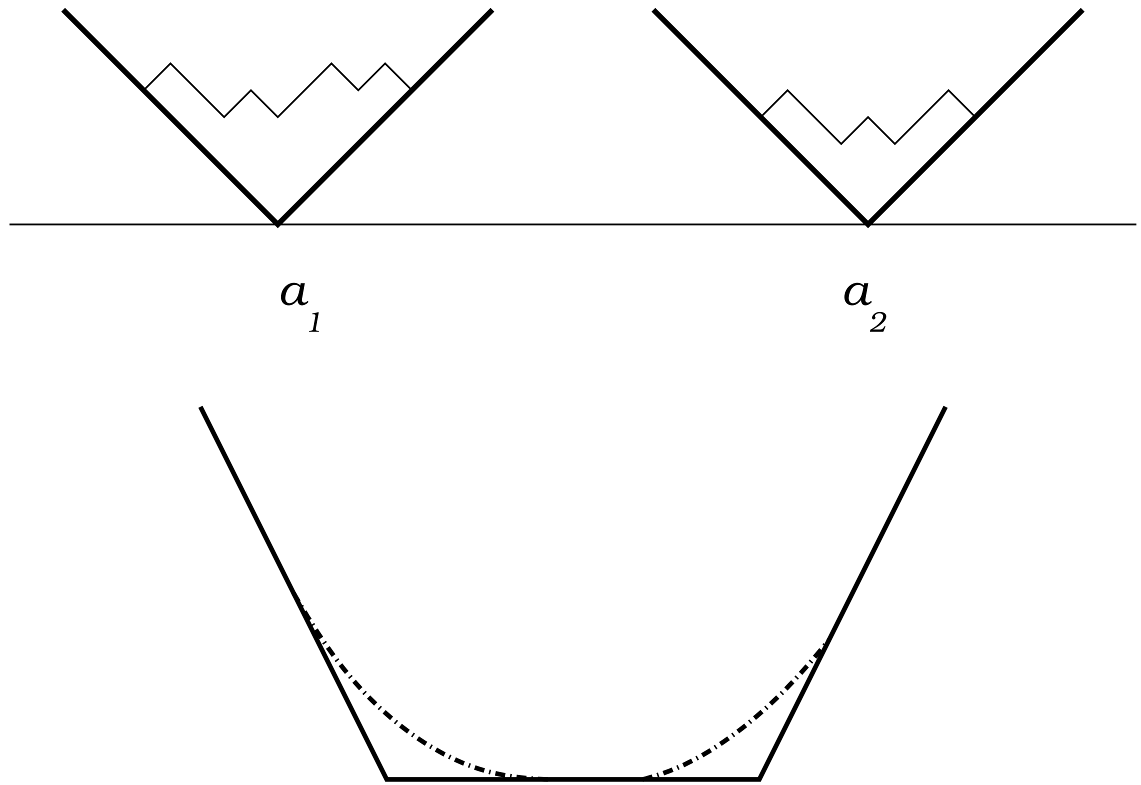


Shape function for partitions (Young diagrams)

$$f(x) = |x - a| + \Delta f(x)$$



Extremal shape for large partition



Non-Abelian theory: extremal shape for $N_c = 2$

From the functional one gets for $S(z) = \frac{d}{dz}\frac{\delta \mathcal{F}}{\delta f''(z)}$

$$S(z) = \sum_{k>0} t_k z^k - \int dx f''(x)(z-x)\left(\log(z-x)-1\right) - a^D \tag{17}$$

with vanishing real part

$$\text{Re } S(x) = \frac{1}{2}\left(S(x+i0)+S(x-i0)\right) = 0 \tag{18}$$

on the cut, where $\Delta f(x) \neq 0$. On the double cover

$$y^2 = \prod_{i=1}^{N_c}(z-x_i^+)(z-x_i^-) \tag{19}$$

S is odd under $y \leftrightarrow -y$, then $f'(x) \sim \text{jump}\left(\Phi(x) = \frac{dS}{dx}\right)$, and

$$d\Phi = \pm \frac{s(z)dz}{\sqrt{\prod_{i=1}^{N_c}(z-x_i^+)(z-x_i^-)}} \tag{20}$$

If all $t_k = 0$, for $k > 1$, $t_1 = \log \Lambda^{N_c}$, $T_n = \delta_{n,1}$:

$$\Phi \underset{P \to P_\pm}{=} \mp 2N_c \log z \pm 2N_c \log \Lambda + O(z^{-1}) \tag{21}$$

and there exists a meromorphic function $w = \Lambda^{N_c} \exp(-\Phi)$, satisfying

$$w + \frac{\Lambda^{2N_c}}{w} = P_{N_c}(z) = \prod_{i=1}^{N_c} (z - v_i) \tag{22}$$

which restores the SW curve.

To restore the dependence on descendants $\sigma_n(\mathbf{1})$ quasiclassically (influenced by Saito formula)

$$\left.\frac{\partial \mathcal{F}}{\partial T_n}\right|_{\mathbf{t}} = (-)^n n!\,(S_n)_0 \tag{23}$$

where

$$\frac{d^n S_n}{dz^n} = S, \quad n \geq 0 \tag{24}$$

or S_n is the n-th primitive (odd under $w \leftrightarrow \frac{1}{w}$).

For higher $t_k \neq 0$, $\exp(-\Phi)$ has an essential singularity and cannot be described algebraically. Implicitly it is fixed by

$$\begin{gathered}\oint_{A_j} d\Phi = -i\pi \int_{\mathbf{I}_j} f''(x)dx = -2\pi i, \\ \mathrm{res}_{P_\pm} d\Phi = \mp 2N_c, \quad \oint_{B_j} d\Phi = 0\end{gathered} \tag{25}$$

Instanton expansion in 4d gauge theory $\mathcal{F} = \sum_{d \geq 0} q^d \mathcal{F}_d$, $q \sim \Lambda^{2N_c}$, $\log \Lambda \sim t_1$.

Topological string expansion: $\hbar$ is background parameter (IR cutoff) in 4d gauge theory.

Topological string condensate: $\langle \sigma_1(\mathbf{1}) \rangle \neq 0$, $T_n = \delta_{n,1}$ is the simplest possible background, while $a \sim T_0$ is the gauge theory condensate itself.

In the pertirbative limit $\Lambda \to 0$ cuts shrink to the points $z = a_j$, $j = 1, \ldots, N_c$: the curve is

$$w_{\text{pert}} = P_{N_c}(z) = \prod_{i=1}^{N_c} (z - v_i) \tag{26}$$

endowed with ($\mathbf{t}(z) \equiv \sum_{k>0} t_k z^k$; $T(x) \equiv \sum_{n>0} T_n x^n$)

$$\begin{aligned} S(z) &= -2 \sum_{j=1}^{N_c} \sigma(z; v_j) + \mathbf{t}'(z) \\ \sigma(z; x) &= \sum_{k>0} \frac{T^{(k)}(x)}{k!} (z - x)^k (\log(z - x) - c_k) \end{aligned} \tag{27}$$

Logic:

- restrict to the N-th class of backgrounds, with only $T_1, \ldots, T_N \neq 0$;

- the "minimal" theory was with $T_n = \delta_{n,1}$ and $\mathcal{F} = \mathcal{F}(a, \mathbf{t})$; $T_1 = 1$ corresponds to the condensate $\langle \sigma_1(\varpi) \rangle \neq 0$;

- $N + 1$-th derivative of S becomes single-valued.

Perturbative solution:

$$a_i^D = S(v_i) = \frac{\partial \mathcal{F}_{\mathsf{pert}}}{\partial a_i} \tag{28}$$

gives rise to

$$\begin{aligned} \mathcal{F}_{\mathsf{pert}}(a_1, \ldots, a_{N_c}; \mathbf{t}, \mathbf{T}) = \sum_{j=1}^{N_c} F_{UV}(a_j; \mathbf{t}, \mathbf{T}) + \\ + \sum_{i \neq j} F(a_i, a_j; \mathbf{T}) \\ a_j = T(v_j), \qquad j = 1, \ldots, N_c \end{aligned} \tag{29}$$

Result: the full functional $\mathcal{F}(a, \mathbf{t}, \mathbf{T})$ is given by solution to:

$$\begin{aligned}\mathcal{F} = -\tfrac{1}{2}\int_{x_1>x_2} & dx_1 dx_2 f''(x_1) f''(x_2) F(x_1, x_2; \mathbf{T}) + \\ & + \int dx f''(x) F_{UV}(x; \mathbf{t}, \mathbf{T}) + \\ & + \sum_i a_i^D \left(a_i - \tfrac{1}{2} \int dx \; x f''(x) \right)\end{aligned} \tag{30}$$

with

$$F_{UV}(x; \mathbf{t}, \mathbf{T}) = \int_0^x \mathbf{t}'(\mathrm{x}) dT(\mathrm{x}) \tag{31}$$

and the kernel

$$\frac{\partial^2 F(x_1, x_2; \mathbf{T})}{\partial x_1 \partial x_2} = T'(x_1) T'(x_2) \log(x_1 - x_2) \tag{32}$$

Nonabelian theory: solve the variational equation

$$\mathbf{t}'(z) - \int dx f''(x)\sigma(z;x) = a^D, \quad z \in \mathbf{I} \tag{33}$$

with $\mathbf{I} = \bigcup$ cuts. The integral

$$S(z) = \mathbf{t}'(z) - a^D - \int dx f''(x)\sigma(z;x) \tag{34}$$

is multivalued, due to the logarithms in $\sigma(z;x)$, but its $N+1$-th derivative

$$d\Phi^{(N-1)} = d\left(\frac{d^N S}{dz^N}\right) \tag{35}$$

can be already decomposed over abelian differentials.

It is determined by singularities at $z(P_\pm) = \infty$ and at the branch points $\{x_j\}$, $j = 1, \ldots, 2N_c$, where it has poles due to $f''(x) \sim (x - x_j)^{-1/2}$ (cf. with matrix models!). In fact $\Phi', \ldots, \Phi^{(N-1)}$ are regular $2-, \ldots, N-$ differentials on the curve.

One writes

$$d\Phi^{(N-1)} = \frac{\phi(z)dz}{y} + \frac{dz}{y}\sum_{j=1}^{2N_c}\sum_{k=1}^{N-1}\left(\frac{q_j^k}{(z-x_j)^k}\right) \tag{36}$$

fix the periods of $d\Phi^{(N-2)}, \ldots, d\Phi'$ by $2N_c$ constraints, ending up, therefore with

$$(2N+1)N_c - 2N_c \cdot N = N_c \tag{37}$$

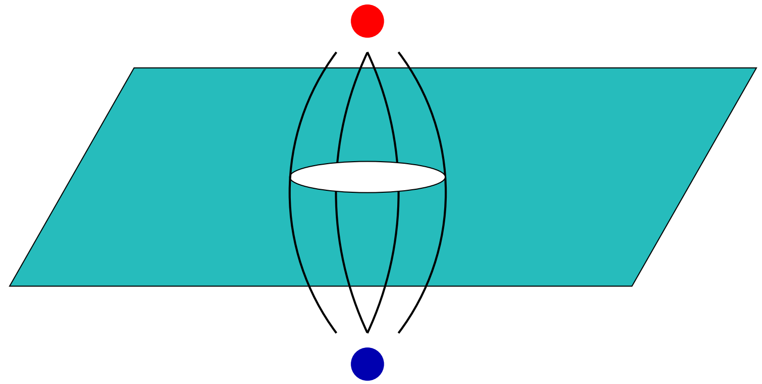
variables, to be absorbed by the Seiberg-Witten periods

$$a_j = \frac{1}{4\pi i}\oint_{A_j}\frac{z^N}{N!}d\Phi^{(N-1)}, \qquad j = 1, \ldots, N_c \tag{38}$$

and define the prepotential by

$$a_j^D = \tfrac{1}{2}\oint_{B_j}\frac{z^N}{N!}d\Phi^{(N-1)} = \frac{\partial\mathcal{F}}{\partial a_j}, \qquad j = 1, \ldots, N_c \tag{39}$$

The Meissner mechanism in superconductor: condensation of electric charge kills magnetic field except for a thin tube, ensuring confinement of magnetic monopoles, if they exist !



To turn into problem of mathematical physics one needs:

- condensates,
- duality between electric and magnetic charges.

- Effective theory near $\mathcal{N} = 2$ singularity or $\mathcal{N} = 1$ vacuum;

- Supersymmetric QCD with large fundamental masses: weak coupling $m \gg \Lambda$ and confinement of monopoles by ANO strings.

- Towards strong coupling: regime of dual theory, $m \ll \Lambda$, change of quantum numbers, but still confinement of monopoles!

New integrable structures:

- Monodromies in "mass moduli space" and KZ equation;

- World-sheet sigma model for ANO string: integrable structure, describing the space of vacua, or quantum numbers in 4d gauge theory!