Disk Scattering of Open and Closed Strings

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Outline

I. Higher point open superstring amplitudes (tree)

St. St., T.R. Taylor 2006–2008.

- Universal properties and relations

II. Open & closed vs. pure open string disk amplitudes

St. St., to appear very soon.

- Sort of generalized KLT on the disk

I. Recent results for $N-$point open superstring amplitudes

$N-$point open string disk amplitudes in background with CFT description

St. St., T.R. Taylor 2006–2008

Motivation: Recent results in YM in spinor basis:
compact expressions, recursion relations, . . .

- Computed $N-$point open superstring disk amplitude involving members of vector multiplets to all orders in α',
- Compact representation to all orders in α',
- Derived SUSY Ward identities to all orders in α'

Universal Properties

- completely model independent
- universal to all string compactifications
- any numbers of supersymmetries

Examples with members of vector multiplets

- 5–gluon MHV amplitude in superstring theory

$$A(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;=\; \mathrm{Tr}(T^1 \dots T^5)\,(\sqrt{2}\, g_{YM})^3\, \alpha'$$
$$\times\;\; \frac{\langle 1\,2\rangle^2}{\langle 3\,4\rangle^2 \langle 4\,5\rangle}\,\big(\,\langle 4\,1\rangle [1\,5]\; \boldsymbol{K}_1 + \langle 4\,2\rangle [2\,5]\; \boldsymbol{K}_2\,\big)$$

- Supersymmetric Ward identities in string theory

$$A(g_1^-, g_2^-, g_3^+, g_4^+, \dots, g_N^+) \;=\; \frac{\langle 12\rangle^2}{\langle 34\rangle^2}\, A(\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+, g_5^+, \dots, g_N^+)$$

- N–gluon MHV amplitude in superstring theory

$$A(g_1^-, g_2^-, g_3^+, g_4^+, \dots, g_N^+; \alpha') \;=\; \left(1 - \alpha'^2 \frac{\zeta(2)}{2} F^{(N)}\right)$$
$$\times\;\; A(g_1^-, g_2^-, g_3^+, g_4^+, \dots, g_N^+) + \mathcal{O}(\alpha'^3)$$

Note:

SUSY transformations within one multiplet (VM) using

- $\mathcal{N}$ conserved SUSY charges $\mathcal{Q}_\alpha^I$, $I=1,\ldots,\mathcal{N}$, with $\mathcal{Q}_\alpha^I = \oint \frac{dz}{2\pi i}\ V_\alpha^I(z)$
- (Space–time) SUSY transformation of open string vertex operator $\mathcal{O}$ on world–sheet disk $[\ Q^I(\eta_I)\ ,\ \mathcal{O}(z)\ ] := \oint\limits_{C_z} \frac{dw}{2\pi i}\ \eta_I^\alpha\ V_\alpha(w)\ \mathcal{O}(z)$

generates SUSY Ward identitites (valid to all orders in α')

c.f. also talk at Strings 2008.

Generalizations and Task

- Include chiral multiplets (N=1)
- Use of world–sheet supercurrent T_F
- Include closed strings to probe brane/bulk couplings

$\longrightarrow$ *Derive relations between different types of amplitudes*

$\longrightarrow$ *Amplitudes of open and closed string moduli*

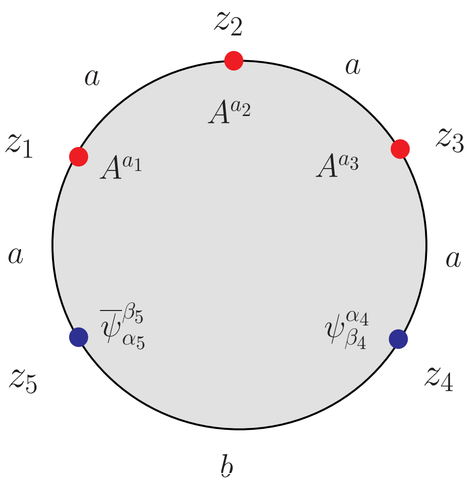
First look: N–point parton amplitudes in $D = 4$

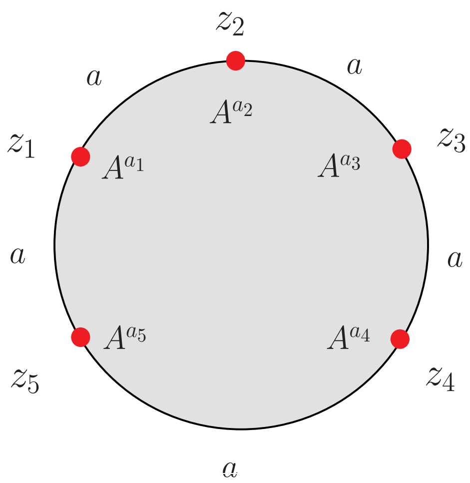
Consider superstring disk amplitudes involving both

VM $\left\{ \begin{array}{l} g = \text{gluon} \\ \chi = \text{gaugino} \end{array} \right.$

CM $\left\{ \begin{array}{l} \psi = \text{fermion} \\ \phi = \text{scalar} \end{array} \right.$ in $D=4$

E.g.:





$$A_\rho(g_1^-, g_2^+, g_3^+, q_4^-, \bar{q}_5^+) = [\, V^{(5)}(s_j) - 2i\, P^{(5)}(s_j)\, \epsilon(1,2,3,4)\,] \times A_\rho^{FT}(g_1^-, g_2^+, g_3^+, q_4^-, \bar{q}_5^+)$$

$$A_\rho(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) = [\, V^{(5)}(s_j) - 2i\, P^{(5)}(s_j)\, \epsilon(1,2,3,4)\,] \times A_\rho^{FT}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+)$$

Striking relation to all orders in α' !

N–point parton amplitudes in $D = 4$

with:

$$A^{FT}_{\rho}(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;=\; i\; g^3_{YM}\; \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$A^{FT}_{\rho}(g_1^-, g_2^+, g_3^+, q_4^-, \bar{q}_5^+) \;=\; 4\; g^3_{YM}\; \frac{\langle 14\rangle^4\langle 15\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

and:

$$V^{(5)}(s_i) \;=\; s_2 s_5\; f_1 + \tfrac{1}{2}\;(s_2 s_3 + s_4 s_5 - s_1 s_2 - s_3 s_4 - s_1 s_5)\; f_2$$

$$P^{(5)}(s_i) \;=\; f_2\;, \qquad \epsilon(i,j,m,n) = \alpha'^2\, \epsilon_{\alpha\beta\mu\nu}\; k_i^\alpha\; k_j^\beta\; k_m^\mu\; k_n^\nu$$

C.f.:

$$A_{\rho}(g_1^-, g_2^-, g_3^+, g_4^+) \;=\; 4\; g^2_{YM}\; V^{(4)}(s_j)\; \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

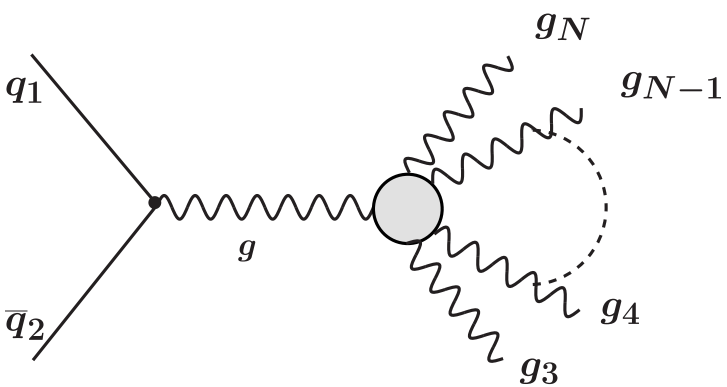
$$A_{\rho}(g_1^-, g_2^+, q_3^-, \bar{q}_4^+) \;=\; 2\; g^2_{YM}\; V^{(4)}(s_j)\; \frac{\langle 13\rangle^4\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

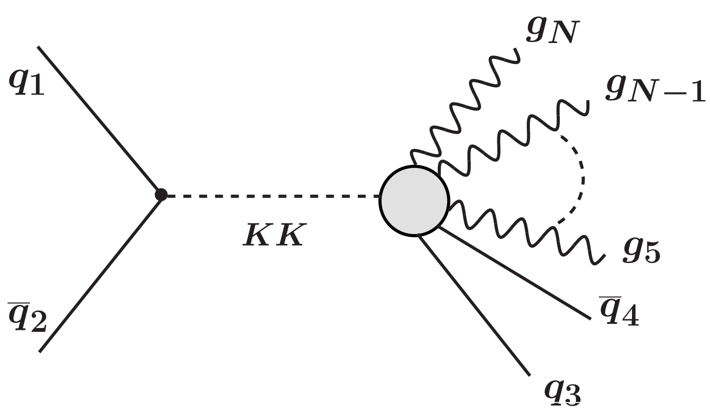
N–point parton amplitudes in $D = 4$

Relations can be generalized to:

$$\begin{cases} A(g^{a_1} \ldots g^{a_N}) \\ A(\chi^{a_1}\overline{\chi}^{a_2} g^{a_1} \ldots g^{a_{N-2}}) \\ A(\psi^{a_1}\overline{\psi}^{a_2} g^{a_1} \ldots g^{a_{N-2}}) \\ A(\phi^{a_1}\overline{\phi}^{a_2} g^{a_1} \ldots g^{a_{N-2}}) \end{cases}$$

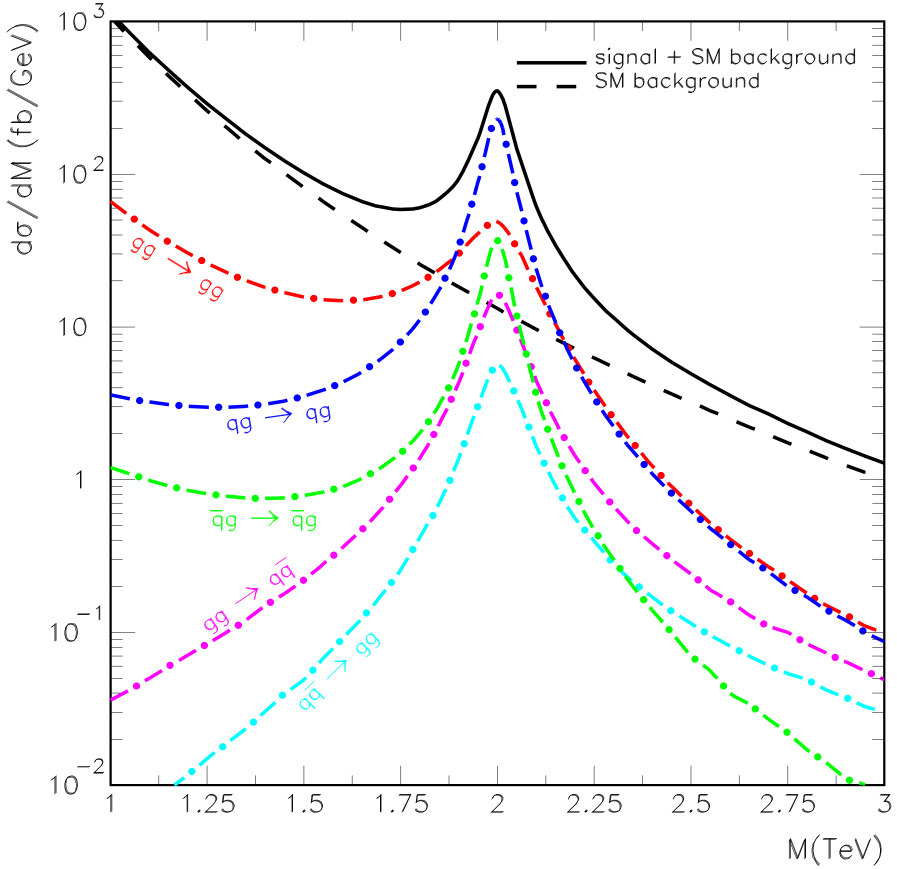
No intermediate exchange of KKs nor windings !





Amplitudes important for low string scale physics

Most relevant for signals from low string scale effects in QCD jets

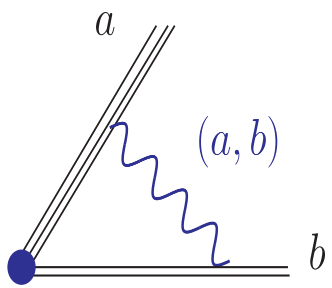


- No intermediate exchange of KKs, windings nor emmission of graviton
- Useful for model–independent low–energy predictions
- Universal deviation from SM in jet distribution

Lüst, St.St., Taylor, arXiv:0807.3333; Anchordoqui, Goldberg, Nawata, Lüst, St.St., Taylor, arXiv:0808.0497, arXiv:0904.3547; Lüst, Schlotterer, St.St., Taylor, to appear

Appendix: Chiral matter vertex operator

Vertex operator of chiral fermion (a,b)



$$V^{(-1/2)}_{\psi^{\alpha}_{\beta}}(z,u,k) = g_{\psi}\ [T^{\alpha}_{\beta}]^{\beta_1}_{\alpha_1}\ \ e^{-\frac{1}{2}\phi(z)}\ \ u^{\lambda}S_{\lambda}(z)\ \ \Xi^{a\cap b}(z)\ e^{ik_{\rho}X^{\rho}(z)}$$

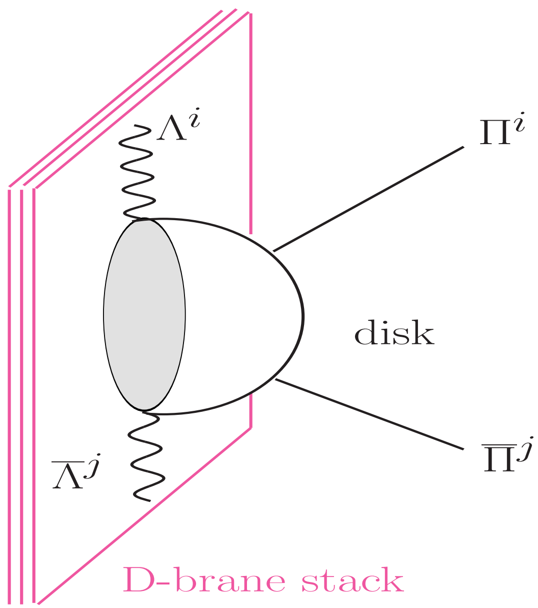
[$g_{\psi}=(2\alpha')^{1/2}\alpha'^{1/4}\ e^{\phi_{10}/2}$]

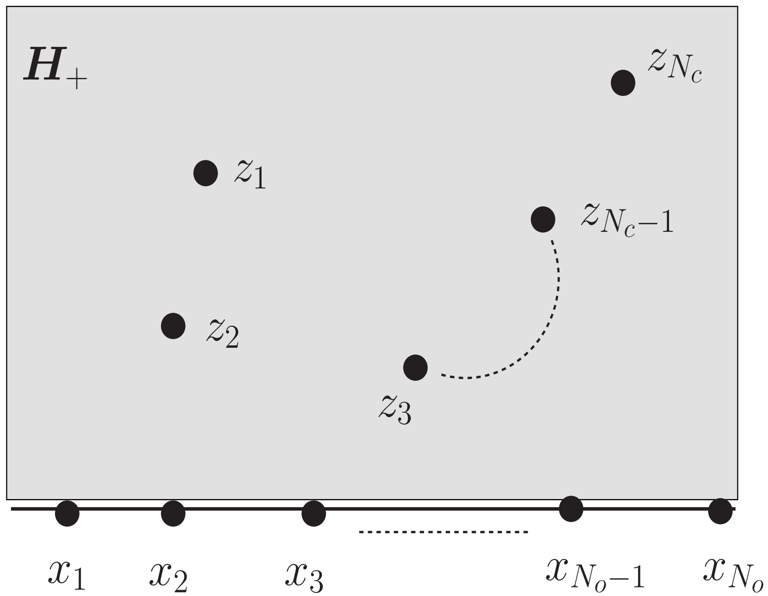
Boundary changing operator $\Xi^{a\cap b}(z)$, with $h=\frac{3}{8}$ and:

$$\langle \Xi^{a\cap b}(z_1)\ \overline{\Xi}^{a\cap b}(z_2)\rangle = \frac{1}{(z_1-z_2)^{3/4}}$$

II. Disk scattering of open and closed strings

$$\mathcal{A} = \sum_{\pi \in S_{N_o}/\mathbf{Z}_2} V_{\mathrm{CKG}}^{-1} \left(\prod_{j=1}^{N_o} \int_{\mathcal{I}_\pi} dx_j \prod_{i=1}^{N_c} \int_{H_+} d^2 z_i \right) \langle \prod_{j=1}^{N_o} : V_o(x_j) : \prod_{i=1}^{N_c} : V_c(\overline{z}_i, z_i) : \rangle$$





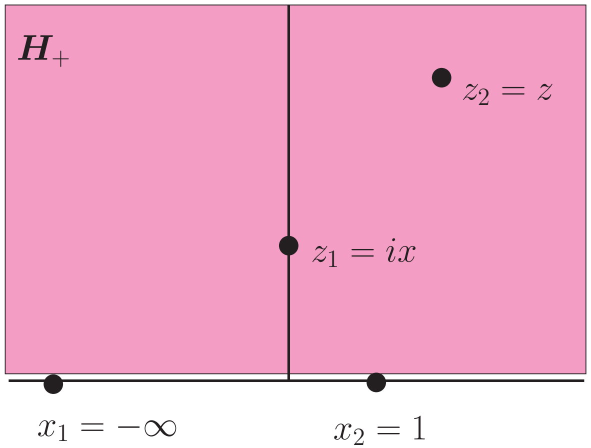
$V_o(x_i)$ = open string vertex operators inserted at x_i on the boundary of the disk

$V_c(\overline{z}_i, z_i)$ = closed string vertex operators inserted at z_i inside the disk

Example: Two open and two closed strings on the disk

With $PSL(2,\mathbf{R})$ transformation three arbitray points $w_1, w_2 \in \mathbf{R}$ and $w_3 \in \mathbf{C}$ may be mapped to the points x_1, x_2 and z_1:

Choice: $x_1 = -\infty$, $x_2 = 1$, $\overline{z}_1 = -ix$, $z_1 = ix$, $\overline{z}_2 = \overline{z}$, $z_2 = z$



with $z \in \mathbf{H}_+$ and $x \in \mathbf{R}^+$

$$\mathcal{A}(1,2,3,4) = \int_{-\infty}^{\infty} dx \, \langle c(-\infty)c(1)c(ix)\rangle$$
$$\times \int_{\mathbf{C}} d^2z \, \langle :V_o(-\infty): :V_o(1): :V_c(-ix,ix): :V_c(\overline{z},z):\rangle$$

Two open & two closed strings versus six open strings on the disk

- generic structure of world–sheet disk amplitude of **two open** & **two closed strings**:

$$W^{(\kappa,\alpha_0)}\begin{bmatrix}\alpha_1,\lambda_1,\gamma_1,\beta_1\\ \alpha_2,\lambda_2,\gamma_2,\beta_2\end{bmatrix} = \int\limits_{-\infty}^{\infty} dx\ x^{\alpha_0}\ (1+ix)^{\alpha_1}\ (1-ix)^{\alpha_2} \int\limits_{\mathbf{C}} d^2z\ (1-z)^{\lambda_1}\ (1-\overline{z})^{\lambda_2}$$
$$\times\ (z-\overline{z})^{\kappa}\ (z-ix)^{\gamma_1}\ (\overline{z}-ix)^{\gamma_2}\ (z+ix)^{\beta_1}\ (\overline{z}+ix)^{\beta_2}$$

- generic structure of world–sheet disk amplitude of **six open strings**: Oprisa St.St., 2005

$$F\begin{bmatrix}n_1,n_2,n_3\\ n_4,n_5,n_6,n_7,n_8,n_9\end{bmatrix} = \int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^1 dz\ x^{p_{23}+n_1}\ y^{p_{23}+k_{24}+p_{34}+n_2}\ z^{p_{16}+n_3}$$
$$\times\ (1-x)^{p_{34}+n_4}\ (1-y)^{p_{45}+n_5}\ (1-z)^{p_{56}+n_6}\ (1-xy)^{p_{35}+n_7}$$
$$\times\ (1-yz)^{p_{46}+n_8}\ (1-xyz)^{p_{36}+n_9}\quad ,\quad n_i\in\mathbf{Z}$$

Two open & two closed strings versus six open strings on the disk

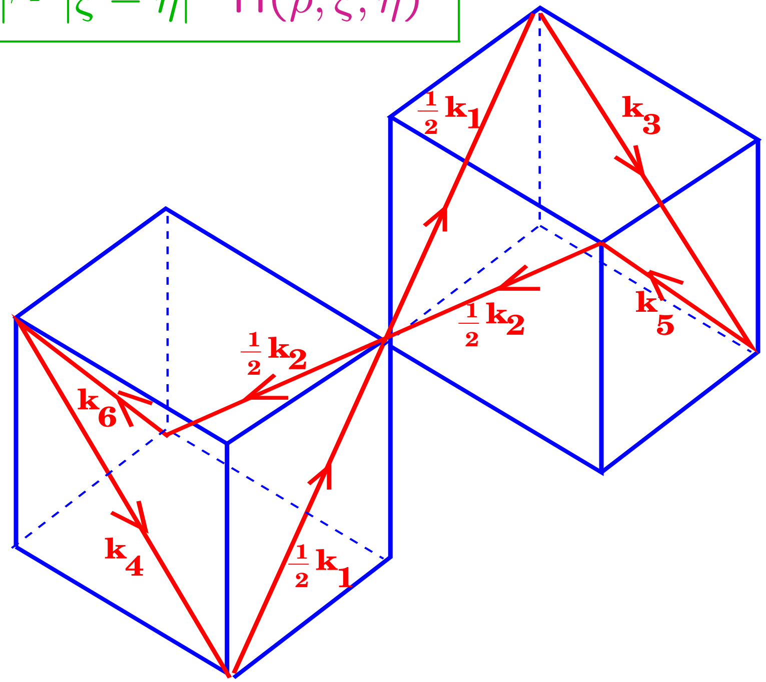
After splitting the complex integral into holomorphic and anti–holomorphic pieces:
Analytic continuation, introduce $\xi = z_1 + iz_2$, $\eta = z_1 - iz_2$, $\rho = ix$, $\quad \rho, \xi, \eta \in \mathbf{R}$.

$$W^{(\kappa,\alpha_0)}\begin{bmatrix}\alpha_1,\lambda_1,\gamma_1,\beta_1\\ \alpha_2,\lambda_2,\gamma_2,\beta_2\end{bmatrix} = \frac{1}{2}\int\limits_{-\infty}^{\infty} d\rho\ |\rho|^{\alpha_0}\ |1+\rho|^{\alpha_1}\ |1-\rho|^{\alpha_2}$$
$$\times \int\limits_{-\infty}^{\infty} d\xi \int\limits_{-\infty}^{\infty} d\eta\ |1-\xi|^{\lambda_1}\ |\xi-\rho|^{\gamma_1}\ |\xi+\rho|^{\beta_1}$$
$$\times\ |1-\eta|^{\lambda_2}\ |\eta-\rho|^{\gamma_2}\ |\eta+\rho|^{\beta_2}\ |\xi-\eta|^{\kappa}\ \Pi(\rho,\xi,\eta)$$

Answer: Six open strings, with:

$$z_1 = -\infty, \quad z_2 = 1, \quad z_3 = -\rho,$$
$$z_4 = \rho, \quad z_5 = \xi, \quad z_6 = \eta$$

$$p_1 = k_1, \qquad p_2 = k_2,$$
$$p_3 = p_4 = \tfrac{1}{2}k_3, \qquad p_5 = p_6 = \tfrac{1}{2}k_4$$



Two open & two closed strings versus six open strings on the disk

$$
\begin{aligned}
& e^{i\pi t} A[1,3,4,5,2,6] + A[1,3,4,5,6,2] + A[1,3,4,6,5,2] + \\
& e^{\frac{i\pi s}{2}+i\pi t} A[1,3,5,4,2,6] + e^{\frac{i\pi s}{2}} A[1,3,5,4,6,2] + \\
& e^{\frac{i\pi s}{2}} A[1,3,6,4,5,2] + e^{i\pi s} \left(A[1,3,5,6,4,2] + A[1,3,6,5,4,2]\right) + \\
& e^{i\pi t} A[1,4,3,5,2,6] + A[1,4,3,5,6,2] + A[1,4,3,6,5,2] + \\
& e^{\frac{i\pi s}{2}+i\pi t} A[1,4,5,3,2,6] + e^{\frac{i\pi s}{2}} A[1,4,5,3,6,2] + e^{\frac{i\pi s}{2}} A[1,4,6,3,5,2] + \\
& e^{i\pi s} \left(A[1,4,5,6,3,2] + A[1,4,6,5,3,2]\right) + e^{i\pi s} A[1,6,3,4,5,2] + \\
& e^{i\pi u} \Big(e^{\frac{i\pi s}{2}} A[1,3,2,5,4,6] + e^{i\pi s} \left(A[1,3,2,5,6,4] + A[1,3,2,6,5,4]\right) + \\
& \quad e^{\frac{i\pi s}{2}+i\pi t} A[1,3,5,2,4,6] + e^{i\pi s+i\pi t} A[1,3,5,2,6,4] + \\
& \quad e^{i\pi s+i\pi t} A[1,3,6,2,5,4] + e^{i\pi s} \left(A[1,3,5,6,2,4] + A[1,3,6,5,2,4]\right) + \\
& \quad e^{\frac{i\pi s}{2}+i\pi t} A[1,6,3,2,5,4] + e^{\frac{i\pi s}{2}} A[1,6,3,5,2,4] \Big) + \\
& e^{\frac{i\pi s}{2}} A[1,6,3,5,4,2] + e^{i\pi s} A[1,6,4,3,5,2] + e^{\frac{i\pi s}{2}} A[1,6,4,5,3,2]
\end{aligned}
$$

Two open & two closed strings versus six open strings on the disk

After inspecting phase $\Pi(\rho, \xi, \eta)$:

$$W^{(\kappa,\alpha_0)}\begin{bmatrix}\alpha_1,\lambda_1,\gamma_1,\beta_1\\ \alpha_2,\lambda_2,\gamma_2,\beta_2\end{bmatrix} = \sigma_\gamma\ \sin(\pi\beta_2)\ [A(163542)+A(163524)+A(164532)]$$
$$+\ \sin(\pi\lambda_2)\ [A(134526)+A(143526)]$$
$$+\ \sigma_\lambda\sigma_\gamma\ \sin(\pi\gamma_2)\ A(132546)+R$$

with the six open string orderings

$$\begin{cases} A(163542): & z_1<z_6<z_3<z_5<z_4<z_2\\ A(163524): & z_1<z_6<z_3<z_5<z_2<z_4\\ A(134526): & z_1<z_3<z_4<z_5<z_2<z_6\\ A(132546): & z_1<z_3<z_2<z_5<z_4<z_6\\ A(164532): & z_1<z_6<z_4<z_5<z_3<z_2\\ A(143526): & z_1<z_4<z_3<z_5<z_2<z_6\end{cases}$$

Two open & two closed strings versus six open strings on the disk

After inspecting phase $\Pi(\rho, \xi, \eta)$:

- many different contributions (open string orderings) $A(a, b, c, d, e, f)$

- many striking relations:

$$
\begin{aligned}
A(1,5,3,6,4,2) &= A(1,2,3,5,4,6) \ , \\
A(1,5,4,6,3,2) &= A(1,2,4,5,3,6) \ , \\
A(1,2,3,6,4,5) &= A(1,2,4,5,3,6) \ , \\
A(1,2,4,6,3,5) &= A(1,2,3,5,4,6) \ ,
\end{aligned}
$$

$$
\begin{aligned}
A(1,2,3,6,5,4) &= \frac{\cos\left[\frac{\pi}{2}(s+t)\right]}{\sin\left(\frac{\pi\ t}{2}\right)}\ A(1,2,3,4,6,5) + \frac{\cos\left[\frac{\pi}{2}(s+t)\right]}{\sin\left(\frac{\pi\ t}{2}\right)}\ A(1,2,4,3,6,5) \\
&+ \frac{\cos\left[\frac{\pi}{4}(s+2t)\right]}{\sin\left(\frac{\pi t}{2}\right)}\ A(1,2,4,5,3,6) \\
A(1,2,4,6,5,3) &= \frac{\cos\left[\frac{\pi}{2}(s+t)\right]}{\sin\left(\frac{\pi\ t}{2}\right)}\ A(1,2,3,4,6,5) + \frac{\cos\left[\frac{\pi}{2}(s+t)\right]}{\sin\left(\frac{\pi\ t}{2}\right)}\ A(1,2,4,3,6,5) \\
&+ \frac{\cos\left[\frac{\pi}{4}(s+2t)\right]}{\sin\left(\frac{\pi t}{2}\right)}\ A(1,2,3,5,4,6)
\end{aligned}
$$

$\Longrightarrow$ six–dimensional basis !

Appendix

To obtain canonical form of open string amplitudes given by generalized Euler integrals (along segment $[0,1]$)

requires rather involved transformations:

$$
\begin{array}{llll}
I_1: & \rho \to -1 + \dfrac{2}{1+yz}, & \xi \to 1 - \dfrac{2y}{1+yz}, & \eta \to 1 - \dfrac{2}{x(1+yz)} \\
I_2: & \rho \to \dfrac{1}{1-2yz}, & \xi \to \dfrac{1-2y}{1-2yz}, & \eta \to -\dfrac{2-x}{x(1-2yz)} \\
I_3: & \rho \to \dfrac{xy}{2-xy}, & \xi \to \dfrac{(2-x)y}{2-xy}, & \eta \to \dfrac{2-xyz}{z(2-xy)} \\
I_4: & \rho \to -\dfrac{1}{1-2xy}, & \xi \to \dfrac{1-2y}{1-2xy}, & \eta \to -\dfrac{2-z}{z(1-2xy)}
\end{array}
$$

Open & closed vs. pure open string disk amplitude

General: Disk amplitude involving N_o open and N_c closed strings

is mapped to disk amplitudes of $N_o + 2N_c$ open strings

E.g.:

$N_o = 2,\ N_c = 1 \implies$ four open strings

$N_o = 3,\ N_c = 1 \implies$ five open strings

$N_0 = 4,\ N_c = 1\ ,\ N_o = 2,\ N_c = 2 \implies$ six open strings

$\vdots \qquad \vdots$

E.g.:

$$N_o = 2,\ N_c = 1:\quad G\left[\alpha_0, \alpha_1, \alpha_2\right] = \sin(\pi\lambda)\ A(1234)$$

$$N_o = 3,\ N_c = 1:\quad G^{(\alpha)}\begin{bmatrix}\lambda_1,\gamma_1\\ \lambda_2,\gamma_2\end{bmatrix} = \sin(\pi\lambda_2)\ A(15243) + \sigma_\gamma\ \sin(\pi\alpha)\ A(12345)$$

Non–trivial: $(N_0 + 2N_c - 3)!$–dimensional basis of functions

Open string disk amplitudes

Basic ingredients of open & closed disk amplitude:
$(N-3)!$ (color) ordered open string amplitudes $A(1,\ldots,N)$.

The full open string tree–level N–point amplitude $\mathcal{A}$:

$$\mathcal{A}(1,2,\ldots,N) = g_{YM}^{N-2} \sum_{\sigma\in S_{N-1}} \mathrm{Tr}(T^{a_1}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(N)}})\ A(1,\sigma(2),\ldots,\sigma(N))$$

with $S_{N-1}=S_N/\mathbf{Z}_N$ and states all in the adjoint representation

$A(1,2,\ldots,N)$ tree–level color–ordered N–leg partial amplitude (helicity subamplitude)

The $(N-1)!$ subamplitudes are not all independent.
In addition to **cyclic symmetries** by applying **reflection** and **parity symmetries**

$$\begin{aligned} A(1,2,\ldots,N) &= A(1,N,\ldots,2)\ , \\ A(1,2,\ldots,N) &= (-1)^N\ A(N,\ldots,2,1) \end{aligned}$$

reduce the number of independent partial amplitudes from $(N-1)!$ to $\frac{1}{2}(N-1)!$

Field–theory $D = 4$

Moreover in $D = 4$ FT <u>further</u> relations found by:

- Kleiss, Kuijf, 1989 $(N-2)!$
  Del Duca, Dixon, Maltoni, 2000
- Bern, Carrasco, Johanson, 2008 $(N-3)!$

E.g.: Subcyclic property (photon-decoupling identity)

$$\sum_{\sigma \in S_{N-1}} A_{FT}(1, \sigma(2), \sigma(3), \ldots, \sigma(N)) = 0$$

In STTH these relations **do not** hold beyond FT order !

World–sheet derivation of amplitude relations

However:

By applying world–sheet string techniques

$\Longrightarrow$ new algebraic identities

- proof does not rely on any kinematic properties of the subamplitudes
- these relations hold in any space–time dimensions D
- for any amount of supersymmetry

World–sheet derivation of amplitude relations

E.g. $N=4$:

$$\frac{A(1,2,4,3)}{A(1,2,3,4)}=\frac{\sin(\pi u)}{\sin(\pi t)} \quad , \quad \frac{A(1,3,2,4)}{A(1,2,3,4)}=\frac{\sin(\pi s)}{\sin(\pi t)}$$

As a result these relations allow to express all six partial amplitudes in terms of **one**, say $A(1,2,3,4)$:

$$\begin{aligned} A(1,4,3,2) &= A(1,2,3,4) \ , \\ A(1,2,4,3) &= A(1,3,4,2)=\frac{\sin(\pi u)}{\sin(\pi t)}\, A(1,2,3,4) \ , \\ A(1,3,2,4) &= A(1,4,2,3)=\frac{\sin(\pi s)}{\sin(\pi t)}\, A(1,2,3,4) \ . \end{aligned}$$

Clearly, in the field–theory limit the relations simply reduce to the well–known identities:

$$\frac{A_{FT}(1,2,4,3)}{A_{FT}(1,2,3,4)}=\frac{u}{t} \quad , \quad \frac{A_{FT}(1,3,2,4)}{A_{FT}(1,2,3,4)}=\frac{s}{t}$$

Subcyclic property $A_{FT}(1,2,3,4)+A_{FT}(1,3,4,2)+A_{FT}(1,4,2,3)=0$

World–sheet derivation of amplitude relations

E.g. $N = 5$: Relations:

$$\sin[\pi(s_2-s_4)]\ A(1,2,3,4,5)+\{\sin[\pi(s_1+s_2-s_4)]-\sin(\pi s_1)\}\ A(1,3,4,5,2)$$
$$+\ \sin[\pi(s_2-s_4)]\ A(1,4,5,2,3)+\{\sin(\pi s_5)+\sin[\pi(s_2-s_4-s_5)]\}\ A(1,5,2,3,4)=0$$

$$[\sin(\pi s_1)+\sin(\pi s_5)]\ A(1,2,3,4,5)+\sin[\pi(s_1+s_5)]\ A(1,3,4,5,2)$$
$$+\ \{\sin[\pi(s_1+s_2-s_4)]-\sin[\pi(s_2-s_4-s_5)]\}\ A(1,4,5,2,3)+\sin[\pi(s_1+s_5)]\ A(1,5,2,3,4)=0$$

As a result these relations allow to express all six partial amplitudes in terms of **two**, say $A(1,2,3,4,5)$ and $A(1,3,2,4,5)$

$$\begin{aligned} A(1,2,5,4,3) &= -A(1,3,4,5,2)=\sin[\pi(s_3-s_1-s_5)]^{-1} \\ &\times \{\ \sin[\pi(s_3-s_5)]\ A(1,2,3,4,5)+\sin[\pi(s_2+s_3-s_5)]\ A(1,3,2,4,5)\ \}\ , \\ A(1,3,4,2,5) &= -A(1,5,2,4,3)=\sin[\pi(s_3-s_1-s_5)]^{-1} \\ &\times \{\ \sin(\pi s_1)\ A(1,2,3,4,5)-\sin[\pi(s_1+s_2)]\ A(1,3,2,4,5)\ \}\ ,\ldots \end{aligned}$$

Clearly, in the field theory limit, these two relations boil down to the subcyclic identity $A_{FT}(1,2,3,4,5)+A_{FT}(1,3,4,5,2)+A_{FT}(1,4,5,2,3)+A_{FT}(1,5,2,3,4)=0.$

World–sheet derivation of amplitude relations

- These relations allow for a **complete reduction** of the full string subamplitudes to a **minimal basis of** $(N-3)!$ **subamplitudes** just like in field–theory
- **Reproduce** Kleiss–Kuijf and Bern–Carrasco–Johanson identitities in field–theory limit

$\Longrightarrow$ Basic ingredients of **open & closed** disk amplitude are $(N-3)!$ (color) ordered **open** string amplitudes $A(1,\ldots,N)$

Open & closed vs. pure open string disk amplitudes

Sort of generalized KLT on the disk

$$V_{\text{closed}}(\bar{z}_i, z_i) \simeq V_{\text{open}}(\bar{z}_i)\ V_{\text{open}}(z_i) \simeq V_{\text{open}}(\eta_i)\ V_{\text{open}}(\xi_i)$$

$$z_i \in \mathbf{C} \qquad\qquad \eta_i,\ \xi_i \in \mathbf{R}$$

E.g.:

$$\langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ G_{\mu_3\mu_4}(\bar{z}_1, z_1)\ G_{\mu_5\mu_6}(\bar{z}_2, z_2)\rangle$$

$$\simeq \langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ A_{\mu_3}(\eta_1)\ A_{\mu_4}(\xi_1)\ A_{\mu_5}(\eta_2)\ A_{\mu_6}(\xi_2)\rangle$$

$$\langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ F_{\alpha\dot{\beta}}(\bar{z}_1, z_1)\ F_{\gamma\dot{\delta}}(\bar{z}_2, z_2)\rangle$$

$$\simeq \langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ \chi_\alpha(\eta_1)\ \chi_{\dot{\beta}}(\xi_1)\ \chi_\gamma(\eta_2)\ \chi_{\dot{\delta}}(\xi_2)\rangle$$

Open & closed vs. pure open string disk amplitudes

This map reveals

important relations between

open & closed string disk amplitudes

and pure open string disk amplitudes !