Space-time Supersymmetry with Vector and Scalar Super Charges[1]

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[1]Based on arXiv:0801.2702, arXiv:0812.1982,arXiv:0901.4862, in collaboration with R. Casalbuoni, F. Elmetti, K. Kamimura, G. Longhi, L. Tamassia and A. Van Proeyen.

Outline

Green-Schwarz (GS) string

Basic variables, 10 scalar fields, coordinates of string and 32 space-time super-partners , 10d Majorana spinors

$$X^m(\tau,\sigma), \theta_\alpha(\tau,\sigma), m = 0, 1, \cdots 9, \alpha = 1, \cdots 32$$

Classical action Super Poincare invariant.

Generators

$$M_{\mu\nu}, P_\mu, Q_\alpha$$

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^\mu)_{\alpha\beta} P_\mu$$

$\#$ Bosons= $\#$ Fermions, On-shell degrees of freedom

Ramond-Neveu-Schwarz (RNS) string

Basic variables, 10 scalar fields, coordinates of string and 10 worldsheet 2d Majorana spinors

$$X^m(\tau,\sigma), \psi_i^m(\tau,\sigma), \quad i = 1, 2$$

No Super-Poincare invariance at level of the action.

After GSO projection spectrum, Super-Poincare invariant spectrum
$\#$ Bosons= $\#$ Fermions.

Can we find an space-time supersymmetry for RNS string?

Is there any relation with Super Poincare supersymmetry?

Quantization of strings in RR backgrounds

Difficulties covariant quantization of GS string

How to covariant quantize GS string in the presence of RR background?

Lightcone quantization

Pure spinor formalism

Others?

Can Vsusy help to solve these difficulties?

Superparticle and Spinning particle

Let us consider first the zero mode of string theory:

IIA GS string $\longrightarrow$ Superparticle
RNS string $\longrightarrow$ Spinning particle

Quantization of massive superparticle gives chiral multiplet[2]

Quantization of massive spinning particle produces the Dirac equation[3]

[2] R. Casalbuoni,1976; L. Brink, R. Schwarz, 1981.
[3] A. Barducci, R. Casalbuoni and L. Lusanna, 1976;L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. S. Howe,1976; F. A. Berezin and M. S. Marinov,1976

Space-time Vector Supersymmetry (Vsusy)

Vsusy was first introduced to get a pseudo-classical description, spinning particle, of the Dirac equation[4].

The odd generators of the algebra are a pseudovector and pseudoscalar

$$G_\mu , G_5$$

The even generators contain the Poincare generators and two central charges

$$M_{\mu\nu} , P_\mu , Z , \tilde{Z}$$

In order to reproduce the Dirac equation the authors introduced a constraint that breaks Vsusy

[4] Barducci, Casalbuoni, Lusanna, 76

Vsusy algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\rho}M_{\nu\sigma},$$

$$[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu, \qquad [M_{\mu\nu}, G_\rho] = i\eta_{\mu\rho}G_\nu - i\eta_{\nu\rho}G_\mu,$$

$$[G_\mu, G_\nu]_+ = \eta_{\mu\nu}Z, \quad [G_5, G_5]_+ = \tilde{Z},$$

$$[G_\mu, G_5]_+ = -P_\mu.$$

Relation to N=2 topological algebra

N=2 Susy algebra in 4d has 8 charges. The euclidean formulation has symmetry group

$$SU_L(2) \otimes SU_R(2) \otimes SU_{Rsymmetry}(2)$$

$$\begin{array}{ccc} Q_{i\alpha} & \longrightarrow & (1/2, 0, 1/2) \\ \bar{Q}^{i\dot{\alpha}} & \longrightarrow & (0, 1/2, 1/2) \end{array}$$

if we perform a twisting[5] identifying the R-symmetry index, i, index with the space-time index,α, the new supersymmetric generators are

$$\begin{array}{ccc} Q_{i\alpha} & \longrightarrow & (1,0) \oplus (0,0) \longrightarrow H_{\alpha\beta} \oplus Q \\ \bar{Q}^{i\dot{\alpha}} & \longrightarrow & (1/2, 1/2) \longrightarrow G_{\alpha\dot{\beta}} \end{array}$$

[5] E. Witten 1988

Topological Susy algebra

$$\{Q,Q\}=Z, \qquad \{H_{\alpha\beta},H_{\gamma\delta}\}=C_{(\alpha|(\gamma}C_{\delta)|\beta)}Z, \tag{1}$$

$$\{G_{\alpha\dot\beta},G_{\gamma\dot\delta}\}=C_{\alpha\beta}C_{\dot\beta\dot\delta}Z, \qquad \{Q,H_{\alpha\beta}\}=0, \tag{2}$$

$$\{Q,G_{\alpha\dot\beta}\}=P_{\alpha\dot\beta}, \qquad \{H_{\alpha\beta},G_{\gamma\dot\delta}\}=C_{\alpha(\gamma}P_{\beta)\dot\delta}, \tag{3}$$

$$[Q,P_{\gamma\dot\delta}]=0, \qquad [H_{\alpha\beta},P_{\gamma\dot\delta}]=0, \tag{4}$$

$$[G_{\alpha\dot\beta},P_{\gamma\dot\delta}]=0, \qquad [P_{\alpha\dot\beta},P_{\gamma\dot\delta}]=0, \tag{5}$$

$$[J_{\alpha\beta},Q]=0, \qquad [J_{\dot\alpha\dot\beta},Q]=0, \tag{6}$$

$$[J_{\alpha\beta},H_{\gamma\delta}]=\frac{i}{2}C_{(\alpha|(\gamma}H_{\delta)|\beta)}, \qquad [J_{\dot\alpha\dot\beta},H_{\gamma\delta}]=0, \tag{7}$$

$$[J_{\alpha\beta},G_{\gamma\dot\delta}]=\frac{i}{2}C_{(\alpha|\gamma}G_{\beta)\dot\delta}, \qquad [J_{\dot\alpha\dot\beta},G_{\gamma\dot\delta}]=\frac{i}{2}C_{(\dot\alpha|\dot\delta}G_{\gamma\dot\beta)}, \tag{8}$$

$$[J_{\alpha\beta},P_{\gamma\dot\delta}]=\frac{i}{2}C_{(\alpha|\gamma}P_{\beta)\dot\delta}, \qquad [J_{\dot\alpha\dot\beta},P_{\gamma\dot\delta}]=\frac{i}{2}C_{(\dot\alpha|\dot\delta}P_{\gamma\dot\beta)}, \tag{9}$$

$$[J_{\alpha\beta},J_{\gamma\delta}]=\frac{i}{2}C_{(\alpha|(\gamma}J_{\delta)|\beta)}, \qquad [J_{\dot\alpha\dot\beta},J_{\dot\gamma\dot\delta}]=\frac{i}{2}C_{(\dot\alpha|(\dot\gamma}J_{\dot\delta)|\dot\beta)}, \tag{10}$$

$$[J_{\alpha\beta},J_{\dot\gamma\dot\delta}]=0, \qquad [Z,\text{any}]=0. \tag{11}$$

Vector Superspace

Minkowski space $\sim$ Translations=$\frac{Poincare}{S0(3,1)}$

$$g = e^{iP_\mu x^\mu}$$

x^μ Minkowski coordinates

Vector superspace $\sim \frac{Vsusy}{S0(3,1)}$

$$g = e^{iP_\mu x^\mu} e^{iG_5 \xi^5} e^{iG_\mu \xi^\mu} e^{iZc} e^{i\tilde{Z}\tilde{c}}.$$

$x^\mu,\ \xi^\mu,\ \xi^5,\ c,\ \tilde{c}$ are the superspace coordinates.

The supersymmetry transformations are

$$\delta x^\mu = i\epsilon^\mu \xi^5, \quad \delta\xi^\mu = \epsilon^\mu, \quad \delta c = \frac{i}{2}\xi_\mu \epsilon^\mu$$

and

$$\delta\xi^5 = \epsilon^5, \quad \delta\tilde{c} = \frac{i}{2}\xi^5\epsilon^5.$$

Vectorfields

$$X^G_\mu = -i\left(\frac{\partial}{\partial\xi^\mu} + i\xi^5\frac{\partial}{\partial x^\mu} - \frac{i}{2}\xi_\mu\frac{\partial}{\partial c}\right), \qquad X^G_5 = -i\left(\frac{\partial}{\partial\xi^5} - \frac{i}{2}\xi^5\frac{\partial}{\partial\tilde{c}}\right),$$

$$X^P_\mu = -i\frac{\partial}{\partial x^\mu}, \qquad X_Z = -i\frac{\partial}{\partial c}, \qquad X_{\tilde{Z}} = -i\frac{\partial}{\partial\tilde{c}}.$$

Algebra

$$[X^G_\mu, X^G_\nu]_+ = -\eta_{\mu\nu}X_Z, \qquad [X^G_5, X^G_5]_+ = -X_{\tilde{Z}}, \qquad [X^G_\mu, X^G_5]_+ = X^P_\mu.$$

Appendix

The MC 1-form associated to Vsusy coset is

$$\begin{aligned}\Omega = -ig^{-1}dg &= P_\mu\left(dx^\mu - i\xi^\mu d\xi^5\right) + G_5 d\xi^5 + G_\mu d\xi^\mu + \\ &+ Z\left(dc - \frac{i}{2}d\xi_\mu\xi^\mu\right) + \tilde{Z}\left(d\tilde{c} - \frac{i}{2}d\xi^5\xi^5\right).\end{aligned}$$

the even differential 1-forms are

$$L_x^\mu = dx^\mu - i\xi^\mu d\xi^5, \quad L_Z = dc + \frac{i}{2}\xi^\mu d\xi_\mu, \quad L_{\tilde{Z}} = d\tilde{c} + \frac{i}{2}\xi^5 d\xi^5$$

and the odd ones by

$$L_\xi^\mu = d\xi^\mu, \quad L_\xi^5 = d\xi^5.$$

Phase Space Lagrangian

$$L^C = p_\mu(\dot{x}^\mu - i\xi^\mu \dot{\xi}^5) - \beta \frac{i}{2}\xi_\mu \dot{\xi}^\mu - \gamma \frac{i}{2}\xi^5 \dot{\xi}^5 - \frac{e}{2}(p^2 + m^2),$$

e is a lagrange multiplier

Second class constraints that allow to construct the Dirac brackets

$$\{p_\mu, x^\nu\}^* = -\delta_\mu{}^\nu, \qquad \{\xi^\mu, \xi^\nu\}^* = \frac{i}{\beta}\eta^{\mu\nu}, \qquad \{\pi_5, \xi^5\}^* = -1$$

Two first class constraints if $\beta\gamma = -m^2$

$$\phi \equiv \frac{1}{2}(p^2 + m^2) = 0, \quad \chi_5 = \pi_5 - \gamma \frac{i}{2}\xi^5 - ip_\mu \xi^\mu = 0$$

$$\{\chi_5, \chi_5\}^* = -\frac{2i}{\beta}\, \phi.$$

The generators of the global supersymmetries are

$$
\begin{aligned}
G_\mu &= i\beta\xi_\mu + ip_\mu\xi^5, \\
G_5 &= \pi_5 + \gamma\frac{i}{2}\xi^5.
\end{aligned}
$$

Algebra

$$
\{G_\mu, G_\nu\}^* = -i\beta\eta_{\mu\nu}, \quad \{G_\mu, G_5\}^* = -ip_\mu, \quad \{G_5, G_5\}^* = -i\gamma.
$$

Quantization

The system is quantized in a covariant manner by requiring the first class constraints to hold on the physical states,

$$[p_\mu, x^\nu] = -i\delta_\mu{}^\nu, \qquad [\xi^\mu, \xi^\nu]_+ = -\frac{1}{\beta}\eta^{\mu\nu}, \quad [\pi_5, \xi^5]_+ = -i.$$

The odd variables define a Clifford algebra C_6

$$\lambda^\mu = \sqrt{-2\beta}\xi^\mu, \quad \lambda^5 = -i\sqrt{\frac{2}{\gamma}}\left(\pi_5 - \frac{i}{2}\gamma\xi^5\right), \quad \lambda^6 = -i\sqrt{\frac{2}{\gamma}}\left(\pi_5 + \frac{i}{2}\gamma\xi^5\right)..$$

$$[\lambda^A, \lambda^B]_+ = 2\bar{\eta}^{AB}, \quad \bar{\eta}^{AB} = (-,+,+,+,+,-), \quad (A, B = 0,1,2,3,5,6).$$

Another representation of C_6 is given by the Dirac representation

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}, \qquad \Gamma^6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

γ_μ 4d Dirac matrices

$$\lambda^A = \begin{array}{ll} \Gamma^A\Gamma^5, & A = 0,1,2,3, \\ \Gamma^5, & A = 5, \\ i\Gamma^5\Gamma^6, & A = 6. \end{array} \quad .$$

Both Clifford algebras have the same automorphism group SO(4,2)

Odd first class constraint becomes

$$\chi_5 = \pi_5 - i\frac{\gamma}{2}\xi^5 - ip_\mu\xi^\mu = \frac{i}{\sqrt{-2\beta}}\Gamma^5(p_\mu\Gamma^\mu + m).$$

χ_5 holds on the physical states which is equivalent to require the Dirac equation on an 8-dimensional spinor [6] Ψ

$$\mathcal{D}\Psi = (p_\mu\Gamma^\mu + m)\Psi = 0.$$

[6] R. Casalbuoni, J. Gomis, K. Kamimura and G. Longhi, 2008

Vector SUSY transformat generators

$$G^\mu = i\beta\xi^\mu + ip^\mu\xi^5 = \frac{i}{\sqrt{2\gamma}}\left(m\lambda^\mu - p^\mu(\lambda^5 - \lambda^6)\right)$$
$$= \frac{i}{\sqrt{2\gamma}}\Gamma^5\left(m\Gamma^\mu - p^\mu(1 - i\Gamma^6)\right)$$

and

$$G_5 = \pi_5 + i\frac{\gamma}{2}\xi^5 = i\sqrt{\frac{\gamma}{2}}\lambda^6 = -\sqrt{\frac{\gamma}{2}}\Gamma^5\Gamma^6.$$

If we impose the constraint

$$\pi_5 + i\frac{\gamma}{2}\xi^5 = 0.$$

In this way the quantization can be done by using only a C_5 algebra which can be realized in a 4-dimensional space.
Note that if we impose this condition the supersymmetry generator G_5 vanishes identically[7]

[7] A. Barducci, R. Casalbuoni and L. Lusanna, 1976

The finite vsusy transformations are

$$e^{i\epsilon^5 G_5}, \quad e^{i\epsilon^\mu G_\mu},$$

ϵ^μ, ϵ^5 anticommutes with G_μ, G_5. Instead at quantum level we have

$$e^{i\alpha_5 EG_5}, \quad e^{i\alpha^\mu EG_\mu},$$

where α_5 and α^μ could be taken as even parameters. The operator E

$$E = \lambda^7 = i\lambda^0\lambda^1\lambda^2\lambda^3\lambda^5\lambda^6 = -\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^6 = i\Gamma^5\Gamma^7, \qquad \Gamma^7 \equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6,$$

anti-commutes with all λ^A's.
We have

$$[EG_\mu, \mathcal{D}] = [EG_5, \mathcal{D}] = 0$$

and the transformations generated by EG_μ and EG_5 leave invariant the action of the theory

$$\int d^4x\ \bar{\psi}\mathcal{D}\psi,$$

where $\bar{\Psi} = \Psi^\dagger\Gamma_*$ is with $\Gamma_* = \Gamma^0\Gamma^6$

Appendix. Relation with Brink et al model

$$L^B = p \cdot \dot{x} - \beta \frac{i}{2} \xi_\mu \dot{\xi}^\mu + \gamma \frac{i}{2} \xi^5 \dot{\xi}^5 - \frac{e}{2}(p^2 + m^2) - i\rho(p \cdot \xi + \gamma \xi^5)$$

where ρ is a Lagrange multiplier and $\beta\gamma = -m^2$ as in L^C.

The second class constraints allow to define Dirac brackets

$$\{\xi^\mu, \xi^\nu\}^* = \frac{i}{\beta} \eta^{\mu\nu}, \qquad \{\xi^5, \xi^5\}^* = -\frac{i}{\gamma}.$$

The first class constraints are $\chi = p \cdot \xi + \gamma \xi^5$ and $\phi = \frac{1}{2}(p^2 + m^2) = 0$.

Casimirs

The Vsusy algebra has four Casimir generator[8], three of them are purely bosonic

$$P^2\,,\quad Z\,,\quad \tilde{Z}\,,$$

there is and extra even one constructed from a genearlization of Pauli-Lubansky four vector

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu(ZM_{\rho\sigma} - G_\rho G_\sigma).$$

given by

$$\hat{W}^2$$

$\hat{W}^\mu$ is the analogue of the superspin of the super-Poincaré group.

$$Z^\mu = W^\mu - \frac{1}{8}\bar{Q}\gamma^\mu\gamma_5 Q$$

[8] R. Casalbuoni, F.Elmetti,J. Gomis, K. Kamimura, L. Tamassia, A. Van Proeyen 2008, 2009

Pauli-Lubansky type vectors

We have three spin four vectors

$$
\begin{aligned}
W^{\mu} &\equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{\nu}M_{\rho\sigma}, \\
W_C^{\mu} &\equiv \epsilon^{\mu\nu\rho\sigma}P_{\nu}G_{\rho}G_{\sigma}. \\
\hat{W}^{\mu} &\equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{\nu}(ZM_{\rho\sigma} - G_{\rho}G_{\sigma}).
\end{aligned}
$$

They are related by

$$ZW^{\mu} = \hat{W}^{\mu} + W_C^{\mu}\,.$$

$$\left[\hat{W}^{\mu}, W_C^{\nu}\right] = 0.$$

$$W_C^2 = Z^2 P^2\,\frac{3}{4}\,. \tag{12}$$

W_C^2 is not independent Casimir, instead W^2 is not a Casimir.

The three vectors $W_*^\mu = \frac{\hat{W}^\mu}{Z}$, W^μ, $\frac{W_C^\mu}{Z}$ all commute with P_μ and G_5 and verify the relation

$$[W_*^\mu, W_*^\nu]_- = \epsilon^{\mu\nu\rho\sigma} P_\rho W_{*\sigma} \,. \tag{13}$$

In the rest frame of the massive states where $P^2 = -m^2$, we have the rotation algebra

$$\left[\frac{W_*^i}{m}, \frac{W_*^j}{m}\right]_- = \epsilon^{ijk}\, \frac{W_{*k}}{m} \,, \tag{14}$$

and define three different spins. The superspin Y labels the eigenvalues $-m^2 Z^2 Y(Y+1)$ of the Casimir $\hat{W}^2$. The spin associated to W_C^2 (C-spin) is fixed to $1/2$. Finally, we denote the usual Lorentz spin by s.
Only $W_*^\mu = \frac{1}{Z}\hat{W}^\mu$ commutes with G_λ, and thus only the superspin Y characterizes a multiplet.

Since $\left[\hat{W}^\mu, W_C^\nu\right]_- = 0.$, A multiplet of superspin Y contains two particles of Lorentz spin $Y \pm 1/2$, for $Y > 0$ integer or half-integer. In the degenerate case of superspin $Y = 0$, the multiplet consists of two spin $1/2$ states.

We observe that a VSUSY multiplet contains either only particles of half-integer Lorentz spin or only particles of integer Lorentz spin.

	eigenvalue			spinning particle
$\frac{1}{Z^2}\hat{W}^2$	$-m^2\ Y(Y+1)$	superspin $= Y$	Casimir	$Y = 0$
$\frac{1}{Z^2}W_C^2$	$-m^2\ \frac{3}{4}$	C spin $= \frac{1}{2}$	Casimir	$C = \frac{1}{2}$
W^2	$-m^2\ s(s+1)$	spin $= s = \lvert Y \pm \frac{1}{2}\rvert$	not Casimir	$s = \frac{1}{2}$

The spinning particle is a realization of the degenerate case $Y = 0$.

Odd 'Casimir'

We will consider representations where $Z, \tilde{Z} \neq 0$, they will be numbers. if

$$Z\tilde{Z} + M^2 = 0\,,$$

we have

$$Q = P \cdot G + ZG_5$$

is an odd casimir for these representations. Theys correspond to BPS representations It verifies Q^2=0

VSUSY as a contraction of $OSp(3,2|2)$

$OSp(3,2|2)$ generators can be organized in a graded anti-symmetric supermatrix as

$$M_{AB}=\begin{bmatrix} M_{\mu\nu} & P_\mu & G_\mu & S_\mu \\ -P_\nu & 0 & S_5 & G_5 \\ G_\nu & S_5 & Z & Z' \\ S_\nu & G_5 & Z' & -\tilde{Z} \end{bmatrix}. \tag{15}$$

$M_{\mu\nu}$, P_μ, Z, $\tilde{Z}$, Z' (bosonic) and G_μ, S_μ, G_5, S_5 (fermionic). The subset of generators $(M_{\mu\nu}, P_\mu, Z, \tilde{Z}, G_\mu, G_5)$ has the correct structure to generate the VSUSY algebra.

The $OSp(3,2|2)$ commutation relations for the sector of interest are

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}]_- = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \,, \\
&[M_{\mu\nu}, P_\rho]_- = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \,, \\
&[P_\mu, P_\nu]_- = M_{\mu\nu} \,, \\
&[M_{\mu\nu}, G_\rho]_- = \eta_{\nu\rho} G_\mu - \eta_{\mu\rho} G_\nu \,, \\
&[P_\mu, G_\nu]_- = -\eta_{\mu\nu} S_5 \,, \qquad [P_\mu, G_5]_- = -S_\mu \,, \qquad [G_\mu, G_\nu]_+ = \eta_{\mu\nu} Z \,, \\
&[G_\mu, G_5]_+ = -P_\mu \,, \qquad [G_5, G_5]_+ = \tilde{Z} \,, \\
&\left[G_\mu, \tilde{Z}\right]_- = 2S_\mu \,, \qquad [G_5, Z]_- = 2S_5 \,, \qquad \left[Z, \tilde{Z}\right]_- = 4Z' \,.
\end{aligned}
\tag{16}
$$

VSUSY is not a subalgebra of $OSp(3,2|2)$, but it could arises after a proper contraction of it.

$$
\begin{aligned}
&M_{\mu\nu} \to M_{\mu\nu} \,, \qquad Z' \to Z' \,, \\
&P_\mu \to \lambda^2 P_\mu \,, \qquad Z \to \lambda^2 Z \,, \qquad \tilde{Z} \to \lambda^2 \tilde{Z} \,, \\
&G_\mu \to \lambda G_\mu \,, \qquad G_5 \to \lambda G_5 \,, \qquad S_\mu \to \lambda S_\mu \,, \qquad S_5 \to \lambda S_5 \,,
\end{aligned}
\tag{17}
$$

with $\lambda \to \infty$. As a result, the commutation relations (16) reduce to the VSUSY algebra.

Representations in terms of a C_6 Clifford algebra

Consider the C_6 algebra

$$[\lambda^A, \lambda^B]_+ = 2\eta^{AB}\,, \qquad A, B = 0, 1, 2, 3, 5, 6$$

$$\eta^{AB} = \Big(-sign(Z),\, sign(Z),\, sign(Z),\, sign(Z),\, sign(\tilde{Z}),\, -sign(\tilde{Z})\Big)\ .$$

1. If $Z > 0$, the Clifford algebra is $C(4,2) = \mathbb{R}(8)$, with real 8-dimensional representations
2. If $Z < 0$, the Clifford algebras is $C(2,4) = \mathbb{H}(4)$, whose minimal representations have 16 real components.

$$G^\mu = \sqrt{\frac{|Z|}{2}}\lambda^\mu - \frac{sign(\tilde{Z})}{\sqrt{2|\tilde{Z}|}}\left(\lambda^5 + \lambda^6\right) P^\mu\,, \qquad G_5 = \sqrt{\frac{|\tilde{Z}|}{2}}\lambda^6\,.$$

Notice $\lambda^5 + \lambda^6$ is a nilpotent matrix.

If $Z\tilde{Z} = -m^2$ the odd Casimir becomes

$$Q = \frac{m}{\sqrt{2|\tilde{Z}|}}\left[P\cdot\lambda + sign(\tilde{Z})m\lambda^5\right]$$

Explicit field theory representations under study[9] .

[9] Work in progress M. Caldarelli, F. Elmetti, S. Knapen, L.Tamassia, A. Van Proeyen,.....

Representations in terms of a C_5 and C_4 Clifford algebras

Consider C_5 Clifford algebra

$$[C_\mu, C_\nu]_+ = +Z\eta_{\mu\nu}, \quad [C_\mu, C_5]_+ = 0, \quad [C_5, C_5,]_+ = \tilde{Z}.$$

If we made an ansatz for the generators

$$G_5 = aC_5 + b(P \cdot C), \qquad G_\mu = C_\mu + d(P \cdot C)P_\mu + eC_5 P_\mu,$$

and Impose the susy algebra we can obtain all the parameters in terms of e

$$a = \pm\sqrt{\frac{Z\tilde{Z} - P^2(2e\tilde{Z}(e\tilde{Z}+1)+1)}{\tilde{Z}(Z - e^2P^2\tilde{Z})}},$$

$$b = -\frac{e\tilde{Z}+1}{\sqrt{Z\left(Z - e^2P^2\tilde{Z}\right)}}, \; d = \frac{-Z + \sqrt{Z\left(Z - e^2P^2\tilde{Z}\right)}}{P^2 Z}$$

It can be shown the representations for any value of e are equivalent. We can always take as a representative

$$G_\mu = C_\mu, \qquad G_5 = \sqrt{1 - \frac{P^2}{Z\tilde{Z}}}\, C_5 - \frac{(P \cdot C)}{Z}.$$

C_4 particle model

$$L = p_\mu \dot{x}^\mu - \frac{i}{2}\xi_\mu \dot{\xi}^\mu - \frac{e}{2}(p^2 + m^2).$$

is invariant under

$$\begin{aligned} \delta\xi^\mu &= \epsilon^\mu, \qquad \delta L = \frac{d}{dt}\left(\frac{i}{2}\xi_\mu \epsilon^\mu\right) \\ \delta x^\mu &= -\frac{i}{m}\epsilon^5 \xi^\mu, \quad \delta\xi^\mu = -\frac{1}{m}\epsilon^5 p^\mu, \quad \delta L = \frac{d}{dt}\left(\frac{i}{2m}\xi_\mu p^\mu \epsilon^5\right) \end{aligned}$$

The generators are

$$\begin{aligned} G_\mu &= -i\xi_\mu, \\ G_5 &= \frac{1}{m}(i\xi_\mu)p^\mu. \end{aligned}$$

The algebra closes on shell,

$$\{G_\mu, G_\nu\}^* = -i\eta_{\mu\nu}, \qquad \{G_\mu, G_5\}^+ = i\frac{p_\mu}{m}, \qquad \{G_5, G_5\}^* = -i\frac{p^2}{m^2}.$$

where

$$\{\xi_\mu, \xi_\nu\}^* = i\eta_{\mu\nu}.$$

The reduced quantization of the model leads to Dirac equation in FW form

The spinning string action in the conformal gauge is

$$S = \frac{1}{\pi}\int d^2\sigma(2\partial_+ X\cdot\partial_- X + i\psi_+\cdot\partial_+\psi_- + i\psi_-\cdot\partial_-\psi_+)$$

is invariant under "5" susy transformation

$$\delta^5_- X^\mu = i\epsilon^5_-\psi^\mu_+, \quad \delta^5_-\psi^\mu_- = -2\partial_+ X^\mu\epsilon^5_-$$

and Vsusy transformation

$$\delta\psi^\mu_- = \epsilon^\mu_-$$

These transformations preserve the Ramond boundary conditions in the case of open string and the boundary conditions of closed string.
If we call by G^5_-, G_μ the generators of those transformations we have

$$\left[G_\mu, G^5_-\right] = P_\mu, \quad [G_\mu, G_\nu] = 0$$

An analogous algebra can be obtained for the "+" sector with $[G_-, G_+] = 0$

Concusions and Outlook

- Spinning particle action invariant under space-time vector supersymmetry
- Quantization preserving supersymmetry gives two uncoupled 4d Dirac equations
- Representation of Vsusy in terms of C_6, C_5, C_4
- Multiplets of Vsusy with $Y \pm 1/2$
- Relation among spinning particle and superparticle
- Massless Representations of Vsusy
- Massless spinning particle and Vsusy
- Spinning string and Vsusy? Yes, work in progress
- Interacting field theory with Vsusy?