

GGI LECTURES ON BPS WALL-CROSSING

G. MOORE, RUTGERS UNIVERSITY
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OUTLINE OF THE MINICOURSE

LECTURE 1: WALL-CROSSING FORMULAE

LECTURE 2: HYPERKÄHLER METRICS &
THE KONTSEVICH-SOIBELMAN WCF

LECTURE 3: M5-BRANES, HITCHIN
SYSTEMS, AND BPS STATES

LECTURE 4: THE SPECTRUM-GENERATING
STOKES MATRIX: FLIPS, TWISTS, & POPS

1. INTRODUCTION & OVERVIEW

THIS MINICOURSE IS ABOUT THE BPS SPECTRUM OF STRING & FIELD THEORIES WITH $N=2, D=4$ SUSY.

THE BPS REPS OF $N=2$ ARE "SMALL" OR "RIGID" AND THEREFORE MORE AMENABLE TO STUDY. IN A SENSE THEY ARE "PIECEWISE CONSTANT" AS A FUNCTION OF PHYSICAL PARAMETERS IN THE THEORY.

THUS, INFORMATION OBTAINED AT WEAK COUPLING CAN BE USED TO OBTAIN INFO AT STRONG COUPLING.

REASONS FOR STUDYING BPS STATES

1. EXACT RESULTS IN FIELD THEORY AND STRING THEORY.
2. MICROSCOPIC ORIGIN OF BLACK HOLE ENTROPY
3. OSV CONJECTURE
4. MATHEMATICS OF "DONALDSON-THOMAS INVARIANTS" AND STABILITY IN DERIVED CATEGORIES.
5. RELATIONS TO AUTOMORPHIC FORMS & ANALYTIC NUMBER THEORY
6. NEW INFINITE DIMENSIONAL ALGEBRAIC STRUCTURES ASSOCIATED TO C.Y. MANIFOLDS ("ALGEBRAS OF BPS STATES")

THE PHYSICAL PARAM'S ARE NOT ONLY COUPLINGS IN THE UV LAGRANGIAN, BUT ALSO THESE THEORIES DO NOT HAVE A UNIQUE QUANTUM VACUUM. MASSLESS SCALAR FIELDS HAVE VEV'S CHARACTERISING THE VAC.

IN THE PATH INTEGRAL ON $\mathbb{R}^4$ THESE ARE BC'S FOR THE FIELDS AT $\vec{x} \rightarrow \infty$.

IT TURNS OUT THAT BPS STATES ARE NOT COMPLETELY INDPT. OF PARAMETERS.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY HOW THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF).

SUMMARY OF 4 LECTURES

This course will explore the wcf of BPS states in field/string theory with an emphasis on field theory.

Lecture 1: We will summarize basic defⁿ's in w.c. and review 3 kinds of formulae: primitive, semiprimitive, and KSWCF. We then give a lightning review of Seiberg-Witten Theory. We end considering naive compactifi on a circle and the resulting "semiflat metric."

Lecture 2: We consider SW compactified on a circle. Low energy physics is a σ -model with target carrying a HK metric. We use twistor techniques to study that HK metric

Using a 1-loop computation and Ward identities for 't Hooft-Wilson-Maldacena loop operators we give a physical proof of the KSWCF.

Lecture 3: We review how M5 branes compactified on a Riemann surface lead to $d=4, N=2$ Theories, and we show how these in turn are related to Hitchin systems. We describe BPS states in this framework.

Lecture 4: Using the connection to Hitchin systems and flat connections we show how the WKB approx. can be used to determine the BPS degeneracies (and not just their changes) for a large class of $d=4, N=2$ Theories.

KEY REFERENCES

1. DENEF + MOORE, "SPLIT STATES, ENTROPY ENIGMAS, HOLES + HALOS,"
hep-th/0702146
2. KONTSEVICH + SOIBELMAN, STABILITY
STRUCTURES, ..., 0811.2435
3. GAIOTTO, MOORE, NEITZKE, FOUR-DIMENSIONAL
WALL-CROSSING, ... arXiv: 0807.4723
4. GAIOTTO, MOORE, NEITZKE, WALL-CROSSING,
HITCHIN SYSTEMS, & WKB, TO APPEAR

2. GENERAL ASPECTS OF $N=2, D=4$ WALL-CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$
WITH $N=2$ SUPERPOINCARÉ SYMMETRY

LET $\mathcal{H}$ BE THE ONE-PARTICLE
HILBERT SPACE.

AS A REPRESENTATION OF THE
 $N=2$ SUPERPOINCARÉ ALGEBRA $\mathcal{A}$, $\mathcal{H}$
DEPENDS ON THE BOUNDARY VALUES
OF FIELDS AT ∞ .

THESE BOUNDARY CONDITIONS ARE VALUED
IN THE MODULI SPACE OF VACUA: $\mathcal{B}$

FOR $u \in \mathcal{B}$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{B}$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r , SO $\mathcal{H}$ IS GRADED BY THE A LATTICE Γ OF FLAVOR, ELECTRIC, $\frac{1}{2}$ MAGNETIC CHARGES. THE LATTICE OF ELEC. $\frac{1}{2}$ MAG. CHARGES HAS RANK $2r$:

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma, u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma, u}$ THE CENTRAL CHARGE OPERATOR

$\mathbb{Z} \in \Delta$ IS A SCALAR.

DENOTE THE VALUE $\mathbb{Z}_\gamma(u)$ OR $\mathbb{Z}(\gamma; u)$.

RECALL THE $N=2, D=4$ SUSY ALGEBRA

$$\Delta = \Delta_0 \oplus \Delta_1$$

$$\Delta_0 = (\underbrace{\text{Spin}(1,3)}_{\hat{M}_{\mu\nu}} \ltimes \underbrace{\mathbb{R}^4}_{\hat{P}_\mu}) \oplus \mathfrak{u}(2) \oplus \underbrace{\mathbb{R}}_{\hat{Z}}$$

$$\Delta_1 = [\text{Spinor} \otimes \mathbb{C}^2]_{\mathbb{R}}$$

$Q_{\alpha I}, \bar{Q}^{\dot{\alpha} I}$

$$\{Q_{\alpha A}, \bar{Q}_{\dot{\beta} B}\} = 2 \hat{P}_\mu \sigma_{\alpha\dot{\beta}}^\mu \delta_A^B$$

$$\{Q_{\alpha A}, Q_{\beta B}\} = 2 \hat{Z} \epsilon_{\alpha\beta} \epsilon_{AB}$$

CONVENTIONS: BAGGER $\frac{1}{2}$ WESS

$$\bar{Q}_{\dot{\alpha} A} = (Q_{\alpha}^A)^+$$

LEMMA: (THE BOGOMOLNYI BOUND)

ON A SUBSPACE $\mathcal{H}_z$ WHERE $\hat{Z} = z$ the energy is bdd by $E \geq |z|$.

PROOF: $D=4, N=2$ IS A REDUCTION OF $D=6, N=1$ ALGEBRA.

$$\text{Spin}(1,3) \times \text{Spin}(2) \hookrightarrow \text{Spin}(1,5)$$

$$2_{1/2} \oplus 2_{-1/2}^* \longleftarrow 4$$

$D=6$ ALGEBRA HAS $SU(2)$ R-SYMMETRY

$$Q^{(6)} \in (4, 2)^+$$

$+$: Symplectic Majorana

$$(Q_{rA}^{(6)})^\dagger = \mathcal{B}_r^s \varepsilon^{AB} Q_{sB}^{(6)} \quad \text{chiral}$$

$$\{Q_{rA}^{(6)}, Q_{sB}^{(6)}\} = -2i (C\Gamma^M)_{rs} \hat{P}_M \varepsilon_{AB}$$

$$\mathcal{B}\Gamma^M\mathcal{B}^{-1} = (\Gamma^M)^*$$

$$C\Gamma^M C^{-1} = -(\Gamma^M)^{\text{tr}}$$

USING A PARTICULAR REPRESENTATION
FOR T^M WE HAVE:

$$\begin{pmatrix} Q_{1A}^{(6)} \\ Q_{2A}^{(6)} \\ Q_{3A}^{(6)} \\ Q_{4A}^{(6)} \end{pmatrix} = \begin{pmatrix} Q_{2A}^{(4)} \\ -Q_{1A}^{(4)} \\ -\bar{Q}_{iA}^{(4)} \\ \bar{Q}_{\dot{2}A}^{(4)} \end{pmatrix}$$

gives the $d=4, N=2$ algebra with

$$\hat{Z} = -i(\hat{P}_4 - i\hat{P}_5)$$

But for unitary physical irreps

$$(M^{(6)})^2 = E^2 - \vec{P}^2 - |Z|^2 \geq 0$$

$$\Rightarrow E^2 \geq \vec{P}^2 + |Z|^2 \geq |Z|^2 \quad \square$$

DEF'N: $\mathcal{H}_{\text{BPS}}$ IS THE SUBSPACE OF $\mathcal{H}$ WHERE $E = |Z|$.

Remark: Equality requires $\vec{P}=0$ and $M^{(6)}=0$
 $M^{(6)}=0 \Rightarrow 1/2$ THE SUSY OPERATORS
ARE REPRESENTED AS ZERO
($N=1$ IS PRESERVED) $\arg Z$
DETERMINES WHICH $N=1$ SUBALGEBRA.

BOUNDSTATES

SOME BPS PARTICLES CAN BE VIEWED AS
BOUNDSTATES OF OTHERS

Cecotti, Fendley, Intriligator Vafa - New SURY
INDEX
 $d=2, N=2$... give ref.

Seiberg + Witten, ... ; key to resolution of ~~prob~~

IN ALL EXAMPLES I KNOW

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$

(DOES IT FOLLOW FROM GENERAL PRINCIPLES?)

SO, BY THE TRIANGLE INEQUALITY

$$E(u) = |Z_\gamma(u)| - (|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|) \leq 0$$

WE ARE DISCUSSING DECAY WHERE WE DON'T TRY TO DISSOCIATE THE BOUNDSTATE BY PUTTING ENERGY IN.

$\Rightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY WHERE THE BINDING ENERGY IS ZERO:

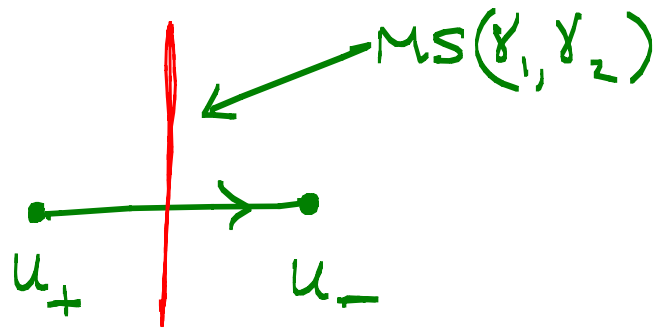
$$\arg Z_{\gamma_1}(u) = \arg Z_{\gamma_2}(u)$$

$$MS(\gamma_1, \gamma_2) := \{u \mid Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+\}$$

- Real Codimension One
- Ends on boundary or on a zero



SO A BOUNDSTATE OF PARTICLES WITH CHARGES
 γ_1, γ_2 MIGHT DECAY



WE WANT TO SAY HOW MANY STATES DECAY.

$$\dim \mathcal{H}_{\gamma, u_+}^{\text{BPS}} - \dim \mathcal{H}_{\gamma, u_-}^{\text{BPS}} = ?$$

$\Rightarrow$ WALL-CROSSING FORMULA

MORE ABOUT SHORT MULTIPLETS

When $E = |\mathbb{Z}|$ A LINEAR COMBINATION OF SUPERCHARGES

$$\{Q, Q^\dagger\} = 0$$

MUST BE REPRESENTED BY ZERO.

AS A $\text{SPIN}(3)$ REP. OF THE FORM

$$\mathcal{H}_{\gamma, u}^{\text{BPS}} = \mathcal{H}_{\frac{1}{2}\text{HM}} \otimes \widetilde{\mathcal{H}}_{\gamma, u}^{\text{BPS}}$$

$$\mathcal{H}_{\frac{1}{2}\text{HM}} = 2(\underline{0}) + (\underline{\frac{1}{2}})$$

IRREP OF SUSY: $\widetilde{\mathcal{H}}_{\gamma, u}^{\text{BPS}} = (j)$ irrep of spin .

FOR MORE DETAILS: BAGGER & WESS, Ch. 1.

$j=0$: HM's 4 real scalars. $\Rightarrow$ HM moduli
 $j=\frac{1}{2}$: VM's 2 real scalars $\Rightarrow$ VM moduli

- HM+VM PARTICLES CAN BECOME MASSLESS IN PAIRS
- THIS CAN EVEN HAPPEN AS A FUNCTION OF HM MODULI
(HARVEY & MOORE, hep-th/9510182, sec. 3.1)

DEFINE THE BPS INDEX

$$\begin{aligned}\Omega(\gamma; u) &:= -\frac{1}{2} \text{Tr}_{\mathcal{H}_{\gamma, u}^{\text{BPS}}} (2J_3)^2 (-1)^{2J_3} \\ &= \text{Tr}_{\tilde{\mathcal{H}}_{\gamma, u}^{\text{BPS}}} (-1)^{2\tilde{J}_3}\end{aligned}$$

exercise:

$$\boxed{\frac{1}{2} \text{HM:} \quad \Omega = +1 \quad \text{VM:} \quad \Omega = -2}$$

- THE BPS INDEX IS PIECEWISE CONSTANT, BUT CAN CHANGE ACROSS WALLS OF MARGINAL STABILITY
- IT IS SOMETIMES USEFUL TO CONSIDER REFINED INVARIANTS

$$\Omega(\gamma; u; x, y) = \text{Tr}_{\mathcal{H}_{\gamma, u}^{\text{BPS}}} (-y)^{2J_3} (+x)^R$$

R : $U(1)$ R -CHARGE OF PRESERVED $N=1$

x DIACONESCU-MOORE, 0706.3193

x DIMOFTE-GUKOV, 0904.1420

Exercise: Coupling just to the spin

$$\Omega(\gamma; u; y) = \text{Tr} (-y)^{2J_3}$$

Compute the order of the zero at $y = 1$.

SYMPLECTIC & POISSON STRUCTURE ON Γ

DIRAC QUANTIZATION $\Rightarrow$ LATTICE OF ELECTRIC + MAGNETIC CHARGES IS SYMPLECTIC

$$\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \rightarrow \mathbb{Z}$$

BILINEAR (AS A $\mathbb{Z}$ -MODULE), ANTISYMMETRIC, NONDEGENERATE.

"DIRAC-SCHWINGER-ZWANZIGER"

HOWEVER, IT SOMETIMES HAPPENS THAT THERE ARE FLAVOR SYMMETRIES IN ADDITION TO GAUGE SYMMETRIES. THEN $\langle \cdot, \cdot \rangle$ IS DEGENERATE:

$$0 \rightarrow \Gamma_{\text{flavor}} \rightarrow \Gamma \rightarrow \Gamma_{\text{gauge}} \rightarrow 0$$

$$\begin{aligned} \Gamma_{\text{flavor}} &= \text{RADICAL OF } \langle \cdot, \cdot \rangle \\ &= \{ \gamma_f \mid \langle \gamma_f, \gamma \rangle = 0, \forall \gamma \in \Gamma \} \end{aligned}$$

IT IS OFTEN USEFUL TO
 THINK OF FLAVOR SYMM'S AS WEAKLY
 GAUGED. IF $\hat{\Gamma}$ IS SYMPLECTIC
 AND $\Gamma_e \subset \hat{\Gamma}$ AN ISOTROPIC SUBLATTICE
 $\Gamma_e^\perp$ WILL HAVE A DEGENERATE FORM,
 THIS IS OUR Γ ABOVE.

CONVERSELY, GIVEN Γ AS ABOVE
AND A SPLITTING $\Gamma \cong \Gamma_{\text{flav}} \oplus \Gamma_{\text{gauge}}$

$$\hat{\Gamma} = \Gamma_{\text{flav}} \oplus \Gamma_{\text{flav}}^* \oplus \Gamma_{\text{gauge}}$$

WILL BE SYMPLECTIC.

3. PRIMITIVE & SEMI-PRIMITIVE WCF

DENEFF & MOORE GAVE FORMULAE FOR $\Delta\Omega$ FOR DECAYS $\gamma \rightarrow \gamma_1 + \gamma_2$ WHERE AT LEAST ONE OF γ_1, γ_2 ARE PRIMITIVE.

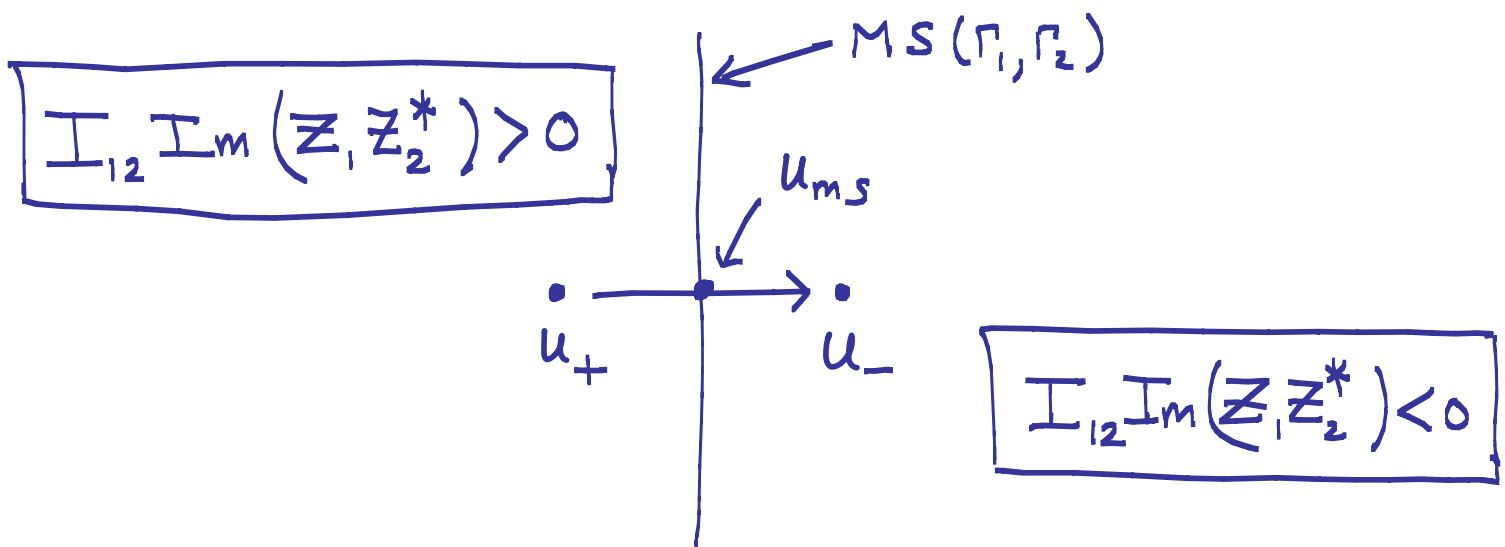
THE DERIVATION IS BASED ON DENEFF'S MULTICENTERED SOLUTIONS OF $\mathcal{N}=2$ SUGRA. (THERE IS ALSO A MICROSCOPIC ARGUMENT BASED ON THE QUIVER QUANTUM MECHANICS ASSOCIATED WITH LIGHT MODES WHICH APPEAR NEAR WMS.)

DENEFF'S MULTI-CENTERED SOLUTIONS HAVE PARAMETERS $\vec{x}_i \in \mathbb{R}^3$, THE CENTERS. NEAR $\vec{x}_i$ THE SOLUTION ASYMPTOTES TO SINGLE-CENTERED BPS SPHERICALLY SYMMETRIC BH. OF CHARGE γ_i .

PRIMITIVE WALL-CROSSING FORMULA:

Γ HAS SYMPLECTIC FORM $\langle \cdot, \cdot \rangle$

LET $I_{12} := \langle \gamma_1, \gamma_2 \rangle$



γ_1, γ_2 PRIMITIVE, u_{ms} GENERIC $\Rightarrow$

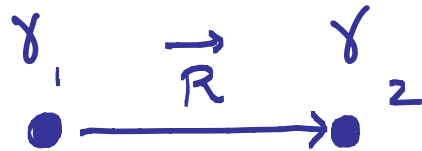
$$\mathcal{H}_+ - \mathcal{H}_- = (J_{12}) \otimes \mathcal{H}(\gamma_1; u_{ms}) \otimes \mathcal{H}(\gamma_2; u_{ms})$$

$$J_{12} = \frac{1}{2}(|I_{12}| - 1)$$

$$\Delta \Omega = (-1)^{|I_{12}|-1} |I_{12}| \Omega(\Gamma_1, t_{ms}) \Omega(\Gamma_2, t_{ms})$$

DERIVATION OF PRIMITIVE WCF:

CONSIDER BOUNDSTATE OF TWO PRIMITIVE CHARGES:

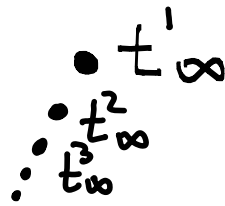
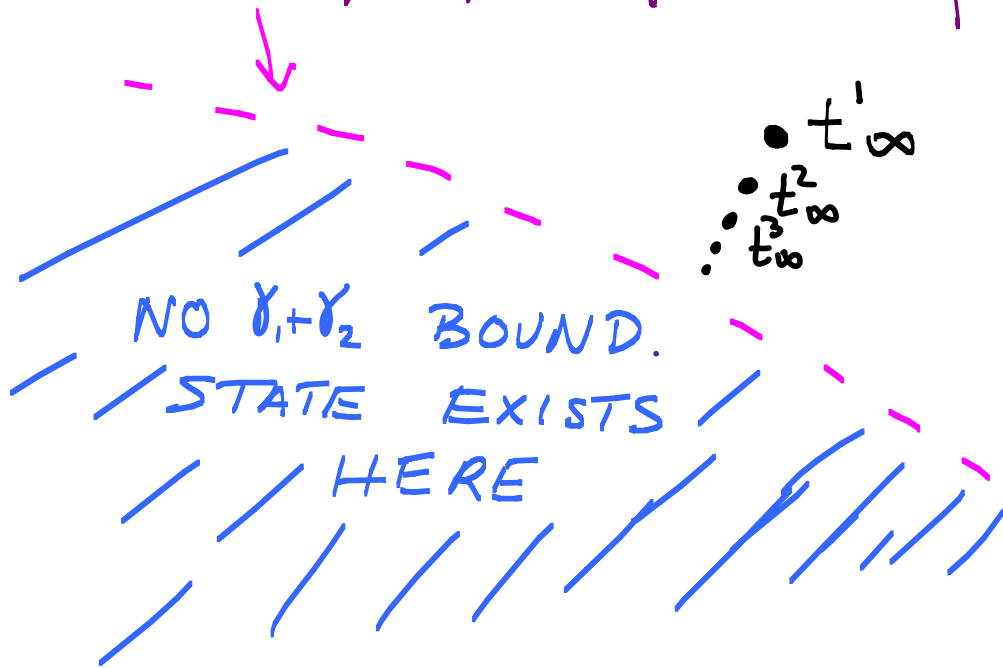


USING EXPLICIT SOLUTIONS TO SUGRA DENEF DERIVED A KEY FORMULA FOR THE BOUNDSTATE RADIUS OF TWO "CONSTITUENT BLACK HOLES:

$$R_{12} = |\vec{x}_1 - \vec{x}_2| = \frac{1}{2} \langle \gamma_1, \gamma_2 \rangle \frac{|z_{\gamma_1}(u) + z_{\gamma_2}(u)|}{\text{Im}(z_{\gamma_1}(u) \overline{z_{\gamma_2}(u)})}$$

- NOTE: $\langle \gamma_1, \gamma_2 \rangle \text{Im}(z_1 \overline{z_2})_\infty > 0$
- NOTE THAT BY CHANGING t_∞ WE CAN MAKE $\text{Im}(z_1 \overline{z_2})|_{t_\infty} \rightarrow 0$ WHILE $|z_1 + z_2|_{t_\infty} \neq 0$

$$MS(\gamma_1, \gamma_2) := \{u \in \mathcal{B} \mid \frac{z_1}{z_2} \in \mathbb{R}_+\}$$



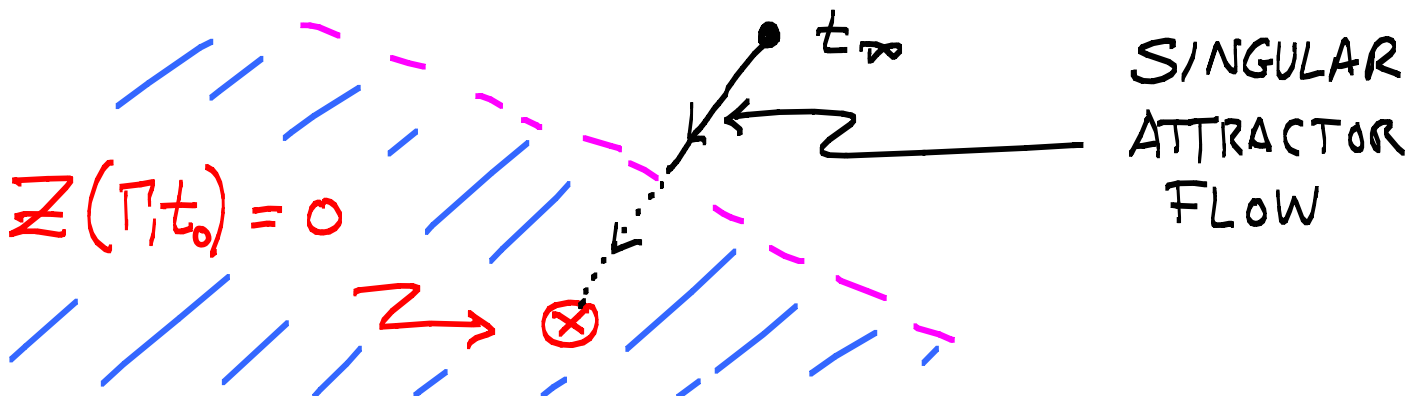
CHANGE BC'S

$$\textcircled{a} \quad r = \infty \Rightarrow$$

$$R_{1,2} \rightarrow \infty$$

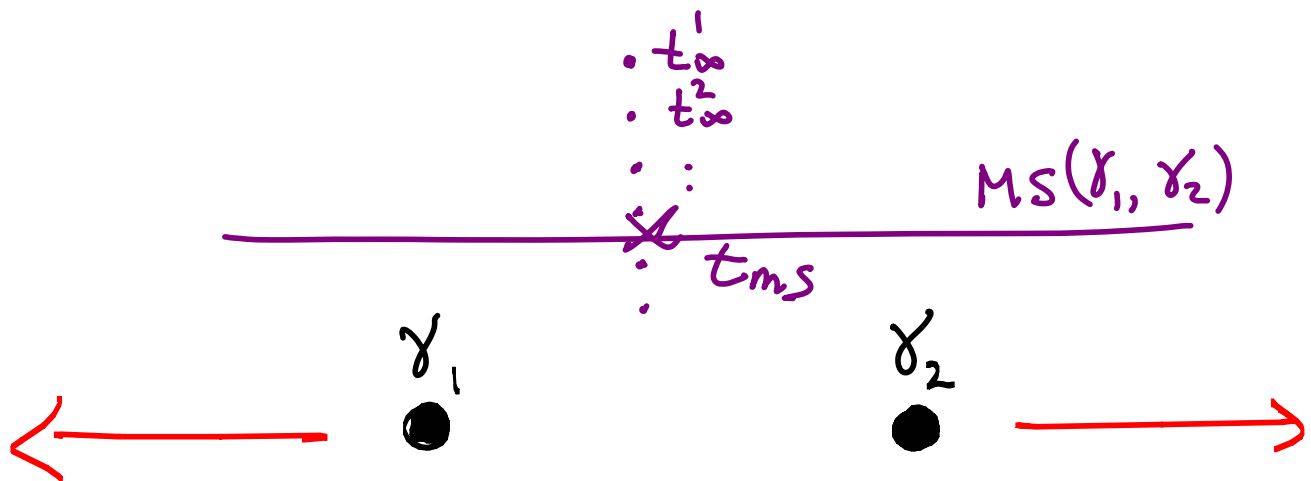
FOR ATTRACTOR EXPERTS:

IF $z(\Gamma; t_0) = 0$ THEN t_0 IS
IN THE BLUE REGION.



MACROSCOPIC ARGUMENT FOR WCF:

$$R_{12} = \frac{1}{2} \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\text{Im}(Z_1 \bar{Z}_2)_\infty}$$



ELECTROMAGNETIC FIELD OF TWO DYONS
HAS SPIN:

$$J_{12} = \frac{1}{2} (K\gamma_1, \gamma_2 > | - 1)$$

quantum
correction

LOCALITY $\Rightarrow$ FOR γ_1, γ_2 PRIMITIVE:

STATES LOST FROM $\mathcal{H}(\gamma; t_\infty)$ ARE

$$(J_{12}) \otimes \mathcal{H}(\gamma_1; t_{ms}) \otimes \mathcal{H}(\gamma_2; t_{ms})$$

THE SEMI-PRIMITIVE WCF

- NOTE THAT $MS(\gamma, \gamma_2) = MS(N_1 \gamma_1, N_2 \gamma_2)$

$N_1, N_2 \in \mathbb{Z}_{>0}$. THUS, ACROSS
THE SAME WALL MANY KINDS
OF BOUNDSTATES CAN DECAY.

- LET US FOCUS ON BOUNDSTATES
WITH ONE CONSTITUENT OF CHARGE
 γ_1 AND N CONSTITUENTS OF
CHARGES PROPORTIONAL TO γ_2

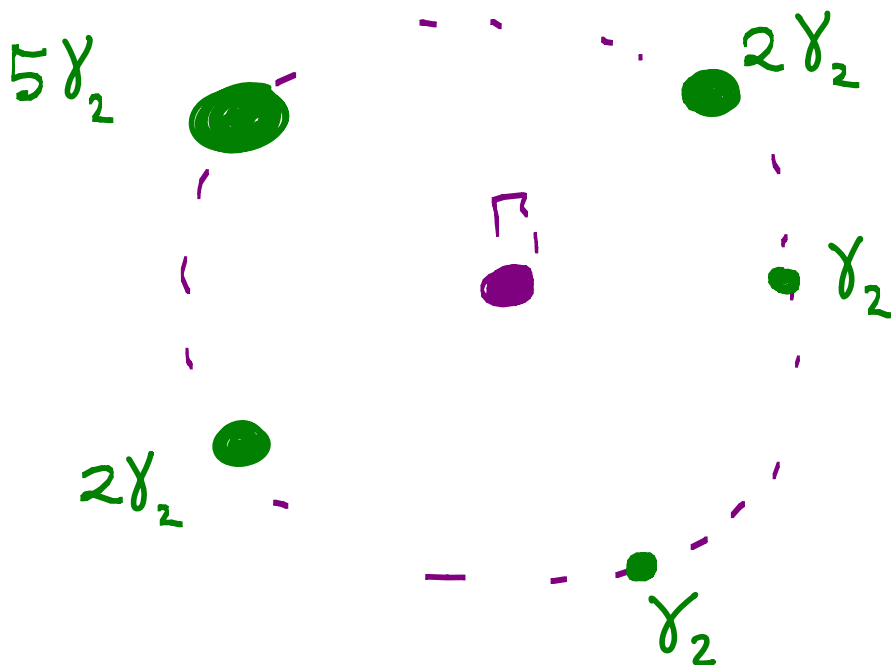
$$\gamma_j = \lambda_j \gamma_2 \quad \lambda_j > 0 :$$

IT TURNS OUT THAT THE
 $N=2$ SUGRA SOLUTIONS SATISFY
 A SIMPLE CONSTRAINT:

$$|\vec{X}_j - \vec{X}_1| = \frac{1}{2} \langle \gamma_1, \gamma_j \rangle \frac{|z_1 + z_j|}{\text{Im}(z_1 \bar{z}_j)}$$

CONVERSELY, ANY SUCH COLLECTION
 OF CENTERS GIVES A SOLUTION.

THESE ARE CALLED HALO SOLUTIONS



THE PARTICLES IN THE HALO ARE
MUTUALLY BPS - HENCE NON-INTERACTING.
THERE IS AGAIN A SPIN-DEGREE
OF FREEDOM FROM THE MUTUALLY
NONLOCAL SOURCE IN CORE + HALO.

QUANTIZATION OF SUCH PARTICLES
GIVES FINITE-DIMENSIONAL VECTOR
SPACES:

$$(J_{\gamma_1, k\gamma_2}) \otimes \mathcal{H}_{k\gamma_2, u}^{\text{BPS}}$$

SINCE WE CAN ADD SUCH PARTICLES
FREELY AND THEY ARE NONINTERACTING
THESE "HALO-STATES" COMPRISE
A FOCK SPACE OF THE FORM

$$\mathcal{H}_{\gamma_1, u} \otimes \bigotimes_{k=1}^{\infty} \widehat{\mathcal{F}} \left(\underbrace{(J_{\gamma_1, k\gamma_2}) \otimes \mathcal{H}_{k\gamma_2, u}^{\text{BPS}}}_{\text{CREATION OPERATORS}} \right)$$

NOW, WHEN u CROSSES THE WALL $MS(\gamma_1, \gamma_2)$ ALL THE HALO STATES DISAPPEAR — THE ENTIRE FOCK SPACE DISAPPEARS FROM THE SPECTRUM.

AT A GENERIC POINT ON THE WALL THERE WILL BE NO OTHER DECAYS, SO:

$$\begin{aligned} \Omega(\gamma_1) &+ \sum_{N \geq 1} \Delta\Omega(\gamma \rightarrow \gamma_1 + N\gamma_2) q^N \\ &= \Omega(\gamma_1) \prod_{k > 0} \left(1 - (-1)^{\langle \gamma_1, k\gamma_2 \rangle} q^k \right)^{|\langle \gamma_1, k\gamma_2 \rangle|} \Omega(k\gamma_2) \end{aligned}$$

UNFORTUNATELY...

THESE METHODS ARE DIFFICULT TO USE WHEN BOTH γ_1, γ_2 ARE NON-PRIMITIVE

4. THE KONTSEVICH-SOIBELMAN FORMULA

KONTSEVICH & SOIBELMAN PROPOSED
A REMARKABLE W.C.F. FOR
AN INDEX $\Omega^{\text{DT}}(\gamma; u)$,

"GENERALIZED DONALDSON-THOMAS
INVARIANT OF A CALABI-YAU 3-FOLD"

WE EXPECT THAT

$$\Omega(\gamma; u) = \Omega^{\text{Ph}}(\gamma; u) = \Omega^{\text{DT}}(\gamma; u)$$

SO THE KS WCF APPLIES TO
PHYSICAL BPS DEGENERACIES.

THEIR FORMULA APPLIES TO ALL
DECAYS $\gamma \rightarrow \gamma_1 + \gamma_2$.

IN THIS COURSE WE PROVE THE KS WCF FOR
 $\Omega^{\text{Ph}}(\gamma; u)$ IN $\mathcal{N}=2$ FIELD THEORIES

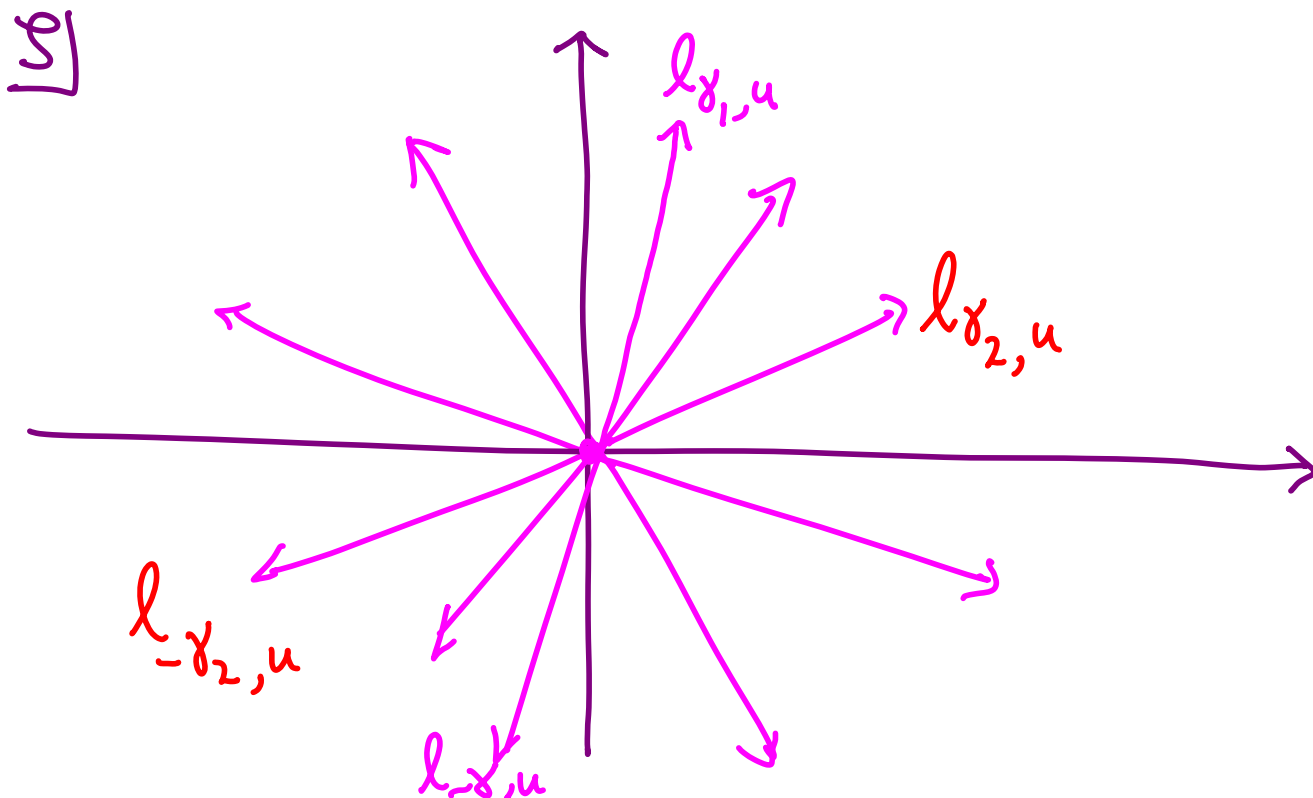
DATA:

1. SYMPLECTIC LATTICE Γ
2. CENTRAL CHARGES $Z_\gamma(u)$, $u \in \mathcal{B}$
3. PIECEWISE CONSTANT $\Omega(\gamma; u) \in \mathbb{Z}$

FIRST INGREDIENT: BPS RAYS :

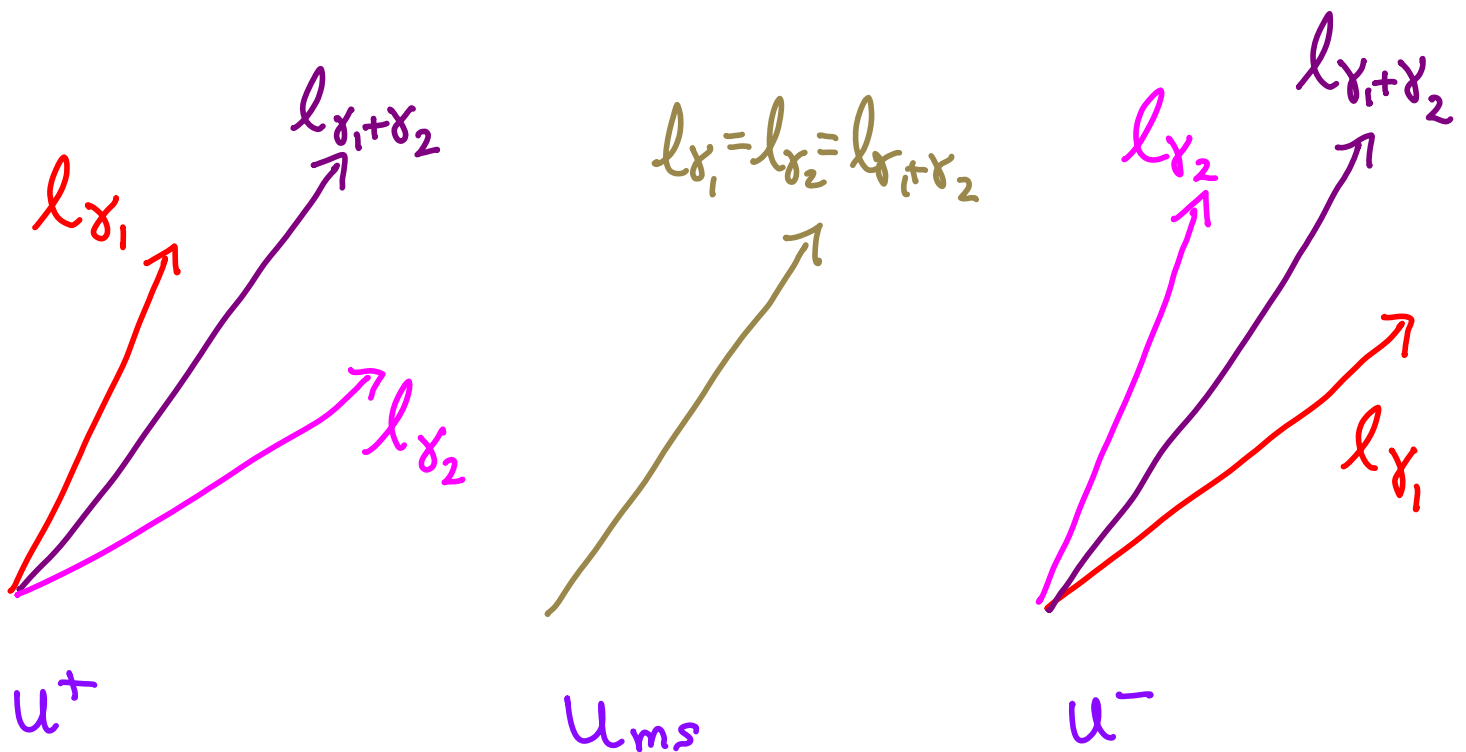
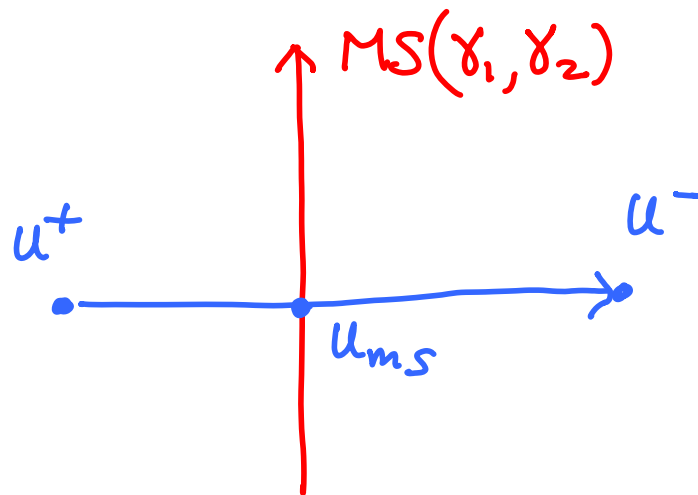
FOR $u \in \mathcal{B}$, $\gamma \in \Gamma$

$$l_{\gamma, u} := Z_\gamma(u) \mathbb{R}_- = \{ \mathcal{S} \mid \mathcal{S}/Z_\gamma(u) \in \mathbb{R}_- \}$$



AS u VARIES THE SLOPES OF
THE BPS RAYS VARY

AS u CROSSES A WALL $MS(\gamma_1, \gamma_2)$
BPS RAYS WILL COALESCE



SECOND INGREDIENT: A SYMPLECTIC TORUS:

- INTRODUCE THE COMPLEX TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \dots \times \mathbb{C}^*}_{2r}$$

$$\gamma \in \Gamma \Rightarrow \text{FUNCTION } X_\gamma : T \rightarrow \mathbb{C}^*$$

"HOLOMORPHIC FOURIER MODES"

CHOOSING A BASIS γ_i FOR $\Gamma \Rightarrow$

$$(e^{\theta_1}, \dots, e^{\theta_{2r}}) \in T, \quad \theta_i \in \mathbb{C}$$

$$X_\gamma = \exp[\gamma \cdot \theta]$$

- HOLOMORPHIC SYMPLECTIC FORM:

$$\bar{\omega}^T := \frac{1}{2} \epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}} \quad \epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

Exercise: Compute the Poisson bracket

$$\begin{aligned} \{X_\gamma, X_{\gamma'}\} &= \langle \gamma, \gamma' \rangle X_\gamma X_{\gamma'} \\ &= \langle \gamma, \gamma' \rangle X_{\gamma+\gamma'} \end{aligned}$$

- FOR EACH $\gamma \in \Gamma$ DEFINE A SYMPLECTOMORPHISM:

$$K_\gamma: X_{\gamma'} \rightarrow X_{\gamma'} (1 - \sigma(\gamma) X_\gamma)^{\langle \gamma', \gamma \rangle}$$

$$X_{\gamma'} \rightarrow X_{\gamma'} \exp[\langle \gamma, \gamma' \rangle \log(1 - \sigma(\gamma) X_\gamma)]$$

$$\sigma(\gamma) = \pm 1$$

LIE ALGEBRA VERSION

$K \ni S$ INTRODUCE A LIE ALGEBRA

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$$

DEFINE A GROUP ELEMENT

$$U_\gamma := \exp \left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)$$

AND WORK WITH U_γ INSTEAD OF K_γ

CHOOSE A QUADRATIC REFINEMENT

$$\frac{\sigma(\gamma_1 + \gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle \gamma_1, \gamma_2 \rangle}$$

$\sigma(\gamma) e_\gamma$ GENERATE THE LIE ALGEBRA
OF SYMPLECTIC VECTOR FIELDS.

Exercise: Show $K_\gamma = \exp \left(\sum \frac{\sigma(n\gamma_0)}{n^2} \{X_{n\gamma_0}, \cdot\} \right)$

EXAMPLE

$$r=1 \Rightarrow \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\langle (a,b), (a',b') \rangle = ab' - a'b$$

$$\Gamma \cong \mathbb{C}^* \times \mathbb{C}^*$$

$$x = X_{1,0} \quad y = X_{0,1}$$

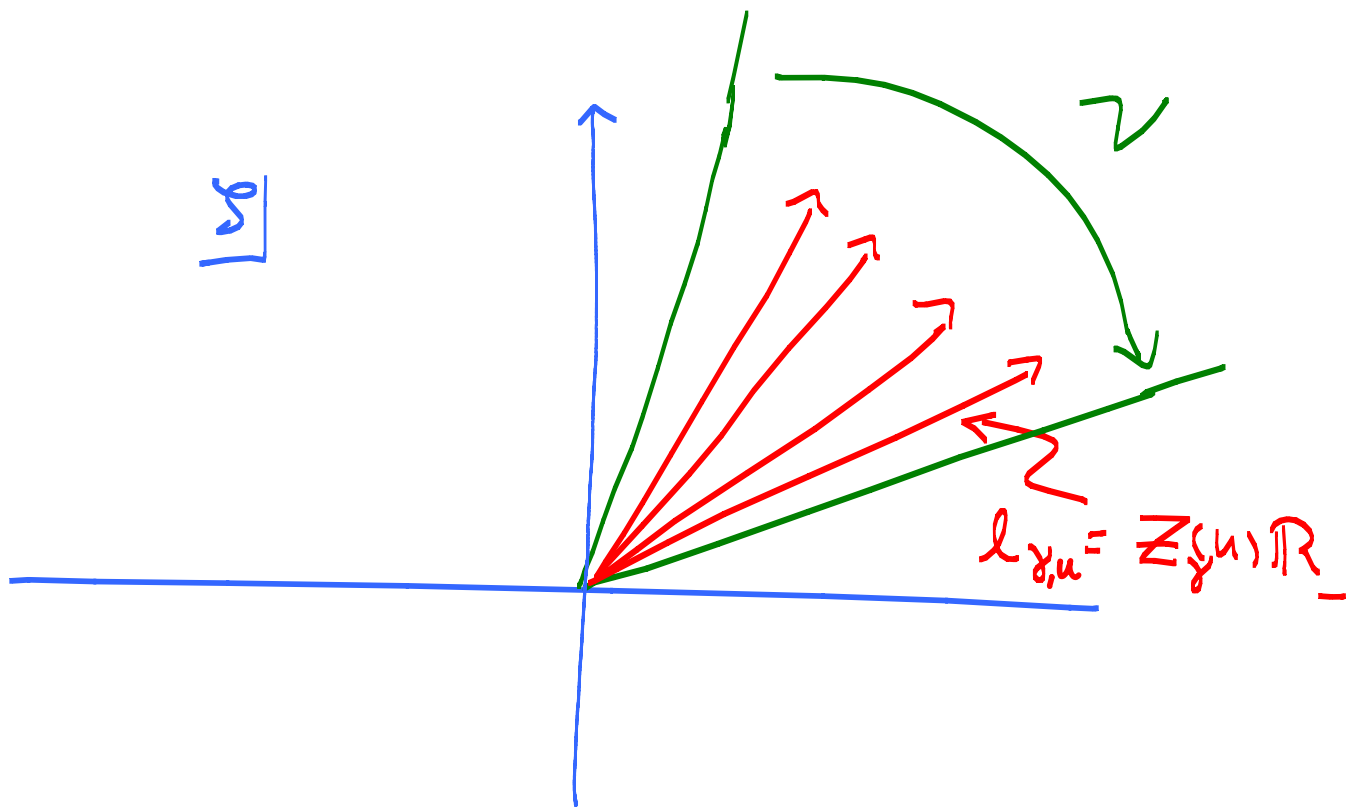
$$\omega^T = \frac{dx}{x} \wedge \frac{dy}{y}$$

$$K_{a,b} : \begin{cases} x \rightarrow x \left(1 - (-1)^{ab} x^a y^b\right)^b \\ y \rightarrow y \left(1 - (-1)^{ab} x^a y^b\right)^{-a} \end{cases}$$

Exercise: PROVE THE "PENTAGON IDENTITY"

$$K_{(1,0)} K_{(0,1)} = K_{(0,1)} K_{(1,1)} K_{(1,0)}$$

THIRD INGREDIENT: CONVEX CONE $\mathcal{V}$



FOR EACH BPS RAY DEFINE

$$S_{\gamma} := \prod_{\gamma' \parallel \gamma} K_{\gamma'}^{\Omega(\gamma'; u)}$$

AND THEN DEFINE :

$$\mathcal{A}_{\mathcal{V}} := \overrightarrow{\prod_{l_{\gamma, u} \in \mathcal{V}} S_{\gamma}}$$

$$A_V := \prod_{\ell_Y \subset V}^{\rightarrow} S_Y = \prod_{-Z_Y \in V}^{\rightarrow} K_Y^{\Omega(Y; u)}$$

THE PRODUCT IS TAKEN OVER
THE RAYS IN THE CLOCKWISE
ORDER (DECREASING SLOPE)

A_V DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS
DEPENDS ON u

2. THE $\Omega(Y; u)$ DEPEND ON u ...

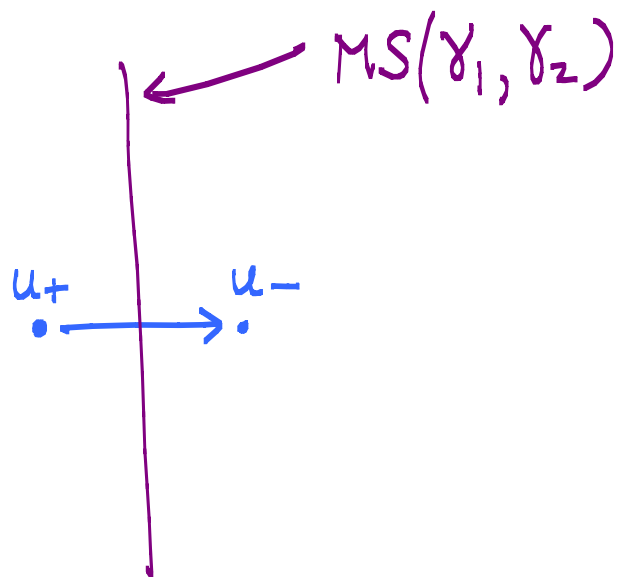
DEFINITION: WE SAY THAT $\Omega(\gamma; u)$
SATISFY THE KS WCF IF,
NEVERTHELESS,

$$A_\gamma = \prod_{\ell_\gamma < \gamma}^{\rightarrow} K_\gamma^{\Omega(\gamma; u)}$$

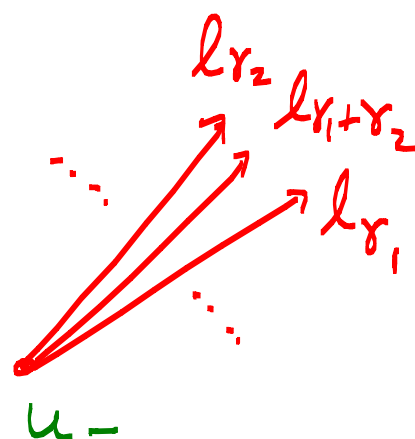
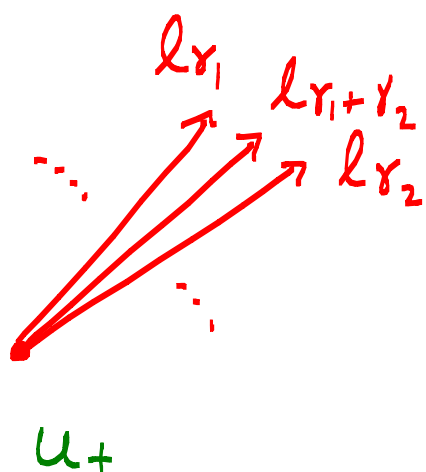
IS CONSTANT IN u AS LONG
AS NO BPS RAY ENTERS OR
LEAVES THE SECTOR γ .

THIS IS A WALL-CROSSING
FORMULA ...

AS u
CROSSES
A WALL



$$l_\gamma = Z_\gamma(u) \cdot R_- \quad \text{ROTATES}$$



$\Rightarrow$ EXCHANGE ORDER IN

$$A_\gamma = \prod_{l_\gamma \in \gamma} K_\gamma^{\Omega(\gamma; u)}$$

$\Omega(\gamma; u)$ MAKES A COMPENSATING CHANGE

UNIQUENESS LEMMA:

SUPPOSE THERE IS A POSITIVE BASIS

γ_i FOR Γ , i.e. $\Omega \neq 0 \Rightarrow$

$$\gamma = \sum n_i \gamma_i \quad \text{with all } n_i \geq 0 \\ \text{or all } n_i \leq 0$$

THEN FOR $[\vartheta_-, \vartheta_+]$ A SECTOR
OF OPENING $\leq \pi$, ANY ELEMENT IN
THE GROUP GENERATED BY K_γ HAS A
UNIQUE DECOMPOSITION:

$$g = \prod_{\vartheta_- < -\arg Z_\gamma(u) < \vartheta_+} K_\gamma^{\Omega(r; u)}$$

PROOF: DEFINE A DEGREE
 $|\gamma| = \sum n_i$. THEN K_γ ARE "UPPER
TRIANGULAR" SO EXP. MAP IS 1-1 & ONTO $\square$

THUS, GIVEN Ω_- ON ONE
SIDE OF THE WALL, Ω_+ ON
THE OTHER SIDE ARE UNIQUELY
DETERMINED.

THIS IS THUS A COMPLETE
SOLUTION OF THE WALL-CROSSING
PROBLEM.

Hilbert spaces?

RECOVERING THE PRIMITIVE WCF

AT A GENERIC POINT $t \in MS(\gamma_1, \gamma_2)$

$$Z(\gamma; t) \parallel Z_1, Z_2 \Rightarrow$$

$$= \gamma_{a,b} = a\gamma_1 + b\gamma_2$$

$(\gamma_1, \gamma_2 \text{ primitive})$

FOR SMALL CONE ANGLE ONLY THE
LIE SUBALGEBRA $Z\langle\gamma_1\rangle + Z\langle\gamma_2\rangle$
CONTRIBUTES:

$$[e_{a,b}, e_{c,d}] = (-1)^{(ad-bc)I_{12}} (ad-bc)I_{12} e_{a+c, b+d}$$

DEFINE:

$$U_{a,b} := \exp\left(\sum_{m=1}^{\infty} \frac{e_{ma, mb}}{m^2}\right)$$

$$\prod_{\substack{a/b \uparrow \\ a \geq 0 \\ b \geq 0}} U_{a,b}^{\bar{\Omega}(\Gamma_{a,b})} = \prod_{\substack{a/b \downarrow \\ a \geq 0 \\ b \geq 0}} U_{a,b}^{\Omega^+(\Gamma_{a,b})}$$

LIE ALGEBRA IS FILTERED $\Rightarrow$
CAN RESTRICT TO

$$\text{Heisenberg Algebra} \left\{ \begin{array}{l} [e_{0,1}, e_{1,0}] = (-1)^{I_{1,2}} I_2 e_{1,1} \\ e_{1,1} \text{ CENTRAL} \end{array} \right.$$

$$U_{0,1}^{\Omega^-(\Gamma_1)} U_{1,1}^{\Omega^-(\Gamma_1+\Gamma_2)} U_{1,0}^{\Omega^-(\Gamma_2)}$$

$$= U_{1,0}^{\Omega^+(\Gamma_2)} U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)} U_{0,1}^{\Omega^+(\Gamma_1)}$$

$$\boxed{U_{0,1} U_{1,0} = U_{1,1}^{\pm I_2} \cdot U_{1,0} U_{0,1}} \Rightarrow$$

$$\begin{aligned} U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2) - \Omega^-(\Gamma_1+\Gamma_2)} &= U_{0,1}^{\Omega(\Gamma_1)} U_{1,0}^{\Omega(\Gamma_2)} U_{0,1}^{-\Omega(\Gamma_1)} U_{1,0}^{-\Omega(\Gamma_2)} \\ &= U_{1,1}^{I_2} \Omega(\Gamma_1) \Omega(\Gamma_2) \end{aligned}$$

PRIMITIVE W.C. FORMULA!

REMARK: WE DID NOT NEED

A SYMPLECTIC STRUCTURE ON Γ ;

A POISSON STRUCTURE WOULD
SUFFICE:

$$\{X_{\gamma_1}, X_{\gamma_2}\} = \langle \gamma_1, \gamma_2 \rangle X_{\gamma_1 + \gamma_2}$$

AND K_γ IS A POISSON-MORPHISM.

5. LIGHTNING SUMMARY OF SEIBERG-WITTEN THEORY

G - COMPACT S.S. GAUGE GROUP, RANK = r

$\Rightarrow D=4, N=2$ FIELD THEORY

(LATER INCLUDE HM'S)

$$\mathcal{B} = (Y_G)^G \cong \mathbb{C}^r : \quad \begin{array}{l} u_2 = \langle \text{Tr } \Phi^2 \rangle \\ u_3 = \langle \text{Tr } \Phi^3 \rangle \\ \vdots \end{array}$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

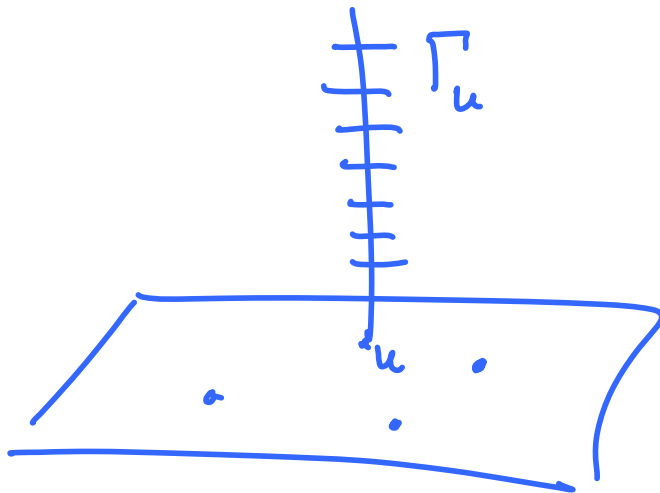
IN TERMS OF

SPECIAL KÄHLER GEOMETRY

REVIEW SPECIAL KÄHLER GEOM:

A USEFUL WAY TO PRESENT THIS FOR US IS TO BEGIN WITH THE LATTICE OF CHARGES Γ . NOW - Γ "DEPENDS ON u " GLOBALLY, BECAUSE THERE CAN BE MONODROMY AROUND COMPLEX COD. 1 DIVISORS IN B .

THIS IS KNOWN AS A "LOCAL SYSTEM"



("LOCAL SYSTEM" IN GENERAL MEANS A BUNDLE WITH A FLAT CONNECTION.)

A.) WE ALSO NEED THE CENTRAL CHARGE FUNCTION Z WHICH IS LINEAR IN γ ,

$$\text{SO: } Z \in \text{Hom}(\Gamma, \mathbb{C})$$

B.) IN REGIONS OF $\mathcal{B}$ CHOOSE A DUALITY FRAME:

$$\begin{aligned} \Gamma &= \Gamma_{el} \oplus \Gamma_{mag}, \quad \Gamma_{mag} = \Gamma_{el}^* \\ &= \text{Span}\{\alpha^I\} \oplus \text{Span}\{\beta_I\} \end{aligned}$$

$$\langle \alpha^I, \alpha^J \rangle = \langle \beta_I, \beta_J \rangle = 0 \quad \langle \alpha^I, \beta_J \rangle = \delta^I_J$$

CHOOSING A DUALITY FRAME:

CENTRAL CHARGE FUNCTION IS:

$$Z_\gamma(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mag}$$

$$a^I = Z_{\alpha_I}(u) \quad a_{D,I} = Z_{\beta_I}(u)$$

C.) NOTE THAT $\text{Hom}(\Gamma, \mathbb{C})$ IS
A SYMPLECTIC VECTOR SPACE
WITH SYMPLECTIC FORM

$$\omega = da^I \wedge da_{D,I}$$

($a^I, a_{D,I}$ INDEPENDENT)

IN OUR PROBLEM $\mathbb{Z}$ IS NOT
AN ARBITRARY ELEMENT OF
 $\text{Hom}(\Gamma, \mathbb{C})$, RATHER, IT IS
PARAMETRIZED BY $\mathcal{B}$, AND
INDEED $\mathcal{B}$ IS COMPLETELY
DETERMINED BY $\mathbb{Z}$.

SO WE MAY REGARD $\mathcal{B}$
AS A SUBSPACE OF $\text{Hom}(\Gamma, \mathbb{C})$,
AND THE KEY POINT IS

THAT IT IS A

LAGRANGIAN SUBSPACE:

$$\langle dZ, dZ \rangle = 0$$

BECAUSE $\mathcal{B}$ IS LAGRANGIAN THE

$$a^I = Z_{\alpha_I}(u) \quad I = 1, \dots, r$$

ARE LOCAL COORD'S ON $\mathcal{B}$

AND MOREOVER THE $a_{D,I}$

ARE FUNCTIONS OF THE a^J .

BECAUSE $\mathcal{B}$ IS LAGRANGIAN,

LOCALLY THERE IS A

GENERATING "PREPOTENTIAL" $\widetilde{\mathcal{F}}$

FOR THIS SUBSPACE:

$$a_{D,I} = \frac{\partial \widetilde{\mathcal{F}}}{\partial a^I}$$

D. THUS

$$\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a^J} = \tau_{JI} .$$

IN TERMS OF THIS DATA THE
EFFECTIVE LAGRANGIAN IS

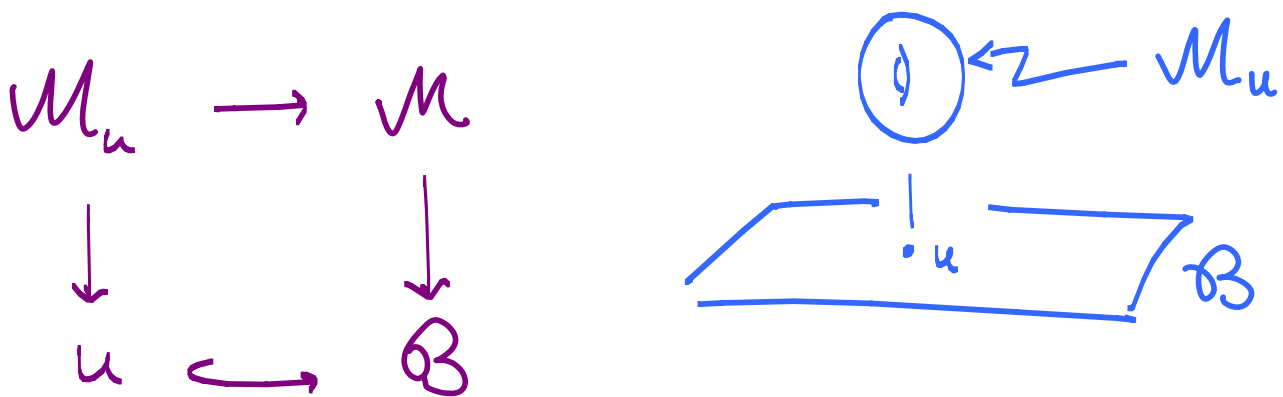
$$\mathcal{L} = \frac{-1}{4\pi} \operatorname{Im} \tau_{IJ} \left(da^I_* d\bar{a}^J + F^I_* F^J \right) \\ + \frac{1}{4\pi} \operatorname{Re} \tau_{IJ} F^I \wedge F^J + \dots$$

(A SUPERSPACE INTEGRAL OF $\mathcal{F}$, HENCE
 $\mathcal{N}=2$ SUSY)

E.) CLEARLY, PHYSICALLY SENSIBLE THEORIES HAVE $\text{Im } \tau_{IJ} > 0$. THUS τ_{IJ} IS A PERIOD MATRIX FOR AN ABELIAN VARIETY

$$\mathcal{M}_u = \Gamma_u^* \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi\mathbb{Z}) \cong U(1)^{2r}$$

THESE FIT TOGETHER TO MAKE A FIBRATION OF TORI:



THE SPACE $\mathcal{M}$ WILL BE OF CENTRAL IMPORTANCE TO US

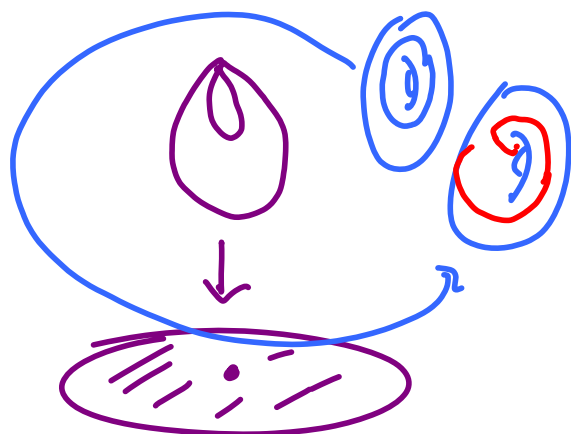
THAT CONCLUDES SPECIAL KÄHLER

IN ORDER TO REALIZE THIS
ABSTRACT STRUCTURE:

$S \ni W$ CONSTRUCT AN EXPLICIT
FAMILY OF RIEMANN SURFACES Σ_u ,
TOGETHER WITH A MEROMORPHIC
(1,0)-FORM λ_u ON Σ_u

- $\Gamma_u := H_1(\bar{\Sigma}_u, \mathbb{Z})$
- $Z_\gamma(u) := \oint_\gamma \lambda_u \quad \gamma \in \Gamma_u$
- $\mathcal{M}_u := \text{JAC}(\bar{\Sigma}_u)$

THE MONODROMY COMES FROM LEFSCHETZ



$$\gamma \rightarrow \gamma + \langle \gamma, v \rangle v$$

BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u: \quad t + \frac{\Lambda^4}{t} = v^2 - 2u$$

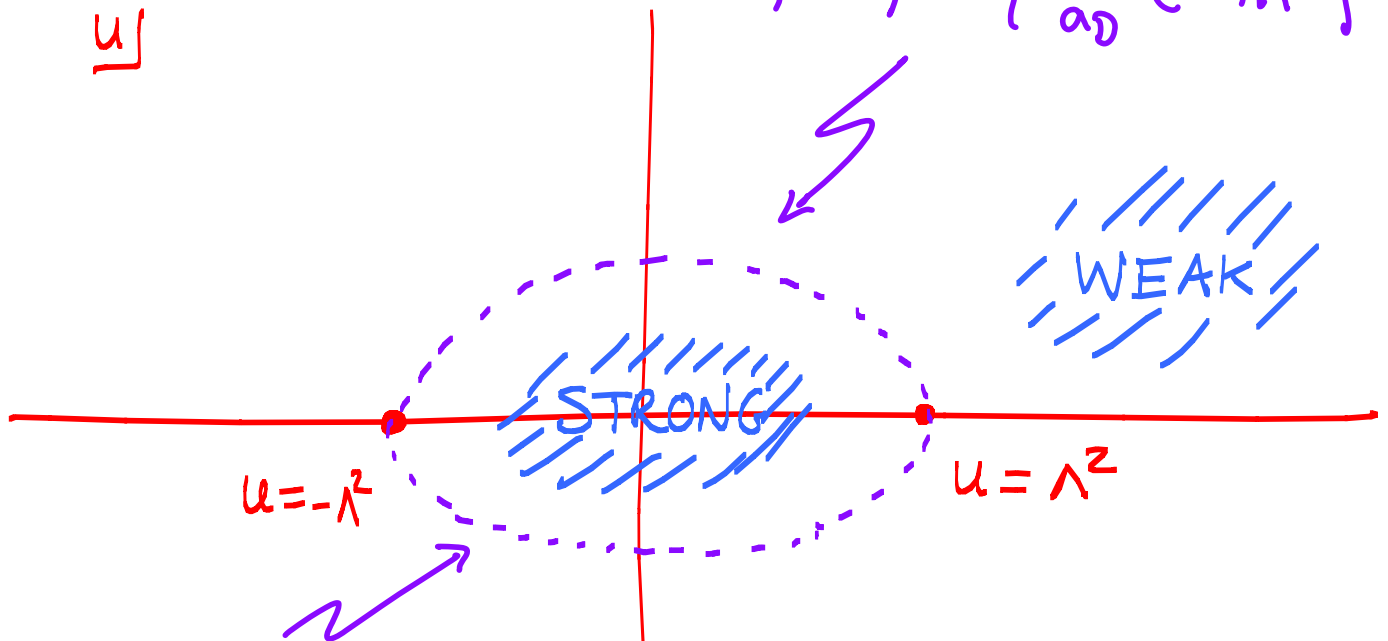
$$\lambda_u = v \frac{dt}{t}$$

$$a = \oint_{\alpha} \lambda_u$$

$$a_D = \oint_{\beta} \lambda_u$$

u

$$MS(\gamma_e, \gamma_m) = \left\{ \frac{a}{a_D} \in \mathbb{R} \right\}$$



$$MS(\gamma_e, \gamma_m) = \left\{ -\frac{a}{a_D} \in \mathbb{R} \right\}$$

AT WEAK COUPLING WE FIND
SOLITONIC STATES OF CHARGE

$(0, 1)$, $Z = a_D$ (MONOPOLE)

AND $(2, -1)$ $Z = 2a - a_D$ (DYON)

THESE VANISH ON THE DISCRIMINANT

LOCUS $u = +\Lambda^2$ $\frac{1}{2}$ $u = -\Lambda^2$ WHERE

THE ELLIPTIC CURVE DEGENERATES.

SPECTRUM:

$$\mathcal{H}_{\text{WEAK}}^{\text{BPS}} = \bigoplus_{n \in \mathbb{Z}} \text{HM}(2n, 1) \oplus \text{VM}(2, 0) \oplus \text{CONJUGATE}$$

$$\mathcal{H}_{\text{STRONG}}^{\text{BPS}} = \text{HM}(2, -1) \oplus \text{HM}(0, 1) \oplus \text{CONJUGATE}$$

[Bilal & Ferrari]

KS IDENTITY:

$$k_{2,-1} k_{0,1} = k_{0,1} k_{2,1} k_{4,1} \dots k_{2,0}^{-2} \dots k_{6,-1} k_{4,-1} k_{2,-1}$$

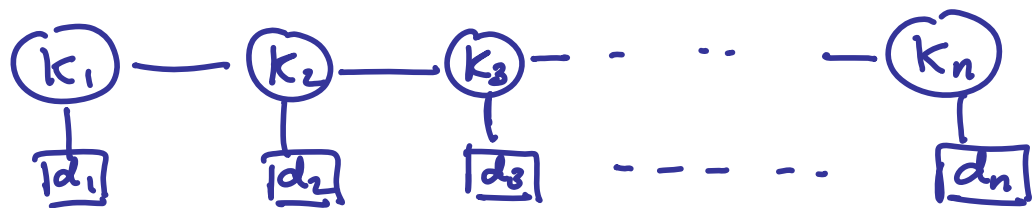
IT IS TRUE !!!

ELEMENTARY PROOF: GMN-1, APPENDIX A.

LATER, WE WILL SHOW HOW A PICTURE WITH 5-BRANES ALLOWS US TO WRITE DOWN THE SW CURVE

$$F(t, v) = 0$$

FOR A LARGE CLASS OF $D=4, N=2$ THEORIES: LINEAR QUIVERS



$$0 \rightarrow \Gamma_f \rightarrow \Gamma \rightarrow \Gamma_g \rightarrow 0$$

Splitting this:

$$Z_{(\gamma_f, \gamma_g)}(u) = Z_{\gamma_g}(u) + \gamma_f^a m_a$$

$\uparrow$
 masses

6. COMPACTIFYING $N=2, D=4$. ON A CIRCLE

- NOW CONSIDER THE THEORY ON $\mathbb{R}^3 \times S^1_R$.

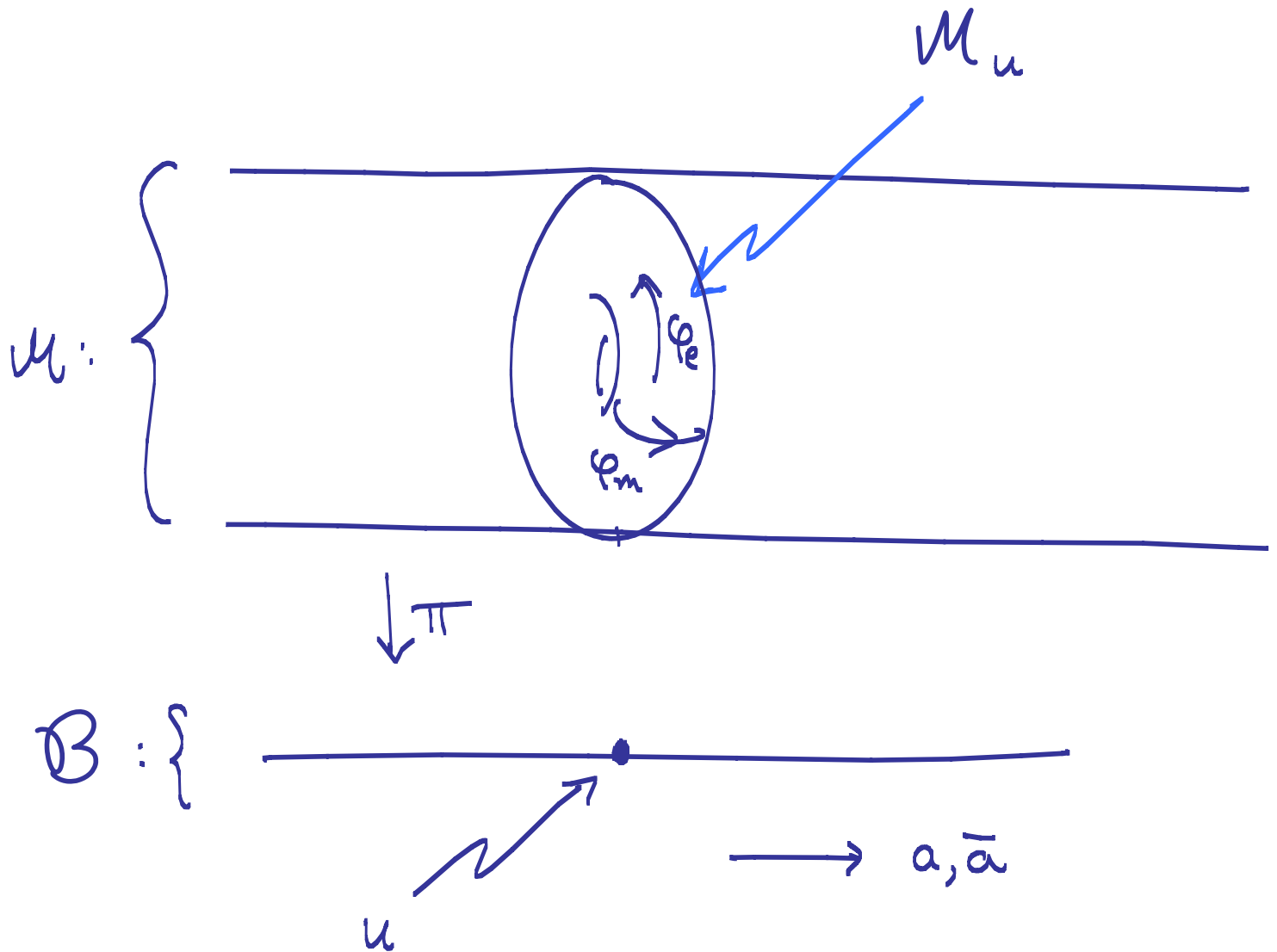
- LOW ENERGY THEORY IS A 3D σ -MODEL : $\mathbb{R}^3 \rightarrow \mathcal{M}$

$$a^I(\vec{x}, x^4) \rightarrow a^I(\vec{x})$$

$$\left. \begin{aligned} \varphi_e^I &= \int_{S^1} A_3^I dx^4 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,3})_I dx^4 \end{aligned} \right\} \text{PERIODIC!}$$

TOPOLOGICALLY $\mathcal{M}$ IS A TORUS
FIBRATION OVER $\mathcal{B}$:

IT IS EXACTLY $\mathcal{M} = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$
 THAT APPEARED ABOVE:



• SUPERSYMMETRY $\Rightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

THE SEMI-FLAT METRIC

LEADING $R \rightarrow \infty$ APPROXIMATION:

USE DIMENSIONAL REDUCTION
+ DUALIZATION OF 3D GAUGE FIELD:

$$\begin{aligned}\mathcal{L}^{(3)} = & -\frac{R}{2} \operatorname{Im} \tau_{IJ} da^I * d\bar{a}^J \\ & - \frac{1}{2R} (\operatorname{Im} \tau)^{-1, IJ} dz_I * d\bar{z}_J\end{aligned}$$

$$dz_I = dy_{m,I} - \tau_{IJ} dy_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{SF} = R (\operatorname{Im} \tau) |da|^2 + R^{-1} (\operatorname{Im} \tau)^{-1} |dz|^2$$

Details of the computation:

$$\mathbb{R}^{1,2} \times S^1$$

0 1 2 3

$$\text{metric } ds^2 = dx^\mu dx_\mu + R^2 (dx^3)^2$$

$$x^3 \sim x^3 + 2\pi$$

KK: $a^I \longrightarrow a^I(x^\mu)$

$$A^I \longrightarrow \varphi_e(x^\mu) dx^3 + \bar{A}^I$$

$$F^I = d\varphi_e dx^3 + \bar{F}^I \quad \varphi_e \sim \varphi_{e+1}$$

$$da^I \underset{4}{*} d\bar{a}^J = R dx^3 da^I \underset{3}{*} d\bar{a}^J$$

$$F^I \underset{4}{*} F^J = \frac{dx^3}{R} d\varphi_e^I \underset{3}{*} d\varphi_e^J + R dx^3 \bar{F}^I \underset{3}{*} \bar{F}^J$$

Integrate over x^3 in action:

$$i \int_{\mathbb{R}^{1,2}} -\frac{R}{2} \text{Im} \tau_{IJ} da^I \underset{3}{*} d\bar{a}^J - \frac{1}{2R} \text{Im} \tau_{IJ} d\varphi_e^I \underset{3}{*} d\varphi_e^J$$

$$- \frac{R}{2} \text{Im} \tau_{IJ} \bar{F}^I \underset{3}{*} \bar{F}^J - \text{Re} \tau_{IJ} \bar{F}^I d\varphi_e^J$$

- Now dualize the gauge field by adding

$$\exp i \int \bar{F}^I \wedge d\varphi_{m,I}$$

$$\varphi_{m,I} \sim \varphi_{m,I} + 1.$$

and doing the Gaussian integral on $\bar{F}^I$ in

$$\exp i \int -\frac{R}{2} \text{Im} \tau_{IJ} \bar{F}^I \wedge \bar{F}^J + \bar{F}^I (d\varphi_{m,I} - \text{Re} \tau_{IJ} d\varphi_e^J)$$

Stationary point

$$\bar{F}^I = -\frac{1}{R} (\text{Im} \tau)^{-1, IJ} (d\varphi_{m,J} - \text{Re} \tau_{JK} d\varphi_e^K)$$

Stationary action is $\frac{1}{2}$ The linear term $\square$

7. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1 .
- THEREFORE THE QUANTUM CORRECTIONS DEPEND ON THE BPS SPECTRUM.
- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON M AWAY FROM THE LOCUS IN $\mathcal{B}$ WHERE BPS PARTICLES BECOME $M=0$.
- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.
CLAIM: IT IS THE KS WCF.