Higher Spin Gauge Theories

Lecture 1

Introduction

Main topic – HS gauge fields

Generalization to higher tensor gauge fields of

SPIN 1 **Y-M gauge potential** A_n :

SPIN 2 **metric field** g_{nm} :

SPIN $\frac{3}{2}$ **gravitino** $\psi_{n\alpha}$:

Goal: non-Abelian HS gauge symmetries

$=$ **nonlinear HS gauge interactions**

Gauge symmetries guarantee consistency both for massless and massive theories like HS gauge theory and String Theory

String theory via spontaneous breaking of HS gauge symmetries!?

HS Theory evolves to a nonlocal theory with emergent concepts of space-time dimension, metric and local event

Example: $4d$ **massless fields live on a delocalized 3-brane in ten dimensions**

Some Reviews

A.Sagnotti, D.Sorokin, P.Sundell, MV: to never appear

A. Fotopoulos and M. Tsulaia, 0805.1346

X. Bekaert, S. Cnockaert, C. Iazeolla and MV, hep-th/0503128

MV, hep-th/0401177; 9910096; 9611024

A. Sagnotti, E. Sezgin and P. Sundell, hep-th/0501156

N. Bouatta, G. Compere and A. Sagnotti, hep-th/0409068

D. Sorokin, arXiv:hep-th/0405069

Plan

Lecture I a

Introduction:

1. Free symmetric fields

2. Structure of HS interactions

Lecture I b

1. Gravity as a gauge theory

2. Frame-like formulation of massless HS fields

3. Free action

4. Central-On-Shell Theorem

Lecture II a

1. Weyl algebra
2. Star product
3. Simplest HS algebra
4. Properties of HS algebras
5. Singletons and AdS/CFT

Lecture II b

1. Cubic HS action
2. Unfolded dynamics
3. Equations of motion in all orders
4. $4d$ HS fields in ten-dimensional space-time

HS fields

Symmetric massless HS fields - main subject of these lectures

- $m \neq 0$ symmetric fields of any spin: Singh-Hagen (1974)

  Traceless symmetric tensors $\phi_{n_1 \ldots n_s}, \; \underbrace{\phi_{n_1 \ldots n_{s-2}}, \; \phi_{n_1 \ldots n_{s-3}}, \ldots, \; \phi}_{\text{supplementary fields}}$

- $m = 0$ symmetric fields of any spin: Fronsdal (1978)

$\phi_{n_1 \ldots n_s}, \phi_{n_1 \ldots n_{s-2}} \sim \varphi_{n_1 \ldots n_s}$ double traceless $\eta^{n_1 n_2} \eta^{n_3 n_4} \varphi_{n_1 \ldots n_s} = 0$

Mixed symmetry fields

- $m = 0$ of any symmetry in flat space Labastida (1989), Skvortsov (2008), Campoleony, Francia, Mourad, Sagnotti (2008)

- $m = 0$ of any symmetry in AdS Brink, Metsaev, MV (2000), Alkalaev, Shaynkman, MV (2003) , N.Boulanger,C.Iazeolla and P.Sundell (2008) , Skvortsov (2009)

A lot of particular examples in the literature

String

String Field Theory:

Massive fields of all symmetry types

$$|\Psi\rangle \;=\; \sum \psi_{m_1\ldots m_{s_1},\,n_1\ldots n_{s_2},\,\ldots} a_{-1}^{m_1}\ldots a_{-1}^{m_{s_1}} a_{-2}^{n_1}\ldots a_{-2}^{n_{s_2}}\ldots|0\rangle$$

$$Q|\Psi\rangle \;=\; 0 \qquad \text{equations + constraints}$$

$$\delta|\psi\rangle \;=\; Q|\varepsilon\rangle \qquad \text{gauge symmetries:} \qquad \text{true+Stueckelberg}$$

Mass scale $m^2 \sim 1/\alpha'$

Tensionless limit $\alpha' \to \infty :$ **All fields become massless**

High-energy symmetries?!

A HS symmetric String Theory = HS gauge theory

Fronsdal theory

$\varphi_{n_1\dots n_s}$ - **rank s double traceless symmetric tensor**

Gauge transformation:

$$\delta\varphi_{k_1\dots k_s} = \partial_{(k_1}\varepsilon_{k_2\dots k_s)}\,, \qquad \varepsilon^m{}_{mk_3\dots k_{s-1}} = 0$$

$(\dots)$- **symmetrization:** $A_{((a_1\dots a_n))} = A_{(a_1\dots a_n)}\,.$

$\varepsilon_{k_1\dots k_{s-1}}$ **is symmetric traceless**

Comment : $\delta\varphi_n{}^n{}_m{}^m{}_{k_5\dots k_s} = 0$

Field equations

$$G_{k_1\dots k_s}(x) \;=\; 0\,,$$

$$G_{k_1\dots k_s}(x) = \Box\varphi_{k_1\dots k_s}(x) - s\partial_{(k_1}\partial^n\varphi_{k_2\dots k_s n)}(x) + \frac{s(s-1)}{2}\partial_{(k_1}\partial_{k_2}\varphi^n{}_{k_3\dots k_s n)}(x)$$

Problem 1.1. Check that $G_{k_1\dots k_s}$ is gauge invariant.

Analysis of Fronsdal equations

$$\delta\varphi_n{}^{nm_1\dots m_{s-2}} \sim \partial_n\varepsilon^{nm_1\dots m_{s-2}}$$

choose a partial gauge

$$\varphi_n{}^{nm_1\dots m_{s-2}} = 0 \qquad \partial_n\varepsilon^{nm_1\dots m_{s-2}} = 0$$

By field equation: $\partial_n\partial_m\,\varphi^{nm\dots} = 0$

Taking into account $\delta\partial_n\varphi^{nm_1\dots m_{s-1}} = \Box\varepsilon^{m_1\dots m_{s-1}}$

choose the gauge $\partial_n\varphi^{nm_1\dots m_{s-1}} = 0$

Leftover gauge symmetry parameter $\varepsilon^{m_1\dots m_{s-1}}$ **satisfies**

$$\Box\varepsilon^{m_1\dots m_{s-1}} = 0 \quad \partial_{m_1}\varepsilon^{m_1\dots m_{s-1}} = 0 \quad \varepsilon^n{}_n{}^{m_1\dots m_{s-3}} = 0.$$

Field equations

$$\Box\varphi^{m_1\dots m_s} = 0 \quad \varphi^n{}_{nm_1\dots m_{s-2}} = 0 \quad \partial_n\varphi^{nm_1\dots m_{s-1}} = 0\,.$$

Fronsdal action

$$S = \int_{M^d} \left(\frac{1}{2} \varphi^{m_1 \ldots m_s} G_{m_1 \ldots m_s}(\varphi) - \frac{1}{8} s(s-1) \varphi_n{}^{n\, m_3 \ldots m_s} G^p{}_{p\, m_3 \ldots m_s}(\varphi) \right)$$

Important property $\forall \varphi \,, \delta\varphi$:

$$\delta S \;=\; \int_{M^d} \left(\delta\varphi^{m_1 \ldots m_s} G_{m_1 \ldots m_s}(\varphi) - \frac{1}{4} s(s-1) \delta\varphi_n{}^{n\, m_3 \ldots m_s} G^p{}_{p\, m_3 \ldots m_s}(\varphi) \right)$$

$$= \; \int_{M^d} \left(\varphi^{m_1 \ldots m_s} G_{m_1 \ldots m_s}(\delta\varphi) - \frac{1}{4} s(s-1) \varphi_n{}^{n\, m_3 \ldots m_s} G^p{}_{p\, m_3 \ldots m_s}(\delta\varphi \right)$$

Problem 1.2. prove

Gauge variation $\delta S = 0$ **because** $\delta G_{nm} = 0$.

$s = 0$ φ **scalar**

$s = 1$ φ_n **Maxwell potential**

$s = 2$ φ_{nm} **linearized metric**

Various formulations of massless fields:

frame-like,

unrestricted,

BRST,

etc,

differ by

adding auxiliary fields that are expressed algebraically by their field equations via derivatives of dynamical fields
and/or Stueckelberg fields along with Stueckelberg shift gauge symmetries.

Interactions as the most crucial test: frame-like formulation

Yang-Mills theory

${A_n}^i{}_j$ -**elements of a Lie algebra** l

$$G_{nm} = \partial_n A_m - \partial_m A_n + g[A_n, A_m],$$

$$\delta A_n = \partial_n \varepsilon + g[A_n, \varepsilon]\,, \qquad \delta G_{nm} = g[G_{nm}, \varepsilon], \qquad \varepsilon^i{}_j(x) \in l.$$

Yang-Mills Action

$$S = -\frac{1}{4}\int tr(G_{mn}G^{mn})\,, \quad S = S^{Maxw} + g\int A^2\partial A + g^2\int A^4\,,$$

$$\delta S = -\frac{1}{4}g\int tr[G_{mn}G^{mn}, \varepsilon] = 0.$$

- **The coupling constants are fine tuned**
- **field spectra are distinguished:** $A^i{}_j-$ **elements of a Lie algebra: not any set of fields** A_n **is allowed**
- **interactions to other fields are restricted, requiring covariant derivatives** $\partial_n \chi^\alpha \to D_n \chi^\alpha = \partial_n \chi^i + A_n{}^\alpha{}_\beta \chi^\beta$

χ^α –**some** l**-module**

- **Cubic vertex contains one derivative**

Gravity

Spin 2: g_{nm} − **gauge field**

Riemann tensor $R_{nm,\,kl}$ **transforms homogeneously under diffeomorphisms**

$$\delta g_{nm} = \partial_n(\xi^k(x))g_{km} + \partial_m(\xi^k(x))g_{kn} + \xi^k(x)\partial_k g_{nm}$$

for $g_{nm} = \eta_{nm} + \kappa\varphi_{nm}$ **diffeomorphisms provide a nonlinear deformation of the Fronsdal transformation** $\delta\varphi_{nm}$

Einstein action $S = -\frac{1}{4\kappa^2}\int\sqrt{g}\,R$ **is a nonliner deformation of the Fronsdal action for spin two.**

Highly restricted field spectrum: only one spin-2 field.

Two derivatives in interactions.

Interactions via covariant derivatives.

$$\partial \;\rightarrow\; D = \partial - \Gamma \quad - \text{ Christoffel connection}$$

Goal

To find a nonlinear HS theory such that

(i) Fronsdal (or Labastuda) theory in the free field limit

(ii) HS gauge symmetries related to HS parameters $\varepsilon^{m_1 \dots m_{s-1}}$ deform to non-Abelian

These conditions were believed for a long time to admit no solution.

$S-$ matrix argument Coleman, Mandula (1967)

If symmetry is larger than usual (super)symmetries in Minkowski space-time + inner symmetries the scattering is trivial: no interaction.

HS Problem

HS-gravity interaction problem

Aragone, Deser (1979)

$$\partial_n \to D_n = \partial_n - \Gamma_n \qquad [D_n\, D_m] = \mathcal{R}_{nm}\ldots$$

Riemann tensor $\mathcal{R}_{nm,kl} \neq 0$ in a curved background.

$$\delta\varphi_{nm\ldots} \to D_n \varepsilon_{m\ldots}$$

$$\delta S_s^{cov} = \int \mathcal{R}_{\ldots}(\varepsilon_{\ldots} D\varphi_{\ldots}) \neq 0 \qquad ?!$$

$\uparrow$

Weyl tensor for $s > 2$

For $s \leq 2$, δS_s^{cov} contains only the Ricci tensor to be compensated by the variation of the Einstein action, allowing nonlinear gravity and supergravity.

For $s > 2$, Weyl tensor contributes to δS_s^{cov} : difficult to achieve HS gauge symmetry at the nonlinear level.

Higher Derivatives in HS Interactions

A.Bengtsson, I.Bengtsson, Brink (1983)

Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots$$

$$S^3 = \sum_{p,q,r} (D^p \varphi)(D^q \varphi)(D^r \varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

String: $\rho \sim \sqrt{\alpha'}$

HS Gauge Theories ($m = 0$): Fradkin, M.V. (1987)

$$AdS_d : \ (X^0)^2 + (X^d)^2 - (X^1)^2 - \dots - (X^{d-1})^2 = \rho^2 \,, \qquad \rho = \lambda^{-1}$$

$$[D_n, D_m] \sim \rho^{-2} = \lambda^2$$

The $\rho \to \infty$ limit is ill-defined at the interaction level both in string theory and in HS gauge theory

HS Fields in AdS Background

Anti-de Sitter space:

$$R_{mn,kl} = 0\,, \qquad R_{mn,kl} = \mathcal{R}_{mn,kl} - \lambda^2 (g_{mk}g_{nl} - g_{ml}g_{nk})$$

$\rho = \lambda^{-1}$ **is** AdS_d **radius.**

Symmetry: $o(2, d-1)$

To preserve HS gauge symmetries of massless fields, mass-like terms have to be adjusted in terms of λ

$$L^{flat} = \partial\varphi\partial\varphi \to L^{AdS} = D\varphi D\varphi + \lambda^2\varphi\varphi$$

For general mixed symmetry fields it is impossible to keep all flat space HS gauge symmetries unbroken in AdS **background**

Metsaev (1995), Brink, Metsaev, M.V. (2000)

Role of AdS Background in HS Theories

Near AdS: expansion in powers of the shifted Riemann tensor $R_{mn,kl}$ (which is zero in the AdS space) rather than in powers of the Riemann tensor $\mathcal{R}$

$$[D_n, D_m] \sim \lambda^2 \sim O(1) + O(R).$$

The action is modified by cubic terms

$$S^{int} = \int_{M^4} \sum_{p,q} \lambda^{-(p+q)} D^p(\varphi) D^q(\varphi) R$$

which contain higher derivatives along with negative powers of λ.

There exists such S^{int} that its HS gauge variation compensates the nonzero gauge variation of the free covariantized action. (Fradkin, M.V. (1987))

For given spin, a highest order of derivatives in a vertex is finite increasing linearly with spin.

Spin 3 example

MV ((≤ 1986)), unpublished , Zinoviev (2008)

$$\delta\varphi_{mnk} = D_{(m}\epsilon_{nk)} + \dots$$

$$S_{33} = \int (D\varphi D\varphi + \lambda^2\varphi^2)$$

$$S_{332} = \lambda^{-2} S^2_{332} + S^0_{332}$$

$$S^2_{332} = \int D^2(\varphi\varphi R)\,, \qquad S^0_{332} = \int \varphi\varphi R$$

$$\delta S^2_{332} = \int \Big([D,D] D(\varphi\epsilon R) \sim \lambda^2 D(\varphi\epsilon R) \Big)$$

$$[D_n, D_m] \sim \lambda^2 \sim O(1) + O(R)$$

$\lambda^{-2}\delta S^2_{332}$ compensates $\delta S_{33} + \delta S^0_{332}$

For analogous analysis for $s = 5/2$ see Sorokin (2004)

Compensation mechanism

$$S^{int} = \sum_{k=0}^{s-1} S_k^{int}\,, \qquad S_k^{int} = \lambda^{-2k} \int_{M^4} \sum_{p+q=2k} D^p(\varphi) D^q(\varphi) R$$

The highest derivative term S_{s-1}^{int} is gauge invariant in the flat limit.

Since $[D_n, D_m] \sim \lambda^2 \sim O(1) + O(R)$ its variation with $\lambda \neq 0$ gives

$$\delta S_{s-1}^{int} = \lambda^{2(1-k)} \int_{M^4} \sum_{p+q=2s-3} D^p(\varphi) D^q(\varepsilon) R$$

This term compensates δS_{s-2}^{int} modulo terms of order $\lambda^{-(2s-6)}$.

The process goes on unless one is left with the λ-independent terms

$$\delta S^{int} = \int_{M^4} \sum_{p+q=1} D^p(\varphi) D^q(\varepsilon) R$$

which just compensates the variation of the covariantized free action

$\delta S^{cov} + \delta S^{int} = 0\,.$ To understand miraculous cancellations:

Geometric approach

Nonlinear HS gauge theories

- Full nonlinear equations of motion are known in any d for symmetric boson HS fields (2003) and in $4d$ for supersymmetric systems of HS fields (1990)
- Once a spin $s > 2$ field appears, a consistent HS theory contains an infinite set of HS fields with infinitely increasing spins
- Different spin one Yang-Mils symmetries $g = u(n), o(n)$ or $usp(2n)$.

Odd spins: adjoint representation of g.

Even spins: the opposite symmetry second rank representation of g, that contains a singlet for a colorless graviton

$o(1)$: $s = 0, 2, 4, 6, \dots$, $u(1)$: $s = 0, 1, 2, 3 \dots$

Fermions: bifundamental.

Cubic actions

in $4d$ Fradkin, MV (1987), $d = 5$ MV (2001), Alkalaev, MV (2002);

particular spins in $d > 4$ Beckaert, Boulanger, Cnockaert (2005), Fotopoulos, Irges, Petkou, Tsulaia (2007), Boulanger, Leclercq, Sundell (2008).

partially gauge fixed approach Metsaev (2005,2007)

Cartan formulation of gravity

Diffeomorphisms without a distinguished spin two metric tensor: exterior algebra calculus

$$g_{mn}\,,\Gamma_{p,mn}\to e_n^a\,,\omega_n{}^{ab}\,,\qquad g_{mn}=e_m^a e_{n\,a}\,,$$

$e^a = dx^n e_n^a$ frame one-form (vielbein)

$\omega^{ab} = dx^n \omega_n{}^{ab}$ Lorentz connection

$a, b\ldots = 0, 1\ldots d-1$ 'flat' tangent space indices.

Extra $\frac{d(d-1)}{2}$ components in e_n^a are compensated by the $o(d-1,1)$ local Lorentz symmetry

$$\delta e^a(x) = \varepsilon^{ab}(x) e_b(x) \qquad \varepsilon_{ab}(x) = -\varepsilon_{ba}(x)\,, \qquad \delta g_{mn}(x) = 0\,.$$

Gravity as a gauge theory

e^a, ω^{ab} **are gauge fields of** AdS **algebra** $o(d-1,2)$ **or its Poincarè contraction** $iso(d-1,1)$

$$W = e^a P_a + \frac{1}{2}\omega^{ab} M_{ab}$$

The YM curvature two-form is

$$R = dW + W \wedge W \equiv T^a P_a + \frac{1}{2} R^{ab} M_{ab}\,,$$

$$T^a = D^L e^a \equiv de^a + \omega^a{}_b \wedge e^b\,,$$

$$R^{\,ab} = \mathcal{R}^{\,ab} - \lambda^2 e^a \wedge e^b\,, \qquad \mathcal{R}^{\,ab} = d\omega^{ab} + \omega^{\,a}{}_c \wedge \omega^{\,cb}$$

$\lambda^{-1} = \rho$ **is the** AdS **radius. Flat limit:** $\lambda \to 0$

Zero torsion condition

$$T^a = 0 \rightarrow \omega = \omega(e, \partial e)$$

$\mathcal{R}_{mn,kl} = e_k^a e_l^b \mathcal{R}_{mn\,,ab}(\omega(e), e)$ **is Riemann tensor for** $T^a = 0$.

AdS_d : $\qquad R^{ab} = 0\,, \qquad R^a = 0\,.$

Minkowski: $\mathcal{R}_{mn,kl} = 0, \qquad R^a = 0$

MacDowell − Mansouri Action

$$S^{MM}[\,e,\omega] = -\frac{1}{4\kappa^2\lambda^2}\int_{\mathcal{M}^4} R^{a_1a_2}\wedge R^{a_3a_4}\epsilon_{a_1a_2a_3a_4}\,, \qquad R^{\,ab} = \mathcal{R}^{\,ab} - \lambda^2 e^a\wedge e^b$$

Three terms:

$\mathcal{R}\wedge e\wedge e$: **Einstein action without cosmological constant,**

$e\wedge e\wedge e\wedge e$: **cosmological term,**

$\mathcal{R}\wedge\mathcal{R}$: **Gauss-Bonnet term that contains higher derivatives but does not contribute to the equations of motion**

$$\delta\int_{\mathcal{M}^4}\mathcal{R}^{a_1a_2}\wedge\mathcal{R}^{a_3a_4}\epsilon_{a_1a_2a_3a_4} \equiv 0$$

Problem 1.1. Prove

Vacuum Global Symmetry

Any solution of

$$T^a(W_0) = 0\,, \qquad R^{ab}(W_0) = 0\,, \qquad \mathrm{rank}(e^a_n) = d$$

describes local AdS_d geometry. W_0 satisfies the equations of motion of the MM action.

To describe a gauge model that has a global symmetry h it is useful to reformulate it in terms of the gauge connections W and curvatures R of h in such a way that the zero curvature condition $R(W_0) = 0$ solves the field equations providing a h-symmetric vacuum solution W_0.

Other way around: if a symmetry h is not known, reformulate dynamics à la MacDowell-Mansouri to guess the structure of an appropriate curvature R and thereby the nonAbelian algebra h.

Frame-like formulation of HS fields

$g_{nm} \longrightarrow e^a_n \longrightarrow \{e^a_n, \omega^{ab}_n\}$

admits a natural generalization to $s \geq 2$

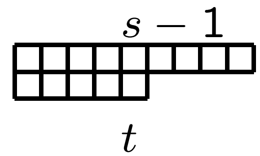
$\varphi_{n_1 \ldots n_s} \rightarrow e_n{}^{a_1 \ldots a_{s-1}} \rightarrow \{e_n{}^{a_1 \ldots a_{s-1}}, \omega_n{}^{a_1 \ldots a_{s-1}, b_1 \ldots b_t}\}$

A set of HS 1–form connections labeled by the index $0 \leq t \leq s-1$ **for a spin** s

$$\omega^{a_1 \ldots a_{s-1}, b_1 \ldots b_t} = dx^m \omega_m{}^{a_1 \ldots a_{s-1}, b_1 \ldots b_t}, \qquad (\omega|_{t=0} = e)$$

symmetric in the fiber indices a_i **and (separately) in** b_j **and satisfy the (anti)symmetry condition**

$$\omega^{(a_1 \ldots a_{s-1}, a_s) b_2 \ldots b_t} = 0 \; :$$



$\omega^{\;a_1 \ldots a_{s-1}, b_1 \ldots b_t}$ **is traceless in** a **and** b.

Identification

$$\varphi_{n_1 \ldots n_s} = e_{(n_1; n_2 \ldots n_s)} \quad \longrightarrow \quad \varphi^{kl}{}_{kl n_5 \ldots n_s} = 0$$

Higher spin curvatures

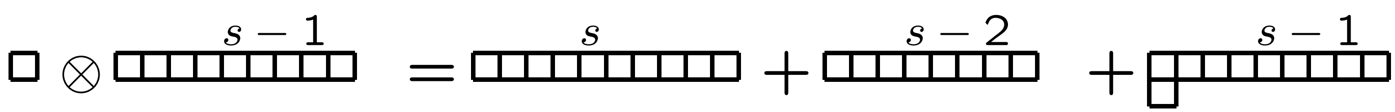
$$R_{a_1\dots a_{s-1},\,b_1\dots b_t} = (D_M^{ad}\omega)_{a_1\dots a_{s-1},\,b_1\dots b_t} = d\omega_{a_1\dots a_{s-1},\,b_1\dots b_t} - h^q\,\omega_{a_1\dots a_{s-1},\,b_1\dots b_t\,q}$$

are invariant under the gauge transformation

$$\delta\omega_{a_1\dots a_{s-1},\,b_1\dots b_t} = D_M^{ad}\epsilon_{a_1\dots a_{s-1},\,b_1\dots b_t}\,, \qquad \left(D_M^{ad}\right)^2 = 0.$$

Additional components in

$e_{n;a_1\dots a_{s-1}}$:

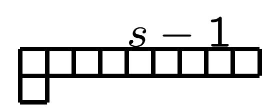


are gauged away by the generalized HS Lorentz gauge parameter $\xi_{a_1\dots a_{s-1},}$

in

$$\delta e_{m a_1\dots a_{s-1}} = \partial_m \xi_{a_1\dots a_{s-1}} - \delta_m^b \xi_{a_1\dots a_{s-1},b}\,, \qquad \delta_m^b : \quad \textbf{flat frame}$$

$\xi_{a_1\dots a_{s-1},b}$ **is a traceless tensor of the symmetry**



: $\xi_{(a_1\dots a_{s-1},a)} = 0.$

$\xi_{a_1\dots a_{s-1}}$: symmetric traceless parameter of the Fronsdal theory

Free Action in Minkowski Space

$$S=\int_{M^d} E_{pqr}(d\, e^{n_1\dots n_{s-2}p} - \frac{1}{2}\, dx_m\, \omega^{n_1\dots n_{s-2}p,m})\, \omega_{n_1\dots n_{s-2}}{}^{q,r}\,.$$

$$E_{pqr} = dx^{a_1}\dots dx^{a_{d-3}}\varepsilon_{a_1\dots a_{d-3}pqr}$$

Important property that makes HS gauge symmetry manifest

$$E_{pqr}dx_m\, \omega^{n_1\dots n_{s-2}p,m}\, \delta\omega_{n_1\dots n_{s-2}}{}^{q,r} = E_{pqr}dx_m\, \delta\omega^{n_1\dots n_{s-2}p,m}\, \omega_{n_1\dots n_{s-2}}{}^{q,r}$$

since it implies that

$$\delta S=\int_{M^d} E_{pqr}(\delta R^{n_1\dots n_{s-2}p}\, \omega_{n_1\dots n_{s-2}}{}^{q,r} + e^{n_1\dots n_{s-2}p}\, \delta R_{n_1\dots n_{s-2}}{}^{q,r})$$

Problem 2.1. Prove

Problem 2.2. Prove that

$$\delta S=\int_{M^d} E^{pqr}\,(R_{n_1\dots n_{s-2}p}\, \delta\omega^{n_1\dots n_{s-2}}{}_{q,r} - \delta e_{n_1\dots n_{s-2}p} R^{n_1\dots n_{s-2}}{}_{q,r})$$

EOM for ω,

$$R^{n_1\dots n_{s-1}} = de^{n_1\dots n_{s-1}} - dx_m\,\omega^{n_1\dots n_{s-1},\,m} = 0$$

expresses ω in terms of derivatives of e modulo a pure gauge part

Problem 2.3. Prove and find $\omega^{n_1\dots n_{s-1},\,m}(e)$.

EOM for e is

$$dx^m E_{qr}{}^{(p}\,R^{n_1\dots n_{s-2})q,\,r} = 0\,,$$

or in the Einstein–like form

$$R_{m(n_1;n_2\dots n_{s-1})}{}^{[p,\,m]} = 0$$

Gauge invariance implies equivalence to the Fronsdal action

$\omega^{a_1 \dots a_{s-1}, b_1 \dots b_t}$ different t : different dynamical roles

$t = 0$: frame-like HS field

$t = 1$: Lorentz connection-like auxiliary field

$t > 1$: extra fields appear for $s > 2$

By virtue of constraints t is an order of derivatives

$$\omega_{a_1 \dots a_{s-1}, b_1 \dots b_t} = \Pi\Big(\partial_{b_1} \dots \partial_{b_t} e_{a_1 \dots a_{s-1}}\Big)$$

Extra field decoupling condition:

independence of the free action of extra fields = absence of higher derivatives.

Extra fields do contribute at the interaction level : should be expressed in terms of the dynamical fields (modulo pure gauge degrees of freedom) by constraints (Lopatin, M.V. (1988))

First On-shell Theorem

by virtue of constraints and field equations, most of the HS field strengths are zero

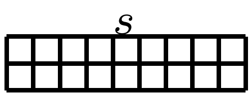
$$R_1{}^{a_1\dots a_{s-1},\,b_1\dots b_t} = \mathbf{X}^{\mathbf{a_1\dots a_{s-1},b_1\dots b_t}}\left(\frac{\delta \mathbf{S_2^s}}{\delta \omega_{\mathbf{dyn}}}\right), \qquad t < s-1$$

$$R_1{}^{a_1\dots a_{s-1},\,b_1\dots b_{s-1}} = h_c \wedge h_d C^{a_1\dots a_{s-1}\,c,\,b_1\dots b_{s-1}\,d} + \mathbf{X}^{\mathbf{a_1\dots a_{s-1},b_1\dots b_{s-1}}}\left(\frac{\delta S_2^s}{\delta \omega_{dyn}}\right)$$

$X^{a_1\dots a_{s-1},b_1\dots b_t}\left(\frac{\delta S_2^s}{\delta \omega_{dyn}}\right) = 0$ **by field equations.**

The generalized Weyl tensor $C^{a_1\dots a_s,b_1\dots b_s}$

$$C^{\{a_1\dots a_s,a_s+1\}b_2\dots b_s} = 0\,, \qquad C^{a_1\dots a_{s-2}cd,b_1\dots b_s}\,\eta_{cd} = 0\,,$$



parametrizes on-shell nontrivial components of the HS field strengths. For $s = 2$ **it parametrizes the on-shell nonzero components of the Riemann tensor, i.e. the Weyl tensor.**

Spin two

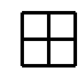
Fields: e^a and ω^{ab}.

Zero-torsion constraint and Einstein equations

$$T^a = 0\,, \qquad R^{ab} = e_c \wedge e_d\, C^{ac,\,bd}$$

$C^{ac,\,bd}$ **is the Weyl tensor in the symmetric basis**

$$C^{ac,\,bd} = C_W^{[ab],\,[cd]} + C_W^{[cb],\,[ad]}\,, \qquad C^{(ac,\,b)d} = 0\,, \qquad \eta_{ac} C^{ac,\,bd} = 0$$

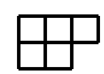


The restrictions on the derivatives of $C^{ac,\,bd}$ result from the Bianchi identities

$$D^L R^{ab} \equiv 0 \Rightarrow e_c \wedge e_d \wedge (D^L C^{ac},{}^{bd}) = 0\,, \qquad \Rightarrow \qquad D^L C^{ac},{}^{bd} = e_f C^{acf},{}^{bd}$$

where D^L is Lorentz covariant derivative and

$$C^{(abc},{}^{d)f} = 0\,, \qquad \eta_{ab} C^{abc,\,de} = 0$$



The process goes on by analyzing Bianchi.

For simplicity, Minkowski background: $h^a = dx^a, \, , \qquad \omega^{ab} = 0$

$$dC^{ac,\,bd} = h_f C^{acf,\,bd} \Rightarrow$$

$$dC^{abf,\,cd} = h_g(3C^{abfg,\,cd} + C^{abfc,\,gd} + C^{abfd,\,gc})$$

Continuation gives

$$dC^{a_1 \dots a_{2+k},\,b_1 b_2} = h_c((2+k)C^{a_1 \dots a_{2+k}\,c,\,b_1 b_2} + 2C^{a_1 \dots a_{2+k}\,(b_1,\,c\,b_2)})$$

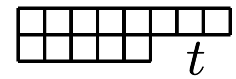
Combined with linearized Einstein equations gives unfolded spin two equations

Analogously for any spin

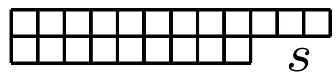
Central On-Shell Theorem

$$0 \le t \le s\,, \qquad \delta(n) = 1(0) \quad n = (\neq)0$$

$$\begin{cases} {R_1}^{a_1\dots a_{s-1},\,b_1\dots b_t} = \delta(t-(s-1))\,h_c \wedge h_d C^{a_1\dots a_{s-1}\,c,\,b_1\dots b_{s-1}\,d} \\[2em] \widetilde{D} C^{a_1\dots a_{s+k},\,b_1\dots b_s} = 0 \end{cases}$$



$s-1$



$s+k$

$$0 \le k \le \infty$$

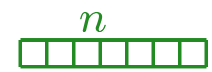
$$\tilde{\mathcal{D}} C^{a_1\dots a_{s+k},\,b_1\dots b_s} = h_c\Big((2+k) C^{a_1\dots a_{s+k}c,\,b_1\dots b_s} + s\, C^{a_1\dots a_{s+k}b_1,\,b_2\dots b_s\,c} + \lambda^2 h^a \dots C \dots$$

$$\tilde{\mathcal{D}}^2 = 0$$

Infinite set of 0−**forms** $C^{a_1\dots a_{s+k},\,b_1\dots b_s}$ **describe all gauge invariant on-shell nontrivial derivatives for a massless field of spin** s.

Klein-Gordon equation

Minkowski space in Cartesian coordinates: $h^a_m = \delta^a_m$

To unfold spin-0 massless field, introduce infinite set of 0-forms 

$$C_{a_1 \dots a_n} = C_{(a_1 \dots a_n)}\,, \qquad \eta^{bc} C_{bca_3 \dots a_n} = 0\,.$$

Unfolded KG equation

$$dC_{a_1 \dots a_n} = h^b C_{a_1 \dots a_n b}$$

This system is consistent: **since** $h^b \wedge h^c = -h^c \wedge h^b$

$$d^2 C_{a_1 \dots a_n} = -h^b \wedge h^c C_{a_1 \dots a_n bc} = 0 \quad (n = 0, 1, \dots)$$

The first two equations

$$\partial_n C = C_n\,, \qquad \partial_n C_m = C_{mn}\,,$$

imply $C_{nm} = \partial_n \partial_m C$.

Tracelessness of C_{nm} :

$$\Box C(x) = 0.$$

All other equations:

$$C_{a_1 \ldots a_n} = \partial_{a_1} \ldots \partial_{a_n} C$$

$C_{a_1 \ldots a_n}$: **set of all on-mass-shell nontrivial derivatives of** $C(x)$.

$d = 1$: **two independent components** $q(t) = C(t)$, $p(t) = C_n(t)$

rank $r > 1$ traceless tensors are zero

Any coordinates in Minkowski space

$$d \to D_0 = d + \omega_0 \,, \qquad D_0^2 = 0 \,, \qquad D_0(h) = 0 \,.$$