# GGI Lectures on the
# Pure Spinor Formalism of the Superstring

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**Outline**

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## 1. Introduction

### 1.1. Ramond-Neveu-Schwarz formalism

The superstring in the RNS formalism has four different sectors. In the NS GSO(+) sector, there are the massless vector and massive states while in the NS GSO(−) there are the tachyon and massive modes. On the other hand, in the R GSO(+) sector, there are massless Weyl and massive states, while in the R GSO(−) there are anti-Weyl massless and massive states. Although the GSO projection projects out the GSO(−) part of the spectrum, some processes (such as tachyon condensation) involve this sector.

The RNS formalism in the NS GSO(+) and NS GSO(−) sectors is supersymmetric at the worldsheet level. For the open string, it can be described by a superfield in two dimensions

$$\mathbf{X}^m(z, \kappa) = X^m(z) + \kappa\psi^m(z). \tag{1.1}$$

In this formalism, can write vertex operators for the massless field in the GSO(+) sector

$$V = \int dz d\kappa (D\mathbf{X}^m) A_m(\mathbf{X}), \tag{1.2}$$

where the derivative is $D = \frac{\partial}{\partial\kappa} + \kappa\frac{\partial}{\partial z}$. The tachyon in the GSO(−) sector can be described by the vertex operator

$$V_T = \int dz d\kappa T(\mathbf{X}) = \int dz (\psi \cdot \frac{\partial}{\partial X})T. \tag{1.3}$$

For the R sector, a vertex operator can be written, but it is more complicated, is not manifestly worldsheet supersymmetric, and involves the spin field [1]

$$\Sigma^\alpha = e^{\frac{1}{2}\int \psi\psi} e^{\frac{1}{2}\int \beta\gamma}. \tag{1.4}$$

Because of the complicated nature of the Ramond vertex operator, scattering amplitudes using the RNS formalism have been computed up to 6 fermions at tree level [2], up to 4 fermions at one loop [3] and, for 2-loops, the only RNS computations involve 4 bosons and no fermions [4].

For curved backgrounds, in the bosonic string case, the action can be written as

$$S = \int d^2 z g_{mn} \partial X^m \overline{\partial} X^n \tag{1.5}$$

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or with an antisymmetric field coupling $b_{mn}(X)$

$$S = \int d^2 z (g_{mn} + b_{mn}) \partial X^m \overline{\partial} X^n. \tag{1.6}$$

There is an obvious generalization for the RNS formalism

$$S = \int d^2 z d^2 \kappa [g_{mn}(\mathbf{X}) + b_{mn}(\mathbf{X})] D \mathbf{X}^m \bar{D} \mathbf{X}^n \tag{1.7}$$

where $\bar{D} = \frac{\partial}{\partial \bar{\kappa}} + \bar{\kappa} \frac{\partial}{\partial \bar{z}}$. This action for the NS-NS sector can be obtained at the linearized level as the product of two massless vector states. But if one tries to describe the R-R sector by naively introducing a term $\Sigma^\alpha \bar{\Sigma}^\beta F_{\alpha\beta}(X)$ to the action, where $\Sigma^\alpha$ is the fermionic vertex operator introduced above, this term would require picture changing operators since the back-reaction of the R-R term would not be in the same picture as the NS-NS term. Since picture-changing is related to worldsheet superconformal invariance and is only understood in on-shell NS-NS backgrounds, it is unclear how to describe the RNS formalism in an R-R background.

If one computes amplitudes in the RNS formalism where all external states are in the NS sector, there could be internal R states in the loops. This means one has to sum over spin structures, which complicates the computation of loop amplitudes. However, if one computes amplitudes where all external states are in the GSO(+) sector, all internal states in the loops will also be GSO(+). This suggests one should try to describe the superstring in a space-time supersymmetric way in which one only has the GSO(+) sector.

The natural variables for the GSO(+) sector are $X^m(z)$ for $m = 0, \ldots .9$ and $\theta^\alpha(z)$ for $\alpha = 1, \ldots .16$, and the vertex operators will be functions of $X^m$ and $\theta^\alpha$. Space-time supersymmetry transforms

$$\delta\theta^\alpha = \epsilon^\alpha, \quad \delta X^m = (\epsilon\gamma^m\theta). \tag{1.8}$$

It will be important to fix the notation used. $\gamma^m_{\alpha\beta}$ and $(\gamma^m)^{\alpha\beta}$ denotes $16 \times 16$ symmetric matrices which are the off-diagonal components of the $32 \times 32$ $\Gamma^m$ matrices. Thus, the $\gamma^m$ matrices are the analog of the Pauli matrices in 10 dimensions. They satisfy the algebra $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta_\alpha{}^\gamma$. By antisymmetrizing the product of 3 gamma matrices, one can check that $(\gamma^{mnp})_{\alpha\beta} = -(\gamma^{mnp})_{\beta\alpha}$, while by antisymmetrizing the product of 5 gamma matrices, one can check that $(\gamma^{mnpqr})_{\alpha\beta} = (\gamma^{mnpqr})_{\beta\alpha}$.

## 1.2. Green-Schwarz formalism

There is a classical description for the superstring using these variables known as the Green-Schwarz formalism [5]. In order to compute the spectrum one must impose the light-cone gauge. On the other hand, the light-cone gauge choice makes difficult scattering amplitude computations, since some unphysical singularities appear in the worldsheet diagrams. Because of the hidden Lorentz invariance, these unphysical singularities must cancel, however, this is difficult to show explicitly. In any case, up to now only 4-point tree and one loop amplitudes have been explicitly computed using this formalism [6].

## 1.3. Pure spinor formalism

In these lectures, a new formalism for the superstring [7] will be presented which has made progress on both computing scattering amplitudes and describing backgrounds in a manifestly spacetime-supersymmetric manner.

1. Scattering amplitude computations:

It has been computed $N$-point tree amplitudes with an arbitrary number of fermions [8], 5-point one-loop amplitudes with up to four fermions [9], and 4-point two-loop amplitudes with up to four fermions [10][11].

Beyond 2-loops there are vanishing (non-renormalization) theorems stating that beyond a certain loop order, the effective action will not get contributions containing a certain number of derivatives of $R^4$ [12]. The proof relies on the counting of fermionic zero modes which are related to space-time supersymmetry. For $g \leq 6$, $\partial^{2g} R^4$ is the lowest order term which appears at genus $g$. If this statement could be extended for all $g$, it would imply that $N = 8$ $d = 4$ supergravity is finite [13] [14]. However, it naively appears that $\partial^{12} R^4$ terms are present for all $g \geq 6$, which implies by dimensional arguments that $N = 8$ $d = 4$ sugra is divergent starting at 9 loops [14].

2. Ramond-Ramond backgrounds:

In the pure spinor formalism, these backgrounds are no more complicated than NS-NS backgrounds. They are necessary to study the string in $AdS_5 \times S^5$. Some work has been done in the GS formalism and $PSU(2, 2|4)$ invariance in $AdS_5 \times S^5$ plays the same role as super-Poincare invariance in a flat background. So quantization in the GS formalism requires breaking the manifest $PSU(2, 2|4)$ invariance whereas quantization in the pure spinor formalism preserves this symmetry.

Using the pure spinor formalism it has been shown that strings in the $AdS_5 \times S^5$ background are consistent at the quantum level to all orders in $\alpha'$ [15]. Non-local conserved currents were constructed [16] [17][18] and shown to exist to all orders in $\alpha'$. This suggests integrability to all orders in $\alpha'$.

## 2. $d = 10$ Super Yang-Mills and Superparticle.

The aim of this section is to describe SYM by performing a first quantization of the superparticle.

### 2.1. Review of the ten-dimensional superparticle

The action for a scalar particle in 10 dimensions can be written as

$$S = \int d\tau (\dot{X}^m P_m + eP^2). \tag{2.1}$$

This action has reparametrization invariance, as well as Lorentz invariance. The indices $m$ goes from $0, \ldots 9$, $X^m(\tau)$ denote the particle coordinates and $P_m$ its momentum conjugate. $e$ is a Lagrange multiplier which ensures the mass-shell condition $P^2 = 0$. There is a supersymmetrical version of this action [19] which can be obtained from (2.1) replacing $\dot{X}^m$ by a supersymmetric combination involving coordinates for the superspace $\theta^\alpha$, with $\alpha = 1, \ldots 16$: $\dot{X}^m \to \Pi^m = \dot{X}^m - \theta\gamma^m\dot{\theta}$ obtaining

$$S = \int d\tau [\Pi^m P_m + eP^2]. \tag{2.2}$$

Since $\Pi^m$ is invariant under the supersymmetry transformation $\delta X^m = \epsilon\gamma^m\theta$, $\delta\theta^\alpha = \epsilon^\alpha$ with constant paramenter $\epsilon^\alpha$, then (2.2) is also invariant. By computing the canonical momentum to $p_\alpha$ one obtains

$$p_\alpha = P_m (\gamma^m\theta)_\alpha. \tag{2.3}$$

Since the momentum is given in term of the coordinates, one has constraints. By defining the Dirac constraints

$$d_\alpha = p_\alpha - P_m (\gamma^m\theta)_\alpha, \tag{2.4}$$

one can check using the canonical Poisson bracket $\{p_\alpha, \theta^\beta\} = \delta_\alpha^\beta$ that the constraints satisfy the algebra $\{d_\alpha, d_\beta\} = -2\gamma^m_{\alpha\beta}P_m$. In order to covariantly quantize one should covariantly separate the first and second-class constraints, but because of the mass-shell condition $P^2 = 0$, there are eight first-class and eight second-class constraints. In order to deal with the second class constraint one can use the light-cone gauge, therefore breaking the manifest Lorentz invariance. However, since the aim is to have a covariant description one should explore another possibility.

## 2.2. Pure spinor superparticle

In 1985, Siegel [20] proposed the following action for the superparticle

$$S = \int d\tau (\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha + eP^2), \tag{2.5}$$

which is invariant under supersymmetry as can be easily checked by writing it in terms of supersymmetry invariant objects

$$S = \int d\tau (\Pi^m P_m + \dot{\theta}^\alpha d_\alpha + eP^2), \tag{2.6}$$

where $d_\alpha$ is defined as above. However, this attempt didn't succeed, roughly speaking, because it has too many degrees of freedom. Nevertheless, it was on the right track and it led to a pure spinor version for the superparticle [21] by modifying (2.6) to

$$S = \int d\tau (\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha \omega_\alpha) \tag{2.7}$$

where $\lambda^\alpha$ is a bosonic pure spinor ghost and $\omega_\alpha$ its conjugate momentum. Pure spinors made their first appearance in d=10 super-Yang-Mills in [22], and Paul Howe was the first to recognize that pure spinors simplify the description of the super-Yang-Mills (and supergravity) equations of motion and gauge invariances [23][24].

An unconstrained spinor in ten dimensions has 16 degrees of freedom, but $\lambda$ is constrained to satisfy the pure spinor condition $\lambda\gamma^m\lambda = 0$. Because of this constraint one has 11 degrees of freedom. Naively counting, one should have 12 bosonic ghosts since, if one counts the 8 fermionic second-class constraints as 4 fermionic first-class constraints, one has a total of 12 fermionic first-class constraints. The fact that $\lambda$ only has 11 components is because one of the 12 bosonic ghosts is cancelled by the fermionic ghost which comes from the $P^2 = 0$ constraint. To see why a pure spinor has 11 independent (complex) components, note that a $U(5)$ subgroup of the (Wick-rotated) Lorentz group leaves invariant a pure spinor up to a complex phase. So pure spinors parameterize the space $C \times \frac{SO(10)}{U(5)}$ which is an eleven-dimensional complex space. Because of the pure spinor condition, the worldsheet action is invariant under $\delta\omega_\alpha = \Lambda^m(\gamma_m\lambda)_\alpha$ which means that $\omega_\alpha$ has 11 gauge-invariant components.

Pure spinors were first defined by Cartan [25]. A product of two bosonic spinors in even dimension $d = 2D$ can be written (up to coefficients) as

$$\lambda^\alpha\lambda^\beta = (\lambda\gamma^{m_1...m_D}\lambda)(\gamma_{m_1...m_D})^{\alpha\beta} + (\lambda\gamma^{m_1...m_{D-4}}\lambda)(\gamma_{m_1...m_{D-4}})^{\alpha\beta} + ..., \tag{2.8}$$

where $(\gamma^{m_1\cdots m_n})_{\alpha\beta}$ for $n = 1, \ldots D$ denotes the antisymmetrization of the $n$ indices and when $n$ is $D \bmod 4$, $(\gamma^{m_1\cdots m_n})_{\alpha\beta}$ is symmetric in $\alpha\beta$. Cartan's definition of pure spinors states that the only nonvanishing component of this decomposition is the one involving the $D$ form. This definition coincides with the 10-dimensional definition of a pure spinor given above.

*2.3. $D = 10$ Super Yang-Mills*

Although it is not known how to write an action for Super Yang-Mills in 10 dimensions invariant under supersymmetry transformations, it is known how to write the equations of motion for SYM in a manifestly covariant way. To write this equation of motion, one can use intuition and modify the ordinary derivatives $\partial_m$ and supersymmetric derivatives $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\gamma^m\theta)_\alpha\partial_m$ which commutes with space-time supersymmetry and satisfy $\{D_\alpha, D_\beta\} = 2\gamma^m_{\alpha\beta}\partial_m$; by

$$\partial_m \rightarrow \nabla_m = \partial_m + A_m(X, \theta), \tag{2.9}$$

$$D_\alpha \rightarrow \nabla_\alpha = D_\alpha + A_\alpha(X, \theta), \tag{2.10}$$

where $A_\alpha$ and $A_m$ are superfields. The covariant derivatives now satisfy $\{\nabla_\alpha, \nabla_\beta\} = 2\gamma^m_{\alpha\beta}\nabla_m$. The equations of motion for the superfield $A_\alpha$ is

$$\nabla_\alpha A_\beta + \nabla_\beta A_\alpha + \{A_\alpha, A_\beta\} = 2\gamma^m_{\alpha\beta}A_m, \tag{2.11}$$

from which one gets

$$A_m = \frac{1}{32}(\gamma_m)^{\alpha\beta}(\nabla_{(\alpha}A_{\beta)} + \{A_\alpha, A_\beta\}) \tag{2.12}$$

and also

$$\gamma^{\alpha\beta}_{mnpqr}(\nabla_{(\alpha}A_{\beta)} + \{A_\alpha, A_\beta\}) = 0. \tag{2.13}$$

There is of course a gauge invariance $\delta A_\alpha(X, \theta) = \nabla_\alpha\Omega(X, \theta)$, $\delta A_m(X, \theta) = \nabla_m\Omega(X, \theta)$ and the first one can be used to gauge fix some of the field components of $A_\alpha(X, \theta)$, such that

$$A_\alpha(X, \theta) = a_m(\gamma^m\theta)_\alpha + \chi^\beta(\gamma^m\theta)_\beta(\gamma_m\theta)_\alpha + \partial_m a_n(\theta\gamma^{pmn}\theta)(\gamma_p\theta)_\beta + \ldots \tag{2.14}$$

where

$$\partial^m(\partial_{[m}a_{n]}) = 0, \quad \partial^m(\gamma_m\chi) = 0. \tag{2.15}$$

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These equations of motion can be obtained as constraints by quantizing the superparticle. If one defines the BRST charge $Q = \lambda^\alpha D_\alpha$, then it is nilpotent since $Q^2 = (\lambda \gamma^m \lambda) \partial_m = 0$ when $\lambda$ satisfies the pure spinor condition $\lambda \gamma^m \lambda = 0$. The vertex operator will be a ghost number one operator, written in terms of the SYM superfield as

$$V = \lambda^\alpha A_\alpha(X, \theta). \tag{2.16}$$

By computing $(Q + V)^2 = 0$ one encounters that $A_\alpha(X, \theta)$ is on-shell. The BRST operator also generates the gauge invariance for the vertex operator $\delta V = Q\Omega(X, \theta)$ which implies $\delta A_\alpha(X, \theta) = D_\alpha \Omega(X, \theta)$.

## 3. Pure Spinor Superstring and Tree Amplitudes

### 3.1. Worldsheet variables

The action for the flat space superstring using the pure spinor formalism is written as

$$S = \int d^2 z \left( \frac{1}{2} \partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial} \theta^\alpha + \omega_\alpha \overline{\partial} \lambda^\alpha + \hat{p}^{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}} + \hat{\omega}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} \right), \tag{3.1}$$

where for the open string case one would have the boundary conditions $\theta^\alpha = \hat{\theta}^\alpha$, $\lambda^\alpha = \hat{\lambda}^\alpha$. For the Type IIA string, the $\hat{\alpha}$ spinor index has the opposite chirality from the $\alpha$ spinor index, while for the Type IIB string it is of the same chirality. The left-moving BRST charge is given by $Q = \oint \lambda^\alpha d_\alpha$, where now $d_\alpha$ stands for

$$d_\alpha = p_\alpha + \partial X^m (\gamma_m \theta)_\alpha + \frac{1}{8}(\gamma^m \theta)_\alpha (\theta \gamma_m \partial \theta), \tag{3.2}$$

and satisfies the OPE[26]

$$d_\alpha(y) d_\beta(z) \rightarrow \frac{\gamma^m_{\alpha\beta} \Pi^m}{y - z}, \tag{3.3}$$

where $\Pi^m = \partial X^m - \theta \gamma^m \partial \theta$.

### 3.2. Physical states

A physical state at ghost number 1 in the cohomology of $Q$ can be written as

$$V = \lambda^\alpha A_\alpha(X, \theta) \tag{3.4}$$

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for the massless case, while for the lowest massive case can be written as [27]

$$V = \lambda^\alpha \Pi_m A_\alpha^m(X,\theta) + \lambda^\alpha \partial\theta^\beta A_{\alpha\beta}(X,\theta) + \lambda^\alpha d_\beta A_\alpha^\beta(X,\theta) \tag{3.5}$$

$$+ \lambda^\alpha N_{mn} A_\alpha^{mn}(X,\theta) + \partial\lambda^\alpha B_\alpha(X,\theta) + \lambda^\alpha J A_\alpha(X,\theta),$$

where $N^{mn} = \frac{1}{2}\omega\gamma^{mn}\lambda$ and $J = \lambda^\alpha\omega_\alpha$. The central charge has a contribution of 10 coming from the $X$'s, $-32$ coming from $\theta$, and 22 coming from $\lambda$, so the total central charge is zero. Because of the pure spinor condition, the OPE's of $\lambda$ and $\omega$ have to be done with care: One can do a $U(5)$ decomposition, losing manifest ten-dimensional Lorentz covariance, but at the end, the result can be expressed in terms of the Lorentz currents in the following covariant way

$$N^{mn}(y)N^{pq}(z) \to \frac{\eta^{m[p}N^{q]n} - \eta^{n[p}N^{q]m}}{y - z} - 3\frac{\eta^{m[p}\eta^{q]n} - \eta^{n[p}\eta^{q]m}}{(y - z)^2}. \tag{3.6}$$

Note that the OPE for the Lorentz currents corresponding to the matter sector $M^{mn} = \frac{1}{2}(p\gamma^{mn}\theta)$ is

$$M^{mn}(y)M^{pq}(z) \to \frac{\eta^{m[p}M^{q]n} - \eta^{n[p}M^{q]m}}{y - z} + 4\frac{\eta^{m[p}\eta^{q]n} - \eta^{n[p}\eta^{q]m}}{(y - z)^2}. \tag{3.7}$$

So for the total Lorentz current $M^{mn} + N^{mn}$, the double pole is the same as in the RNS formalism where the Lorentz current is $\psi^m\psi^n$.

### 3.3. Tree amplitudes

The simplest case to consider is the scattering amplitude of three open string states

$$\langle V_1(z_1)V_2(z_2)V_3(z_3)\rangle = \langle \lambda^\alpha A_\alpha^1(z_1)\lambda^\beta A_\beta^2(z_2)\lambda^\gamma A_\gamma^3(z_3)\rangle. \tag{3.8}$$

After using the OPE's one is faced with the following integral $\int d^{10}X \int d^{16}\theta \int d^{11}\lambda$ which diverges, so one has to regularize it. One can use intuition from bosonic string theory for deciding which zero modes of $\lambda^\alpha$ and $\theta^\alpha$ need to be present for non-vanishing amplitudes. In bosonic string theory, the zero-mode prescription coming from functional integration is

$$\langle c\partial c\partial^2 c\rangle = 1 \tag{3.9}$$

where $c$ is the worldsheet ghost coming from fixing the conformal gauge. It happens that $c\partial c\partial^2 c$ is the vertex operator of $+3$ ghost-number for the Yang-Mills antighost [28]. It

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is natural to use this ansatz and impose that non-vanishing correlation functions in this formalism must also be proportional to the vertex operator for the Yang-Mills antighost, which is $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ [29]. So, the zero mode prescription for tree amplitudes in the pure spinor formalism is

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1. \tag{3.10}$$

Although there is a generalization of this prescription for computing loop amplitudes which involves picture-changing operators [30], a better method is to introduce a new set of "non-minimal" variables $\overline{\lambda}_\alpha$ and $r_\alpha$, with corresponding conjugate momenta $\overline{\omega}^\alpha$ and $s^\alpha$ [31]. The left-moving contribution to the action for the non-minimal pure spinor formalism [32] is given by

$$S = \int d^2z(\frac{1}{2}\partial X^m\overline{\partial}X_m + p_\alpha\overline{\partial}\theta^\alpha + \omega_\alpha\overline{\partial}\lambda^\alpha + \overline{\omega}^\alpha\overline{\partial}\overline{\lambda}_\alpha + s^\alpha\overline{\partial}r_\alpha). \tag{3.11}$$

$\overline{\lambda}$ is constrained to satisfy the pure spinor condition $\overline{\lambda}\gamma^m\overline{\lambda} = 0$ and one also imposes that $\overline{\lambda}\gamma^m r = 0$. Note that $\overline{\lambda}_\alpha$ and $\overline{\omega}^\alpha$ are bosons, and $r_\alpha$ and $s^\alpha$ are fermions. The BRST charge is now $Q_{nonmin} = \int dz(\lambda^\alpha d_\alpha + \overline{\omega}^\alpha r_\alpha)$ so that the cohomology is not modified and all physical states can be chosen to be independent of the new variables.

Non-minimal pure spinor variables are useful because one can now construct a regulator $exp(\{Q,\Lambda\})$ which makes finite the measure of integration. Note that the regulator is equal to $1+Q\Omega$, so it does not affect BRST-invariant amplitudes. If one defines $\Lambda = -\overline{\lambda}_\alpha\theta^\alpha$ so that $Q\Lambda = -\overline{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha$ and inserts the regulator $exp(\{Q,\Lambda\})$, the integral becomes

$$\int d^{10}X \int d^{16}\theta \int d^{11}\lambda \int d^{11}\overline{\lambda} \int d^{11}rf(X,\theta,\lambda) \rightarrow \tag{3.12}$$

$$\int d^{10}X \int d^{16}\theta \int d^{11}\lambda \int d^{11}\overline{\lambda} \int d^{11}re^{\{Q,\Lambda\}}f(X,\theta,\lambda)$$

$$= \int d^{10}X \int d^{16}\theta \int d^{11}\lambda \int d^{11}\overline{\lambda} \int d^{11}re^{-\lambda^\alpha\overline{\lambda}_\alpha-r_\alpha\theta^\alpha}f(X,\theta,\lambda).$$

If $\overline{\lambda}_\alpha$ is interpreted as the complex conjugate to $\lambda^\alpha$, this choice of $\Lambda$ regularizes the integration over $\lambda$. Since $r$ does not appear in $f(X,\theta,\lambda)$, one can show that (3.12) is equal to

$$T^{\alpha\beta\gamma\delta_1...\delta_5} \int d^{10}X \int (d^5\theta)_{\delta_1...\delta_5}(\frac{\partial}{\partial\lambda})^3_{\alpha\beta\gamma}f(X,\theta,\lambda) \tag{3.13}$$

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where the tensor $T_{\alpha\beta\gamma\delta_1...\delta_5}$ (the inverse of $T^{\alpha\beta\gamma\delta_1...\delta_5}$) is a Lorentz-invariant tensor defined by

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) = T_{\alpha\beta\gamma\delta_1...\delta_5}\lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\delta_1}...\theta^{\delta_5}. \qquad (3.14)$$

To obtain (3.13), one uses that $\bar{\lambda}\gamma^m r = 0$ implies that

$$\int d^{11}r = T^{\alpha\beta\gamma\delta_1...\delta_5}\epsilon_{\delta_1...\delta_{16}}\frac{\partial}{\partial r_{\delta_6}}...\frac{\partial}{\partial r_{\delta_{16}}}\bar{\lambda}_\alpha\bar{\lambda}_\beta\bar{\lambda}_\gamma. \qquad (3.15)$$

So (3.12) reproduces the ansatz of (3.10).

The four-point amplitude at tree level is given by considering three unintegrated vertex operator and one integrated vertex operator

$$A_4 = \langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4 U(z_4)\rangle. \qquad (3.16)$$

To find the form of the integrated vertex operator $U$, start with the superparticle action

$$\int d\tau(\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha + \omega_\alpha\dot{\lambda}^\alpha), \qquad (3.17)$$

and consider a super Yang-Mills background

$$\int d\tau(\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha + \omega_\alpha\dot{\lambda}^\alpha + e(A_m\dot{X}^m + A_\alpha\dot{\theta}^\alpha + ...)) \qquad (3.18)$$

where ... is determined from BRST invariance. In RNS, the integrated operator is $\int d\tau(A_m\partial X^m + \psi^m\psi^n\partial_n A_m)$ where the last term is determined by worldsheet superconformal invariance. In the pure spinor formalism, the integrated vertex operator is determined by BRST invariance and is given by

$$U = A_m\Pi^m + A_\alpha\partial\theta^\alpha + W^\alpha d_\alpha + F^{mn}N_{mn}, \qquad (3.19)$$

where $W^\alpha$ and $F_{mn}$ are superfield strengths. The lowest component of $W^\alpha$ is the gaugino $\chi^\alpha$ and the lowest component of $F_{mn}$ is the fieldstrength $\partial_{[m}a_{n]}$. One can check that $QU = \partial(\lambda^\alpha A_\alpha)$ so $\int dzU$ is BRST invariant.

The N-point tree level amplitude

$$\langle V^1(z_1)V^2(z_2)V^3(z_3)\int U_4...\int U_N\rangle \qquad (3.20)$$

can be computed by first integrating out the non-zero modes by evaluating the OPE's. To integrate the zero modes, use

$$\langle f(X,\theta,\lambda)\rangle = T\int d^{10}X(\frac{\partial}{\partial\lambda})^3(\frac{\partial}{\partial\theta})^5 f \qquad (3.21)$$

where $T$ is the tensor of (3.14). From the three point tree level amplitude $\langle\lambda A\lambda A\lambda A\rangle$ one gets the usual cubic term in the SYM amplitude $\int d^{10}X(aa\partial a + \chi a\chi)$.

## 4. Loop Amplitudes

### 4.1. b ghost

In the pure spinor formalism there is no analog of the $c$ ghost, but there is an analog of the $b$ ghost which is necessary for the computation of string loop amplitudes. For example, the closed string one loop amplitude requires a $b$ and $\bar{b}$ ghost integrated over the Beltrami differential of the torus as

$$\int d^2\tau \langle V_1 \int b \int \bar{b} \int U_2 \ldots \int U_N \rangle \tag{4.1}$$

where in the case of the closed string, $V = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} A_{\alpha\hat{\beta}}(X, \theta, \hat{\theta})$. Note that at the linearized level, BRST invariance of this vertex operator implies that $A_{\alpha\hat{\beta}}$ satisfies the supergravity equations of motion.

It will be shown that a composite operator for the $b$ ghost can be written in terms of the other worldsheet fields in such a way that $\{Q, b\} = T$. To construct this operator, note that after adding the non-minimal variables of the previous section, the energy momentum tensor is given by

$$T_{nonmin} = \frac{1}{2}\Pi^m \Pi^m + d_\alpha \partial\theta^\alpha + s^\alpha \partial r_\alpha + T_\lambda + T_{\bar{\lambda}} \tag{4.2}$$

where $T_\lambda$ and $T_{\bar{\lambda}}$ are the stress tensors for $\lambda^\alpha$ and $\bar{\lambda}_\alpha$. If one would start with $b^\alpha = \frac{1}{2}\Pi^m(\gamma^m d)^\alpha$, then $Qb^\alpha = \frac{1}{2}\Pi^2 \lambda^\alpha$ up to terms involving $\partial\theta^\alpha$. So, naively, one should "divide" $b^\alpha$ by $\lambda^\alpha$. With the help of the non-minimal variables, this is possible by defining

$$b = \frac{\frac{1}{2}\bar{\lambda}_\alpha(\Pi^m \gamma_m d)^\alpha}{\lambda^\beta \bar{\lambda}_\beta} + \ldots \tag{4.3}$$

where ... is determined by $\{Q_{nonmin}, b\} = T_{nonmin}$ where $Q_{nonmin} = \int dz(\lambda^\alpha d_\alpha + \bar{\omega}^\alpha r_\alpha)$. One finds that the complete expression for the $b$ ghost is

$$b = s^\alpha \partial\bar{\lambda}_\alpha + \frac{\bar{\lambda}_\alpha(2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha)}{4(\bar{\lambda}\lambda)} \tag{4.4}$$

$$+\frac{(\bar{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d + 24N_{mn}\Pi_p)}{192(\bar{\lambda}\lambda)^2} - \frac{(r\gamma_{mnp}r)(\bar{\lambda}\gamma^m d)N^{np}}{16(\bar{\lambda}\lambda)^3} + \frac{(r\gamma_{mnp}r)(\bar{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\bar{\lambda}\lambda)^4},$$

which satisfies $\{Q_{nonmin}, b\} = T_{nonmin}$. From now on, the *nonmin* subscript will be dropped out.

The fact that the $b$ ghost has poles when $\lambda\bar{\lambda} \to 0$ means there are subtleties in defining the Hilbert space of allowable states in the pure spinor formalism. If one allowed states with arbitrary powers of poles, the cohomology would become trivial. This is easy to verify since the operator

$$S = \frac{\theta\bar{\lambda}}{\lambda\bar{\lambda} + r\theta},\tag{4.5}$$

satisfies $QS = 1$. Then $QV = 0$ implies $Q(SV) = V$, so the existence of $S$ in the Hilbert space would trivialize the BRST cohomology. Expanding $S$, one finds a pole of 11th order when $(\lambda\bar{\lambda}) = 0$. So if one allowed operators with this pole behavior in $\lambda\bar{\lambda}$, the cohomology would become trivial. One therefore forbids states in the Hilbert space which diverge faster than $(\lambda\bar{\lambda})^{-10}$ when $\lambda \to 0$. This allows the above operator for the $b$ ghost but forbids the $S$ operator.

### 4.2. Loop amplitude computations

For $g$-loop amplitudes, one needs to insert $3g - 3$ $b$ ghosts. So for $g \geq 3$, the number of poles in the $b$ ghost could add up to more than 11. This would make the functional integral $\int d^{11}\lambda \int d^{11}\bar{\lambda}$ diverge near $\lambda\bar{\lambda} = 0$. This difficulty is overcome with an appropriate definition of a regulator [33] which smooths out the poles of the different $b$ ghosts so that the total divergence is slower than $(\lambda\bar{\lambda})^{-11}$. However, the form of this regulator is complicated and its explicit contribution has only been worked out in simple cases [34]. Nevertheless, there are several multiloop amplitudes one can compute which do not require this complicated regulator.

In the non-minimal pure spinor formalism, the integration measure at $g$ loops is

$$A = \int d^{10}X \int d^{16}\theta \int d^{11}\lambda \int d^{11}\bar{\lambda} \int d^{16g}p \int d^{11g}\omega \int d^{11g}\bar{\omega} \int d^{11g}s \int d^{11}r \tag{4.6}$$

where the conformal weight one worldsheet fields contribute with $g$ zero modes. One can separate out the non-zero modes and use the free field OPE's to integrate them out, leaving an integration over bosonic and fermionic zero modes. To account for the bosonic and fermionic zero modes, the zero mode regulator used for tree-level amplitudes must be modified to $\Lambda = -\bar{\lambda}_\alpha\theta^\alpha - \sum_{I=1}^{g}\omega_{I\alpha}s_I^\alpha$ which implies $Q\Lambda = \bar{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha - \sum_{I=1}^{g}(\bar{\omega}_I^\alpha\omega_{I\alpha} - s_I^\alpha d_{I\alpha})$ [32].

As an example, one can compute the four-point massless one-loop and two-loop amplitudes. Using (4.1), the one-loop four-point open superstring amplitude is given by

$$A = \int d^2\tau \int d^{10}X \int d^{16}\theta \int d^{11}\lambda \int d^{11}\bar{\lambda} \int d^{16}p \int d^{11}\omega \int d^{11}\bar{\omega} \int d^{11}s \int d^{11}r \int b$$
$$\tag{4.7}$$

$$(\lambda A)(\int \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + N^{mn}F_{mn})^3 e^{\{Q,\Lambda\}}.$$

To get a non-vanishing amplitude, one needs to absorb $16 d_\alpha$ zero modes from $\int d^{16}p$. One can get 3 from the term $d_\alpha W^\alpha$. The maximum number of $d_\alpha$ zero modes one can get from the regulator is 11, so the remaining two must come from the third term in the $b$ ghost. This third term of $b$ has one $r_\alpha$, so the remaining 10 $r$'s must come from the regulator. Note that $\omega\,\overline{\omega}$ and $\lambda\bar\lambda$ have gaussian integrals, which are easy to compute. So after integrating over the zero modes of $p_\alpha$, $r_\alpha$ and $s^\alpha$, one finds a term proportional to

$$\int d^{16}\theta\; \theta^{10} AWWW \tag{4.8}$$

where the factor of $\theta^{10}$ comes from the regulator, and indices on the superfields in (4.8) are contracted in a Lorentz-invariant manner. The computation of the Lorentz index contractions for the gluon contribution was done in [35], giving as a result $t_8 f^4$ where $t_8$ is a Lorentz-invariant tensor which contracts the 8 indices of $f^4$. For closed strings the analogous result was $t_8 t_8 R^4$. Using the non-minimal pure spinor formalism, the gauge anomaly one loop computation was also performed in [11], and five point one loop computations were performed in [9].

For four point two-loops, the closed string amplitude is given by

$$A = \int (d^2\tau)^3 \langle (\int b)^3 (\int \bar b)^3 \int U_1 \ldots \int U_4 e^{\{Q,\Lambda\}} \rangle. \tag{4.9}$$

Because of the two non-trivial cycles,

$$\Lambda = -\overline{\lambda}_\alpha \theta^\alpha - \sum_{I=1}^{2} \omega_\alpha^{(I)} s^{\alpha(I)}, \tag{4.10}$$

and

$$\{Q,\Lambda\} = -\overline{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha - \sum_{I=1}^{2} (\overline{\omega}^{\alpha(I)}\omega_\alpha^{(I)} - s^{\alpha(I)} d_\alpha^{(I)}). \tag{4.11}$$

One now needs to absorb $32 d_\alpha$ zero modes. The regulator contributes 22, each vertex operator contributes 1 and, because there are three $b$ fields, the third term in (4.4) gives the remaining 6 and also absorbs 3 $r_\alpha$ zero modes. The regulator absorbs the 22 $s^\alpha$ zero modes and also absorbs the remaining 8 $r_\alpha$ zero modes and contributes 8 $\theta^\alpha$ zero modes. So the resulting amplitude is of the form

$$|\int d^{16}\theta\; \theta^8 WWWW\,|^2. \tag{4.12}$$

The Lorentz index contractions for the graviton contribution was shown in [10] to give $t_8 t_8 \partial^4 R^4$, and confirmed the Type IIB S-duality prediction [36] that $\partial^4 R^4$ is the term of lowest order in derivatives at two loops.

### 4.3. Non-renormalization theorems

Now one can ask what is the term of lowest order in derivatives at higher loops. At $g$ loops, the naive expression for the term of lowest order in derivatives which saturates the fermionic zero modes is

$$A = \int d^{16}\theta \int d^{11}r \int d^{16g}p \int d^{11g}s \; (r\theta)^{12-2g}(ds)^{11g}(rdd)^{2g-1}(\Pi d)^{g-2} \, d^4, \qquad (4.13)$$

where $(r\theta)^{12-2g}(ds)^{11g}$ comes from the regulator, $(rdd)^{2g-1}(\Pi d)^{g-2}$ comes from the $3g-3$ $b$ ghosts, and $d^4$ comes from the four vertex operators. This naive formula predicts that the term of lowest order in derivatives at $g$ loops is $| \int d^{16}\theta(\theta)^{12-2g}WWWW |^2$, which corresponds to $\partial^{2g}R^4$. However, this formula clearly breaks down at $g > 6$ because of the $(r\theta)^{12-2g}$ term in (4.13).

The source of this breakdown is the divergence when $\lambda\bar{\lambda} \to 0$. For $g < 6$, one can argue that these divergences are not present since the terms in the $b$ ghost which contribute do not diverge faster than $(\lambda\bar{\lambda})^{-10}$. This is related to the fact that $\partial^{2g}R^4$ is a superspace $F$-term when $g < 6$. However, when $g \geq 6$, the poles from the $b$ ghost diverge faster than $(\lambda\bar{\lambda})^{-10}$ which means one has to use the complicated regulator of [33]. This is related to the fact that $\partial^{2g}R^4$ can be written as a superspace $D$-term when $g \geq 6$. In the presence of the complicated regulator, the zero mode counting of (4.13) is modified. Although a detailed analysis of the zero mode counting has not yet been done in the presence of this complicated regulator, it naively appears that the $\partial^{12}R^4$ term can appear at all loops for $g \geq 6$ [12]. If this naive counting is correct, it would imply (by dimensional arguments) that the first divergence of $N = 8$ d=4 supergravity appears at 9 loops [14].

## 5. Curved Backgrounds

### 5.1. $\alpha'$ corrections to supergravity

The action in a curved background can be obtained by considering the flat background with vertex operators, and then covariantizing. Use the variables $Z^M = (X^m, \theta^\mu)$ for the open string. In this notation, $\partial\theta^\alpha A_\alpha + \Pi^m A_m$ combines to $\partial Z^M A_M$. For the closed superstring, use the coordinates $(X^m, \theta^\mu, \bar{\theta}^{\hat{\mu}})$. One gets the action[37]

$$S = \int dz d\bar{z}(\frac{1}{2}(G_{MN} + B_{MN})\partial Z^M \bar{\partial}Z^N + E^\alpha_M d_\alpha \bar{\partial}Z^M + E^{\hat{\alpha}}_M \bar{d}_{\hat{\alpha}}\partial Z^M + F^{\alpha\hat{\beta}}d_\alpha \bar{d}_{\hat{\beta}} \qquad (5.1)$$

$$+\Omega_M^{ab}\partial Z^M \bar{N}_{ab} + \bar{\Omega}_M^{ab}\bar{\partial}Z^M N_{ab} + C^{\alpha ab}d_\alpha \bar{N}_{ab} + \bar{C}^{\hat{\alpha}ab}\bar{d}_{\hat{\alpha}}N_{ab} + R^{abcd}N_{ab}\bar{N}_{cd} + \omega_\alpha \bar{\partial}\lambda^\alpha + \bar{\omega}_{\hat{\alpha}}\partial\bar{\lambda}^{\hat{\alpha}}).$$

The index notation is $A = (a, \alpha, \hat{\alpha})$ and $E_M^A(Z)$ is the supervielbein. Note that the superspace metric $G_{MN} = E_M^a E_N^b \eta_{ab}$ does not involve the supervielbein with indices $(\alpha, \hat{\alpha})$. So all the components of $E_M^A(Z)$ appear in the action, while in the Green-Schwarz action $E_M^\alpha(Z)$ and $E_M^{\hat{\alpha}}(Z)$ do not appear. In (5.1), the lowest component of $F^{\alpha\hat{\beta}}$ is the Ramond-Ramond field strength. Note that $d_\alpha$ is treated as an independent variable in this action instead of $p_\alpha$.

To compute $\alpha'$ corrections to the supergravity equations of motion using this action, one should compute whether the action is BRST invariant, or equivalently, if the BRST charge $Q$ is nilpotent and conserved. It was shown in [37] that nilpotence of $Q$ and $\bar{\partial}(\lambda^\alpha d_\alpha) = 0$ at the classical level implies the supergravity equations of motion to lowest order in $\alpha'$. These equations of motion imply $\kappa$-symmetry in the Green-Schwarz formalism. Hovever, because $E_M^\alpha$ does not appear explicitly in the action in the GS formalism, it is not true that $\kappa$-symmetry implies the supergravity equations of motion.

At higher loop order, one needs to introduce the dilaton coupling $\alpha' \int d^2 z \Phi(Z)r$ and compute loop corrections to the OPE of $Q$ with $Q$ and the OPE of the stress tensor with $Q$. The one-loop Yang-Mills Chern-Simons corrections have been computed in this manner [38].

### 5.2. $AdS_5 \times S^5$ background

If $F^{\alpha\hat{\beta}}$ is an invertible matrix as in the $AdS_5 \times S^5$ background, one can solve the auxiliary equations of motion of $d_\alpha$ and write $d_\alpha$ in terms of $Z^M$. Because of $PSU(2, 2|4)$ isometry in this background, it is natural to define $E_M^A$ as in [39] in terms of a coset $g(z) \in \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \simeq \frac{SO(4,2)\times SO(6)}{SO(4,1)\times SO(5)} + 32$ fermions. The left-invariant currents are defined by $J = (g^{-1}\partial g)$ and $\bar{J} = (g^{-1}\bar{\partial}g)$ where the global $PSU(2, 2|4)$ isometries act on the left as $g \to \Sigma g$. The action will be defined to be invariant under local transformations by the right $g \to g\Omega(z)$ where $\Omega(z)$ takes values in $SO(4, 1) \times SO(5)$.

The currents can be decomposed into the ten vector elements $J^a$ and $J^{a'}$ (where $a = 0, \ldots 4, a' = 5 \ldots 9$), the 32 fermionic elements $J^\alpha$ and $J^{\hat{\alpha}}$ (where $\alpha, \hat{\alpha} = 1 \ldots 16$), and the 20 bosonic elements $J^{[ab]}$ and $J^{[a'b']}$, where $[ab] \in SO(4, 1)$ and $[a'b'] \in SO(5)$. These currents can also be written in terms of the vielbeins as $J^A = E_M^A \partial Z^M$, where $E_M^{[ab]}$ is

defined to be the spin connection $\Omega_M^{[ab]}$. After using the equations of motion to solve for $d_\alpha$ and $\bar{d}_{\hat{\alpha}}$, the BRST charge can be written as

$$Q = \int dz \lambda^\alpha J^{\hat{\alpha}} \eta_{\alpha\hat{\alpha}} + \int d\bar{z} \overline{\lambda}^{\hat{\alpha}} \overline{J}^\alpha \eta_{\alpha\hat{\alpha}}, \tag{5.2}$$

where $\eta_{\alpha\hat{\alpha}} = (\gamma^{01234})_{\alpha\hat{\alpha}}$ is in the direction of the RR field strength.

The pure spinor action in the $AdS_5 \times S^5$ background can be written as

$$S = R^2 \int d^2z (\frac{1}{2} J^a \overline{J}_a + \frac{1}{2} J^{a'} \bar{J}_{a'} + \delta_{\alpha\hat{\beta}} (J^\alpha \overline{J}^{\hat{\beta}} - 3J^{\hat{\beta}} \overline{J}^\alpha) + \omega_\alpha \overline{\nabla} \lambda^\alpha + \overline{\omega}_{\hat{\alpha}} \nabla \overline{\lambda}^{\hat{\alpha}} \tag{5.3}$$

$$+ (\omega \gamma^{ab} \lambda)(\overline{\omega} \gamma_{ab} \overline{\lambda}) - (\omega \gamma^{a'b'} \lambda)(\overline{\omega} \gamma_{a'b'} \overline{\lambda})),$$

where the last line appears because the space-time curvature of $AdS_5 \times S^5$ is non-vanishing.

To show that this action has BRST symmetry, note that the BRST charges act on the group elements as $Qg = g(\lambda^\alpha T_\alpha + \overline{\lambda}^{\hat{\alpha}} T_{\hat{\alpha}})$ where $T_\alpha$ and $T_{\hat{\alpha}}$ are the 32 fermionic generators of $PSU(2,2|4)$. From this, it is trivial to work out how $Q$ acts on $J$. Note that $Q^2$ acting on $g$ will be zero because of the pure spinor condition satisfied by $\lambda^\alpha$ and $\overline{\lambda}^{\hat{\alpha}}$.

What can be done with this model, which looks rather simple? One interesting question is if there is a version of this action which is BRST invariant to all order in $\alpha'$? This can be answered in the affirmative by using cohomology arguments [15].

Since the BRST operator is nilpotent, one can ask about its cohomology. At the lowest order in $\alpha'$, define the classical action of (5.3) to be $S_0$. This action is BRST invariant since $QS_0 = 0$. In other words, the BRST transformation of the corresponding Lagrangian $L_0$ is a total derivative $QL_0 = d\Lambda_0$. After computing the quantum part of the effective action $S_1$, one can ask if the sum of the classical and quantum action is still BRST invariant? In other words, is $Q(S_0 + \alpha' S_1) = 0$, or equivalently, is $Q(L_0 + \alpha' L_1) = d\Lambda$?. Now, the BRST variation of the quantum effective action should be a local operator, since quantum anomalies come from a short-distance regulator. Therefore, $QL_1 = \Omega_1$ where $\Omega_1$ is some local quantity. Furthermore, $\Omega_1$ is BRST-closed since $Q^2 L_1 = 0$. One can therefore ask if $\Omega_1$ is BRST-exact, that is, does $\Omega_1 = Q\Sigma$ for some local $\Sigma$?. The answer happens to be yes, since the cohomology is trivial at ghost number 1 for operators of non-zero conformal weight. This trivial cohomology is easily confirmed by constructing the most general operator of ghost number 1 which is local and which is invariant under $PSU(2,2|4)$. Since $Q(L_0 + \alpha' L_1) = d\Lambda + \alpha' \Omega_1 = d\Lambda + \alpha' Q\Sigma$, one can always add a local $PSU(2,2|4)$ invariant counter-term $-\alpha'\Sigma$ to the Lagrangian such that $Q(L_0 + \alpha' L_1 - \alpha'\Sigma) = d\Lambda$. So

after including the counter-term, the action $S_0 + \alpha' S_1 - \alpha' \int d^2 z \Sigma$ is BRST-invariant. This type of argument for quantum BRST invariance can be repeated to all perturbative orders in $\alpha'$. However, in principle there could be BRST anomalies which are non-perturbative in $\alpha'$.

The existence of non-local conserved currents is important for integrability. The local $PSU(2, 2|4)$ conserved charges are the Nöether charges for the global symmetry algebra,

$$q^A = \int d\sigma j^A, \tag{5.4}$$

where $A$ is a $PSU(2, 2|4)$ Lie algebra index.

Suppose the theory is on the plane and define the non-local charge

$$k_{(1)}^C = f_{AB}{}^C \int_{-\infty}^{\infty} d\sigma j^A(\sigma) \int_{-\infty}^{\sigma} d\sigma' j^B(\sigma') - \int_{-\infty}^{\infty} d\sigma l^C(\sigma) \tag{5.5}$$

for some $l^C$ where $f_{AB}^C$ are the $psu(2, 2|4)$ structure constants. Note that $Q j^A = \partial_\sigma h^A$ for some $h^A$ because $Q \int_{-\infty}^{\infty} d\sigma j^A(\sigma) = 0$. Therefore,

$$Q k_{(1)}^C = 2 f_{AB}{}^C \int_{-\infty}^{\infty} d\sigma j^A(\sigma) h^B(\sigma) - \int_{-\infty}^{\infty} d\sigma Q l^C. \tag{5.6}$$

So if $l^C(\sigma)$ is defined such that $Q l^C(\sigma) = 2 f_{AB}^C j^A h^B(\sigma)$, then $k_{(1)}^C$ will be BRST invariant. Using cohomology arguments similar to those above, one can prove that there always exists such an $l^C(\sigma)$. Therefore, one can contruct non-local BRST conserved charges. Furthermore, by repeatedly commuting $k_{(1)}^C$ with each other, one can obtain an infinite set of conserved charges and prove that the construction is valid at the quantum level to all orders in perturbation theory [15]. Classical non-local conserved currents have been constructed in [16][17][18] and it would be interesting to compute the algebra of these currents.

## 6. Open Problems

1) Geometrical principles: At the moment, there is no covariant derivation of the pure spinor BRST operator from gauge fixing a more symmetrical formalism. Although there are various procedures [40] [41] [42] for getting the pure spinor BRST operator from gauge-fixing, none of these procedures are Lorentz covariant at all stages in the gauge-fixing. Such a covariant derivation of the BRST operator would probably also provide a

"geometric" explanation of the complicated form of the $b$ ghost [43]. An interesting open question is to compute the cohomology of the $b$ ghost.

2) Superstring field theory: $QV + V * V = 0$ where $*$ is the star product in Witten's action gives the correct open superstring field theory equations of motion. In bosonic string theory, this comes from the action $S = \langle \frac{1}{2} VQV + \frac{1}{3} V * V * V \rangle$ [44]. Although $\langle \ \rangle$ can be defined in the non-minimal formalism using functional integration, the expression

$$\langle f \rangle = \int d^{10} X \int d^{16} \theta \int d^{11} \lambda \int d^{11} \overline{\lambda} \int d^{11} r e^{\{Q, \Lambda\}} f(X, \lambda, \theta) \qquad (6.1)$$

only makes sense if $f$ does not have poles which diverge faster than $(\lambda \overline{\lambda})^{-10}$. One can insert a regulator, but $f$ is not BRST closed since string fields are off-shell. So the action will depend on where one puts the regulator. Furthermore, the regulator breaks manifest spacetime supersymmetry. So although the equations of motion are manifestly spacetime supersymmetric, the action is not. Furthermore, to compute the four-point tree amplitude in string field theory, one needs to introduce the $b$ ghost which contains poles when $\lambda \to 0$. It is unclear how to define the off-shell Hilbert space of allowed string fields in such a way that the product of these string fields never contain poles which diverge faster than $(\lambda \overline{\lambda})^{-10}$.

3) Multiloop amplitudes: Computations beyond two-loops require a complicated regulator since the $b$ ghosts contribute poles which diverge faster than $(\lambda \overline{\lambda})^{-11}$. Up to now, no non-vanishing computations have been performed beyond two loops. A related question is the computation of $N$-point tree amplitudes in a gauge which involves more than 6 $b$ ghosts. These tree amplitude computations will also require the complicated regulator.

4) Unitarity: There is not yet a proof that BRST invariance of the scattering amplitudes implies that the amplitudes are unitary. This could be done either by proving equivalence to the RNS computation or by proving equivalence to the light-cone GS computation.

5) Compactification: Compactifications of the pure spinor formalism on a Calabi-Yau manifold have recently been considered in [45]. One expects that the resulting formalism should be equivalent with the hybrid formalism, however, this has not yet been proven. A related question is if one can construct lower-dimensional versions of the pure spinor formalism [46] [47].

6) M-theory: There is a d=11 version of the pure spinor formalism for the superparticle which describes linearized d=11 supergravity [48]. The $d = 11$ pure spinor is $\lambda^A$, $A =$

$1, \ldots 32$ such that $\lambda \gamma^M \lambda = 0$ for $M = 0, \ldots, 10$. Just as $Q = \lambda^\alpha d_\alpha$ at ghost number 1 gives SYM in 10 dimensions, $Q = \lambda^A D_A$ at ghost number 3 gives linearized $d = 11$ sugra. The vertex operator at ghost number 3 is $\lambda^A \lambda^B \lambda^C B_{ABC}$ where $B_{ABC}$ is the spinor component of the 3-form. This works nicely for the superparticle, but not has yet been generalized for the supermembrane. The main complication is that the constraint $\lambda \gamma^M \lambda = 0$ does not commute with the Hamiltonian and generates secondary constraints.

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