Higher Spin Gauge Theories
Lecture II

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1. $4d$ HS fields in spinor notation
2. Weyl algebra
3. Star product
4. Simplest HS algebra
5. Properties of HS algebras
6. Singletons and AdS/CFT

Lecture II b

1. Cubic HS action
2. Unfolded dynamics
3. Equations of motion in all orders
4. $4d$ HS fields in ten-dimensional space-time

Spinorial and tensorial HS models

Tensorial HS models in any dimension:

HS fields are realized as forms carrying tensor indices.

Spinorial $3d$ and $4d$ HS models:

HS fields are realized as forms carrying spinor indices.

The case of four dimensions

Key fact $\quad 2 \times 2 = 4$

Minkowski coordinates as 2×2 **hermitian matrices**

$$x^n \Rightarrow x^{\alpha\dot{\alpha}} = \sum_{n=0}^{3} x^n \sigma_n^{\alpha\dot{\alpha}}, \qquad \sigma_n^{\alpha\dot{\alpha}} = (I^{\alpha\dot{\alpha}}, \overrightarrow{\sigma}_k^{\alpha\dot{\alpha}})$$

$I^{\alpha\dot{\alpha}}$: **unit matrix**

$\overrightarrow{\sigma}_k^{\alpha\dot{\alpha}}, \quad k = 1, 2, 3$: **Pauli matrices**

$\alpha, \beta, \ldots = 1, 2, \ \dot{\alpha}, \dot{\beta}, \ldots = 1, 2$ **two-component spinor indices**

$$\det |x^{\alpha\dot{\alpha}}| = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

Lorentz symmetry: $sl(2, \mathbb{C}) \sim o(3, 1)$.

Two-component spinors

Two-component indices are contracted by the antisymmetric 2×2 matrix

$$\epsilon_{\alpha\beta}: \quad \epsilon_{12} = \epsilon^{12} = 1\,, \qquad \epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta_\alpha^\beta\,, \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi_\alpha = \psi^\beta\epsilon_{\beta\alpha}$$

Lorentz invariants $\psi^\alpha\chi_\alpha$: **Lorentz Symmetry:** $sl_2(\mathbb{C}) \sim o(3,1)$.

Dictionary between tensors and multispinors by:

$$\sigma^a_{\alpha\dot\alpha}\,, \qquad \sigma^{ab}_{\alpha\beta} = \sigma^{[a}_{\alpha\dot\alpha}\sigma^{b]\dot\beta}_{\beta}\,, \qquad \bar\sigma^{ab}_{\dot\alpha\dot\beta} = \sigma^{[a}_{\alpha\dot\alpha}\sigma^{b]\alpha}{}_{\dot\beta}$$

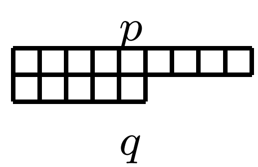
Pair of dotted and undotted indices: vector

Pairs of symmetrized indices of the same type: antisymmetric tensors

Irreducible representations of the Lorentz group: symmetric multispinors

$$A_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m} \oplus \overline{A}_{\beta_1\ldots\beta_m\,,\dot\alpha_1\ldots\dot\alpha_n} \sim \omega_{a_1\ldots a_p\,,b_1\ldots b_q}\,, \qquad p = |n+m|/2\,, \quad q = |n-m|/2$$

Irreducibility: $A_{(a_1\ldots a_p\,,a_{p+1})b_2\ldots b_q} = 0$:



, $\qquad A_{a_1\ldots a_p\,,b_1\ldots b_q}\eta^{a_1a_2} = 0\,.$

Gauge connections

Gauge 1-forms $\omega_{\alpha_1\dots\alpha_n\,,\dot\beta_1\dots\dot\beta_m}\,,\qquad n+m=2(s-1)$

$s=1:\quad \omega(x)=dx^n\omega_n(x)$

$s=2:\quad \omega_{\alpha\dot\beta}(x)\,,\qquad \omega_{\alpha\beta}(x)\,,\qquad \bar\omega_{\dot\alpha\dot\beta}(x)$

$s=3/2:\quad \omega_{\alpha}(x)\,,\qquad \bar\omega_{\dot\alpha}(x)$

Frame-like fields: $|n-m|=0$ **(bosons) or** $|n-m|=1$ **fermions**

Auxiliary Lorentz-like fields: $|n-m|=2$ **(bosons)**

Extra fields: $|n-m|>2$

Gauge invariant field strengths

0-forms $\quad C_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m}\,, \qquad |n-m| = 2s$

(Anti)selfdual Weyl tensors carry only (dotted)undotted spinor indices

$s = 0: \quad C(x)$

$s = 1/2: \quad C_\alpha(x)\,, \qquad \bar{C}_{\dot\alpha}(x)$

$s = 1: \quad C_{\alpha\beta}\,, \qquad \bar{C}_{\dot\alpha\dot\beta}$

$s = 3/2: \quad C_{\alpha\beta\gamma}\,, \qquad \bar{C}_{\dot\alpha\dot\beta\dot\gamma}$

$s = 2: \quad C_{\alpha_1\ldots\alpha_4}\,, \qquad \bar{C}_{\dot\alpha_1\ldots\dot\alpha_4}$

HS multiplets

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2\ldots$

$\omega_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m}$ **and** $C_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m}$ **with all** $n\geq 0$ **and** $m \geq 0$.

Generating functions $\omega(Y|x)$ **and** $C(Y|x)$: **Unrestricted functions of commuting spinor (twistor) variables** $Y = (y_\alpha, \bar y_{\dot\alpha})$

$$A(Y|x) = \sum_{n,m=0}^{\infty} \frac{1}{2n!m!} A_{\alpha_1\ldots\alpha_n\,,\dot\alpha_1\ldots\dot\alpha_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar y^{\dot\alpha_1}\ldots\bar y^{\dot\alpha_m}$$

Fermions require doubling of fields

$$\omega^{ii}(y,\bar y \mid x)\,,\qquad C^{i1-i}(y,\bar y \mid x)\,,\qquad i = 0,1\,,$$

$$\bar\omega^{ii}(y,\bar y \mid x) = \omega^{ii}(\bar y, y \mid x)\,,\qquad \bar C^{i\,1-i}(y,\bar y \mid x) = C^{1-i\,i}(\bar y, y \mid x)\,.$$

Twistor Central On-shell theorem

The full unfolded system for the doubled sets of free fields is

$$R_1^{ii}(y,\overline{y}\mid x) = \overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\; C^{1-i\,i}(0,\overline{y}\mid x) + H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\; C^{i\,1-i}(y,0\mid x)\;,$$

$$\tilde{D}_0 C^{i\,1-i}(y,\overline{y}\mid x) = 0\,,$$

where

$$H^{\alpha\beta} = h^{\alpha}{}_{\dot\alpha}\wedge h^{\beta\dot\alpha}\,,\qquad \overline{H}^{\dot\alpha\dot\beta} = h_{\alpha}{}^{\dot\alpha}\wedge h^{\alpha\dot\beta}\;,$$

$$R_1(y,\bar{y}\mid x) = D^{ad}\omega(y,\bar{y}\mid x)$$

$$D^{ad}\omega = D^L - \lambda h^{\alpha\dot\beta}\left(y_\alpha\frac{\partial}{\partial\bar{y}^{\dot\beta}} + \frac{\partial}{\partial y^{\alpha}}\bar{y}_{\dot\beta}\right)\,,\qquad \tilde{D} = D^L + \lambda h^{\alpha\dot\beta}\left(y_\alpha\bar{y}_{\dot\beta} + \frac{\partial^2}{\partial y^{\alpha}\partial\bar{y}^{\dot\beta}}\right)\,,$$

$$D^L A = d_x - \left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^{\beta}} + \bar{\omega}^{\dot\alpha\dot\beta}\bar{y}_{\dot\alpha}\frac{\partial}{\partial\bar{y}^{\dot\beta}}\right)\,.$$

NonAbelian generalization via star-product algebra

Weyl algebra

Weyl algebra A_n: associative algebra of functions $f(\widehat{Y})$ of n pairs of oscillators

$$[\widehat{Y}_\mu\,, \widehat{Y}_\nu] = 2iC_{\mu\nu}\,, \qquad \mu,\nu = 1,\dots 2n\,.$$

Different types of orderings are equivalent for polynomial $f(\widehat{Y})$ because commutators of oscillators decrease an order of polynomial.

Weyl ordering: totally symmetric

$$f(\widehat{Y}) = \sum_{p=0}^{\infty} f^{\mu_1\dots\mu_p}\widehat{Y}_{\mu_1}\dots\widehat{Y}_{\mu_p}\,,$$

$f^{\mu_1\dots\mu_p}$ **totally symmetric**

Wick (normal) ordering $[\widehat{a}_i^-\,, \widehat{a}^{+j}] = \delta_i^j$

$$f(\widehat{a}^\pm) = \sum_{p,q=0}^{\infty} \chi^{i_1\dots i_p}_{j_1\dots j_q}\widehat{a}^{+j_1}\dots\widehat{a}^{+j_q}\widehat{a}^-_{i_1}\dots\widehat{a}^-_{i_q}$$

Star Product

Weyl symbol $f(Y)$ of the operator $\hat{f}(\hat{Y})$ is a function of commuting variables Y_μ that has the same expansion

$$f(Y) = \sum_{p=0}^{\infty} f^{\mu_1 \dots \mu_p} Y_{\mu_1} \dots Y_{\mu_p}$$

Y_ν is the Weyl symbol of $\hat{Y}_\nu$.

Wick symbol $f(a^\pm)$ of the operator $\hat{f}(\widehat{a^\pm})$ is a function of commuting variables $a^\pm$ that has the same expansion

$$f(a^\pm) = \sum_{p,q=0}^{\infty} \chi^{i_1 \dots i_p}_{j_1 \dots j_q} a^{+j_1} \dots a^{+j_q} a^-_{i_1} \dots a^-_{i_q}$$

Star–product algebra is defined by the rule

Weyl star-product $(f * g)(Y)$ is a symbol of $\hat{f}(\hat{Y})\hat{g}(\hat{Y})$. In particular,

$$[Y_\nu, Y_\mu]_* = 2iC_{\nu\mu}\,, \qquad [a\,, b]_* = a * b - b * a$$

Wick star-product $(f \star g)(a^\pm)$ is a symbol of $\hat{f}(\hat{a}^\pm)\hat{g}(\hat{a}^\pm)$.

Examples

$$Y_\mu * Y_\nu = Y_{(\mu}Y_{\nu)} + iC_{\mu\nu}$$

$$a^{+j} \star a_i^- = a^{+j}a_i^- \,, \qquad a_i^- \star a^{+j} = a^{+j}a_i^- + \delta_i^j$$

Problem 2.1. Prove

$$[Y_\nu\,, f(Y)]_* = 2i\frac{\partial}{\partial Y^\nu}f(Y)\,, \qquad Y^\nu = C^{\nu\mu}Y_\mu$$

$$\{Y_\nu\,, f(Y)\}_* = 2Y_\nu f(Y)$$

$$a^{+i} \star f(a^\pm) = a^{+i}f(a^\pm)\,, \qquad f(a^\pm) \star a_j^- = f(a^\pm)a_j^-$$

$$a_i^- \star f(a^\pm) = \left(a_i^- + \frac{\partial}{\partial a^{+i}}\right)f(a^\pm)\,, \qquad f(a^\pm) \star a^{+\mathrm{J}} = \left(a^{+j} + \frac{\partial}{\partial a_j^-}\right)f(a^\pm)$$

Weyl-Moyal star-product

For the Weyl ordering, star–product is given by the Weyl-Moyal formula

$$(f_1 * f_2)(Y) = f_1(Y)\ \exp\,[i\overleftarrow{\partial^\nu}\overrightarrow{\partial^\mu}C_{\nu\mu}]\, f_2(Y)\ , \qquad \partial^\mu \equiv \frac{\partial}{\partial Y_\mu}$$

Problem 2.2. Prove using Campbell-Hausdorf formula for exponentials exp $J^\nu \widehat{Y}_\nu$

Important properties

- associativity: $(f * g) * h = f * (g * h)$
- regularity: star product of any two polynomials of Y is a polynomial

The Weyl-Moyal star product has integral representation

$$(f_1 * f_2)(Y) = \frac{1}{\pi^{2M}} \int dSdT\ \exp(-iS_\mu T_\nu C^{\mu\nu}) f_1(Y+S)\, f_2(Y+T)$$

Supertrace

$$str(f(Y)) = f(0)$$

Boson-fermion parity for spinorial Y_ν

$$f(Y) = (-1)^{\pi(f)} f(-Y)$$

$$str(f(Y) * g(Y)) = (-1)^{\pi(f)} str(g(Y) * f(Y)) = (-1)^{\pi(g)} str(g(Y) * f(Y))$$

Bilinear form $str(f*g)$ is invariant under $\delta f = [\epsilon\,,f]_*$ provided that fermion fields carry additional Grassmann parity

In components

$$str(A * B) = \sum_{n,m=0}^{\infty} \frac{i^{n+m-1}}{n!m!} A_{\alpha_1\ldots\alpha_n,\,\dot\beta_1\ldots\dot\beta_m} \wedge B^{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_m}\,,$$

for

$$A(Y) = \sum_{n,m=0}^{\infty} \frac{1}{2n!m!} A_{\alpha_1\ldots\alpha_n\,,\dot\alpha_1\ldots\dot\alpha_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot\alpha_1}\ldots\bar{y}^{\dot\alpha_m}$$

NonAbelian HS Algebra

$$R(Y|x) = d\omega(Y|x) + \omega(Y|x) * \wedge\omega(Y|x)$$

$$\omega = \omega_0 + \omega_1 \,, \qquad \omega_0 = \frac{1}{4i}(\omega_0^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_0^{\dot\alpha\dot\beta} \bar{y}_{\dot\alpha} \bar{y}_{\dot\beta} + 2\lambda h^{\alpha\dot\beta} y_\alpha \bar{y}_{\dot\beta})$$

$$R_0 = 0 \,, \qquad R_1 = D_0\omega_1 = d\omega_1 + [\omega_0 \,, \omega_1]_*$$

HS gauge transformation

$$\delta\omega(Y|x) = D\epsilon(Y|x) = d\epsilon(Y|x) + [\omega(Y|x) \,, \epsilon(Y|x)]_*$$

- **The simplest** $4d$ **HS algebra** $hu(1,0|4)$ **is the infinite-dimensional Lie algebra of even polynomials** $f(-Y) = f(Y)$ **with star-commutator** $[f \,, g]_*$ **as Lie product**

• $T^{\nu\mu}$ - generators of $sp(4) \sim (3,2) \subset hu(1,0|4)$: bilinears of Y.

Y^{μ} independent generators correspond to spin one

spin s generators are homogeneous Weyl symbols

$$\omega^s(\nu Y|x) = \nu^{2(s-1)}\omega(Y|x)\,.$$

$hu(1,0|4)$ is a global symmetry algebra of the most symmetric vacuum solution of the nonlinear bosonic HS theory

• HS algebras possess extensions to superalgebras $hu(n,m|2M)$, $ho(n,m|2M$ $husp(2n,2m|2M)$ with fermions and non-Abelian spin one YM gauge algebras $u(n) \oplus u(m)$, $o(n) \oplus o(m)$, $usp(2n) \oplus usp(2m)$

The construction of HS gauge symmetries is analogous Chan-Paton construction in String Theory

Orthogonal and symplectic gauge symmetry result from the construction analogous to orientifolds (Pradisi, Sagnotti) but in the space of auxiliary oscillators rather than in space-time

Properties of HS algebras

Let T_{s_1} be homogeneous polynomial of degree $2(s-1)$

$$[T_{s_1}, T_{s_2}] = T_{s_1+s_2-2m} = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2} \,.$$

Once a gauge field of spin $s > 2$ appears, the HS symmetry algebra requires an infinite tower of HS gauge fields together with gravity: $[T_s, T_s]$ gives rise to generators T_{2s-2}, of a gauge field of spin $s' = 2s-2 > s$ and also gives rise to generators T_2 of $o(3,2) \sim sp(4)$.

The spin-2 barrier separates theories with usual finite-dimensional lower-spin symmetries from those with infinite-dimensional HS symmetries. The maximal finite-dimensional subalgebra of $hu(1,0|4)$ is: $u(1) \oplus o(3,2)$, where $u(1)$ is associated with the unit element.
Even spin generators T_{2p} span a proper subalgebra $ho(1,0|4)$.

Singletons and AdS/CFT

Representations of HS symmetries: HS multiplets

HS algebras in AdS_4 are conformal HS symmetries of

$3d$ massless scalar S and spinor F

Flato-Fronsdal theorem:

$B \otimes B$ **and** $F \otimes F$: $m = 0$, $s = 0, 1, 2, \ldots \infty$ **in** AdS_4

$B \otimes F$: $m = 0$, $s = 1/2, 3/2, 5/2 \ldots \infty$ **in** AdS_4

global HS symmetries are symmetries of free $3d$ and $4d$ fields.

Interactions deform symmetries by field-dependent corrections

Klebanov-Polyakov conjecture: AdS/CFT **duality between** $N \to \infty$ $3d$ $O(N)$ **sigma-model and $4d$ HS gauge theory**

Bianchi, Heslop, Riccioni conjecture: states of String Theory arrange into modules of HS algebras

Cubic Actions

HS generalizations of the MacDowell-Mansouri action for gravity

$$S = -\frac{1}{4\kappa^2} \sum_{n,m=0}^{\infty} \frac{i^{n+m-1}}{n!m!} \epsilon(\mathbf{n-m}) \int_{M^4} R_{\alpha_1 \dots \alpha_n, \dot\beta_1 \dots \dot\beta_m}(x) \wedge R^{\alpha_1 \dots \alpha_n, \dot\beta_1 \dots \dot\beta_m}(x)\,,$$

$R^{\alpha_1 \dots \alpha_n, \dot\beta_1 \dots \dot\beta_m}(x)$ **are components of the HS curvature tensor**

$$R(Y|x) = d\omega(Y|x) + \omega(Y|x) \exp[i\overleftarrow{\partial^\nu}\overrightarrow{\partial^\mu} C_{\nu\mu}] \wedge \omega(Y|x)$$

$$\epsilon(-n) = -\epsilon(n)\,, \qquad \epsilon(n) = 1 \quad n > 0\,.$$

$$S(\epsilon(\mathbf{n-m}) \to \mathbf{1}) = S^{top} = -\frac{1}{4\kappa^2} \int_{M^4} str(R \wedge *R)\,, \qquad \delta S^{top} = 0\,.$$

Free Action in AdS_4

The quadratic part S^2 with $R \to R_1$ is manifestly gauge invariant.

Extra field decoupling condition

$$\frac{\delta S^2}{\delta \omega^{\alpha_1 \ldots \alpha_n \, , \dot{\alpha}_1 \ldots \dot{\alpha}_m}} \equiv 0 \,, \qquad |n-m| > 2$$

S^2 is the free action for all spin $s > 1$ massless fields.

Free massless equations of motion

$$h^{(\alpha_1}{}_{\dot{\beta}} \wedge R^{\alpha_2 \ldots \alpha_{s-1}), \dot{\alpha}_1 \ldots \dot{\alpha}_{s-1} \dot{\beta}} - h_{\gamma}{}^{(\dot{\alpha}_1} \wedge R^{\alpha_1 \ldots \alpha_{s-1} \gamma, \dot{\alpha}_2 \ldots \dot{\alpha}_{s-1})} = 0$$

in the bosonic case and

$$h_{\gamma_1}{}^{\dot{\alpha}_1} \wedge R^{\alpha_1 \ldots \alpha_{s-3/2} \gamma, \dot{\alpha}_2 \ldots \dot{\alpha}_{s-1/2}} = 0 \,,$$

and complex conjugated in the fermionic case.

EOM for the Lorentz-like auxiliary fields: HS "zero-torsion" constraint

$$R_{\alpha_1 \ldots \alpha_{s-1}, \dot{\alpha}_1 \ldots \dot{\alpha}_{s-1}} = 0 \,.$$

Constraints and Cubic Interactions

Extra fields that contribute beyond quadratic approximation have to be expressed via derivatives of the frame-like field by the constraints

$$h_{(\alpha}{}^{\dot\gamma}\wedge R_{\alpha_1\ldots\alpha_n)\alpha\,,\dot\beta_1\ldots\dot\beta_m\dot\gamma}=0\qquad n>m\geq 0\,,\qquad h^{\gamma}{}_{(\dot\beta}\wedge R_{\alpha_1\ldots\alpha_n\gamma\,,\dot\beta_1\ldots\dot\beta_m)_{\dot\beta}}=0$$

To prove HS gauge invariance in the cubic order it suffices to prove that

$$\delta S=\left(\frac{\delta S_2^s}{\delta\omega^{dyn}}\Delta(\omega^{dyn}\epsilon)\right)$$

since such terms can be compensated by a modification of the transformation law

$$\delta'\omega^{dyn}=\delta'\omega^{dyn}-\Delta(\omega^{dyn}\epsilon)$$

Use first on-shell theorem which contains the constraints

$$R_1(y,\overline{y}\mid x)\sim\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\;C(0,\overline{y}\mid x)+H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\;C(y,0\mid x)$$

HS gauge invariance of the cubic HS action

$$\delta S = -\frac{1}{2\kappa^2}\sum_{n,m=0}^{\infty}\frac{i^{n+m-1}}{n!m!}\,\epsilon(\mathbf{n}-\mathbf{m})\int_{M^4}[\epsilon\,,R]_{*\,\alpha_1\ldots\alpha_n,\,\dot\beta_1\ldots\dot\beta_m}(x)\wedge R^{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_m}(x)\,,$$

By Central On-Shell Theorem leaves three options

- **holomorphic:** $R_{\alpha_1\ldots\alpha_n}\wedge R_{\beta_1\ldots\beta_m}\epsilon^{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}$
- **antiholomorphic:** $R_{\dot\alpha_1\ldots\dot\alpha_n}\wedge R_{\dot\beta_1\ldots\dot\beta_m}\epsilon^{\dot\alpha_1\ldots\dot\alpha_n,\dot\beta_1\ldots\dot\beta_m}$
- **mixed:** $R_{\alpha_1\ldots\alpha_n}\wedge R_{\dot\beta_1\ldots\dot\beta_m}\epsilon^{\alpha_1\ldots\dot\alpha_n,\beta_1\ldots\dot\beta_m}$

Holomorphic and antiholomorphic terms vanish because $\epsilon(n-m)=\pm 1$.

The mixed terms vanish because

$$H^{\alpha\beta}\wedge\bar H^{\dot\alpha\dot\beta}\equiv h^{\alpha}{}_{\dot\gamma}\wedge h^{\beta\dot\gamma}\wedge h_{\gamma}{}^{\dot\alpha}\wedge h^{\gamma\dot\beta}\equiv 0$$

in

$$R(y,0)\times R(0,\bar y)=\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\;C^{1-i\,i}(0,\overline{y}\mid x)\wedge H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\;C^{i\,1-i}(y,0\mid x)$$

Central On-Shell Theorem and unfolded dynamics

$$R^{ii}(y,\overline{y}\mid x)=\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\;C^{1-i\,i}(0,\overline{y}\mid x)+H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\;C^{i\,1-i}(y,0\mid x)+\dots$$

$$\tilde{D}C^{i\,1-i}(y,\overline{y}\mid x)+\dots\,,\qquad\qquad \dots=O(C,\omega_1)$$

$$R(y,\overline{y}\mid x)=d\omega(y,\overline{y}\mid x)+\omega(y,\overline{y}\mid x)*\omega(y,\overline{y}\mid x)$$

$$\tilde{D}C(y,\overline{y}\mid x)=dC(y,\overline{y}\mid x)+\omega(y,\overline{y}\mid x)*C(y,\overline{y}\mid x)-C(y,\overline{y}\mid x)*\omega(y,-\overline{y}\mid x)$$

Such field equations are unfolded: exterior differential of any of the differential form field is expressed via the fields themselves

Problem: find nonlinear corrections that guarantee formal consistency = gauge invariance of the system

Unfolded Dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

degrees of freedom = # of dynamical variables

Field theory: infinite number of degrees of freedom = spaces of functions

Maxwell $q \sim \overrightarrow{A}(x)$, $p \sim \overrightarrow{E}(x)$.

Dirac approach is nice and efficient but noncovariant.

Covariant extension $t \to x^n$?

Unfolded dynamics: multidimensional generalization

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\alpha(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W^\alpha_{n_1 \ldots n_p}(x)$$

a set of differential forms

Unfolded equations

$$dW^\alpha(x) = G^\alpha(W(x))\,, \qquad d = dx^n\partial_n$$

$G^\alpha(W)$: **function of "supercoordinates"** W^α

$$G^\alpha(W) = \sum_{n=1}^{\infty} f^\alpha{}_{\beta_1\ldots\beta_n} W^{\beta_1}\wedge\ldots\wedge W^{\beta_n}$$

Covariant first-order differential equations

$d>1$: **Nontrivial compatibility conditions**

$$G^\beta(W)\wedge\frac{\partial G^\alpha(W)}{\partial W^\beta}\equiv 0$$

equivalent to the generalized Jacobi identities

$$\sum_{n=0}^{m}(n+1)f^\gamma{}_{[\beta_1\ldots\beta_{m-n}}f^\alpha{}_{\gamma\beta_{m-n+1}\ldots\beta_m\}} = 0$$

Any solution to generalized Jacobi identities: FDA (Sullivan (1968))

FDA is universal if the generalized Jacobi identity holds independently of space-time dimension. The HS FDAs are universal.

Every universal FDA = some L_∞ algebra

Universal unfolded systems are analogues of one-dimensional Hamiltonian systems

The unfolded equation is invariant under the gauge transformation

$$\delta W^\alpha = d\varepsilon^\alpha + \varepsilon^\beta \frac{\partial G^\alpha(W)}{\partial W^\beta},$$

where the gauge parameter $\varepsilon^\alpha(x)$ is a $(p_\alpha - 1)$-form.

(No gauge parameters for 0-forms W^α)

Properties

- General applicability
- Manifest (HS) gauge invariance
- Invariance under diffeomorphisms

  Exterior algebra formalism
- Interactions: nonlinear deformation of $G^{\alpha}(W)$
- Degrees of freedom are in 0-forms $C^{i}(x_0)$ at any $x = x_0$ (as $q(t_0)$)

  infinite-dimensional module dual to the space of single-particle states

realized as a space of functions of auxiliary variables (like $C(y, \bar{y})$ instead of phase space coordinates in the Hamiltonian approach
- Natural realization of infinite symmetries with higher derivatives
- Independence of ambient space-time

  Geometry is encoded by $G^{\alpha}(W)$
- Lie algebra cohomology interpretation: σ_- cohomology

Unfolding as a covariant twistor transform

Twistor transform

$$
\begin{array}{ccc}
 & C(Y|x) & \\
\eta \swarrow & & \searrow \nu \\
M(x) & & T(Y)\,.
\end{array}
$$

$W^{\alpha}(Y|x)$ **are functions on the "correspondence space"** C.

Space-time M **: coordinates** x. **Twistor space** T **: coordinates** Y.

Unfolded equations describe the Penrose transform by mapping functions on T **to solutions of field equations in** M.

Effective (spinorial HS models):

$W^{\alpha}(Y|x)$ **are unrestricted functions on** $T = R^{n}$ **or some projective space.**

Ineffective (tensorial HS models):

$W^{\alpha}(Y|x)$ **are subject to differential conditions in** T. **The unfolded field equations are still useful to describe interactions**

Idea of Nonlinear Construction

Being possible in a few first orders, straightforward construction of nonlinear deformation quickly gets very complicated.

• Trick: to find a larger algebra g' such that the substitution

$$\star \qquad \omega \to W = \omega + \omega C + \omega C^2 + \dots$$

into g' reconstructs nonlinear equations via a zero-curvature condition

$$dW + W \wedge W = 0$$

To find restrictions on W that reconstructs $\star$ in all orders

Result: no interaction ambiguity modulo field redefinitions in the tensorial models and one arbitrary function in the $4d$ spinorial model.

YM constant $g^2 = |\Lambda|^{\frac{d-2}{2}} \kappa^2$ can be rescaled away in the classical HS model

Doubling of spinors and Klein operators

$$\omega(Y|x) \longrightarrow W(Z;Y;K|x)\,, \qquad C(Y|x) \longrightarrow B(Z;Y;K|x)$$

to be accompanied by equations that determine the dependence on the additional variables Z_ν in terms of "initial data"

$$\omega(Y;K|x) = W(0;Y;K|x) = \sum_{ij=1}^{2} k^i \bar{k}^j \omega^{ij}(Y|x)$$

$$C(Y;K|x) = B(0;Y;K|x) = \sum_{ij=1}^{2} k^i \bar{k}^j \omega^{ij}(Y|x)\,.$$

$S(Z,Y,K|x) = dZ^\nu S_\nu$ **is connection along** Z^ν

Klein operators $K = (k, \bar{k})$ **generate chirality automorphisms**

$$kf(A) = f(\tilde{A})k\,, \quad \bar{k}f(A) = f(-\tilde{A})\bar{k}\,, \qquad A = (a_\alpha\,, \bar{a}_{\dot\alpha}) : \quad \tilde{A} = A = (-a_\alpha\,, \bar{a}_{\dot\alpha})$$

$k\bar{k}$ **is boson-fermion parity generator:** $k\bar{k}f(Y) = f(-Y)k\bar{k}$.

$$P(Y) = P^{\alpha\dot\alpha} y_\alpha \bar{y}_{\dot\alpha} \quad \longrightarrow \quad \tilde{P}(Y) = -P(Y)\,, \qquad \tilde{M}(Y) = M(Y)\,.$$

HS star product

$$(f \star g)(Z,Y) = \int dSdT f(Z+S, Y+S) g(Z-T, Y+T) \exp -iS_\nu T^\nu$$

$$[Y_\nu, Y_\mu]_\star = -[Z_\nu, Z_\mu]_\star = 2iC_{\nu\mu}\,, \qquad Z-Y \;:\; Z+Y$$ **normal ordering**

Inner Klein operators:

$$\kappa = \exp iz_\alpha y^\alpha\,, \qquad \bar\kappa = \exp iz_{\dot\alpha} y^{\dot\alpha}\,, \qquad \kappa \star f = \tilde f \star \kappa\,, \qquad \kappa \star \kappa = 1$$

Nonlinear HS Equations

$$\mathcal{W}\star\mathcal{W} = i(dZ^\nu dZ_\nu + dz^\alpha dz_\alpha F(B)\star k\star\kappa + d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha}\bar{F}(B)\star\bar{k}\star\bar{\kappa}) , \qquad \mathcal{W}\star B = B\star\mathcal{W}$$

Manifest gauge invariance

$$\delta\mathcal{W} = [\varepsilon, \mathcal{W}]_\star , \qquad \delta B = \varepsilon\star B - B\star\varepsilon , \qquad \varepsilon = \varepsilon(Z;Y;K|x)$$

$x - z$ **decomposition**

$$\begin{cases} dW + W\star W = 0 \\ dB + W\star B - B\star W = 0 \\ dS + W\star S + S\star W = 0 \\ S\star B - B\star S = 0 \\ S\star S = i(dZ^\nu dZ_\nu + dz^\alpha dz_\alpha F(B)\star k\star\kappa + d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha}\bar{F}(B)\star\bar{k}\star\bar{\kappa}) \end{cases}$$

Nontrivial equations are free of space-time differential d.

HS equations describe two dimensional fuzzy hyperboloid in noncommutative space of Y_μ **and** Z_μ. **Its radius depends on HS curvature** $B(x)$.

Consistency

I. Compatibility with

a. $d^2 = 0$

$$d^2W = d(W \star W) = (W \star W) \star W - W \star (W \star W) = 0$$

b. $(f \star g) \star h = f \star (h \star g)$

$$(S \star S) \star S = S \star (S \star S)$$

is elementary. The term with B may look problematic because S does not commute with the B-dependent terms but it is zero because $(dz^\alpha)^3 = 0$ and $(\delta \bar{z}^{\dot\alpha})^3 = 0$.

II. No divergences despite non-polynomial inner Klein operators elements: $\kappa = \exp iz_\alpha y^\alpha$ and $\bar\kappa = \exp i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha}$

Less trivial but still elementary

A particular form of star product plays crucial role

Perturbative analysis

Vacuum solution

$$B_0 = 0\,, \qquad S_0 = dZ^\nu Z_\nu\,, \qquad W_0 = \frac{1}{2}\omega_0^{\mu\nu}(x) Y_\mu Y_\nu$$

$$dW_0 + W_0 \star W_0 = 0$$

$\omega_0^{\mu\nu}(x)$: **describes** AdS_d.

First-order fluctuations

$$B_1 = C\,, \qquad S = S_0 + S_1\,, \qquad W = W_0 + W_1$$

$$[S_0\,, f]_\star = -2i d_z f\,, \qquad d_z = dZ^\nu \frac{\partial}{\partial Z^\nu}$$

Central On-Shell Theorem

is reproduced in the lowest order in a few steps:

1. $[S\,,B]_\star = 0$ **implies** $d_z B(Z;Y;K|x) = 0$ **and hence**

$$B(Z;Y;K|x) = C(Y;K|x) + \dots$$

2. $dB + W_0 \star B - B \star W_0 = 0$ **implies**

$$\tilde{D}_0 C = 0$$

3. $S\star S = idz^\alpha dz_\alpha F(B)$ **implies in the lowest order** $\{d_z\,,S_1\}_\star = -\frac{1}{2}F(C)dz^\alpha dz_\alpha$ **and hence reconstructs** S_1 **via** C **up to** Z**-exact terms**

$$S_1 = S_1(C) + d_z\epsilon(Z;Y;K|x)$$

4. $\epsilon(Z;Y;K|x)$ represents infinitesimal HS gauge transformations $\delta\mathcal{W} = [\varepsilon, \mathcal{W}]_\star$. Fixing the gauge ambiguity by setting $d_z\epsilon(Z;Y;K|x) = 0$ leaves leftover symmetry with

$$\epsilon(Z;Y;K|x) = \epsilon(Y;K|x)$$

where $\epsilon(Y;K|x)$ is the HS gauge parameter of the original formulation.

5. Solving $d_x + W\star S + S\star W = 0$ implies in the lowest order $D_0(S_1) = 2id_z W_1$. This gives

$$W_1 = \omega(Y;K|x) + W_1(W_0, C)$$

where $\omega(Y;K|x)$ is an arbitrary function of its arguments to be identified with the original HS gauge field in the frame-like formalism

6. Substitution of W_1 into the zero-curvature equation $dW + W \star W = 0$ gives the equation

$$R = hhC$$

of the Central-On-Shell theorem

HS theory in any dimension

$Y_\nu \to Y_i^A\,, \qquad C_{\nu\mu} \to \epsilon_{ij}\eta^{AB}\,, \qquad i,j = 1,2\,, \qquad A,B = 0,1,\dots d$

$\epsilon_{ij} = -\epsilon_{j\,i}$, $\epsilon_{12} = \epsilon^{12} = 1$: $sp(2)$ **symplectic form**

$\eta_{AB} = \eta_{BA}$: $o(d-1,2)$ **invariant metric**

$$A^A = \eta^{AB}A_B\,, \qquad a^i = \epsilon^{ij}a_j\,, \qquad a_i = a^j\epsilon_{ji}$$

Star-product algebra

$$[Y_i^A, Y_j^B]_* = \epsilon_{ij}\,\eta^{AB}$$

T^{AB} **rotates** $o(d-1,2)$ **vector indices**

$$[T^{AB}\,, Y_i^C]_* = \frac{1}{2}\Big(Y_i^A\eta^{BC} - Y_i^B\eta^{AB}\Big)$$

$t_{ij} = Y_i^A Y_j^B \eta_{AB}$ **rotate** $sp(2)$ **indices**

$$[t_{ij}\,, Y_k^A]_* = \epsilon_{jk}Y_i^A + \epsilon_{ik}Y_j^A$$

T^{AB} and t_{ij} form a Howe dual pair $o(d-1,2) \oplus sp(2)$

$$[T^{AB}, t_{ij}]_* = 0$$

$\mathcal{S}$ subalgebra of the Weyl algebra spanned by $sp(2)$ singlets $f(Y)$

$$\mathcal{S}: \qquad [f(Y), t_{ij}]_* = 0$$

$\mathcal{S}$ is not simple: two-sided ideal

$$g \in \mathcal{I}: \qquad g = t_{ij} * g^{ij} = g^{ij} * t_{ij}\,.$$

Since

$$t_{ij} = Y_i^A Y_j^B \eta_{AB}$$

$\mathcal{I}$ contains traces:, $\mathcal{A} = \mathcal{S}/\mathcal{I}$ consists of traceless tensors $\{T_s\}$ described by two-row rectangular tableaux.

HS algebra results from $\mathcal{A}$

Invariant functionals via Q–cohomology

Equivalent form of compatibility condition

$$Q^2 = 0\,, \qquad Q = G^\alpha(W)\frac{\partial}{\partial W^\alpha}$$

Q-manifolds

Hamiltonian-like form of the unfolded equations

$$dF(W(x)) = Q(F(W(x))\,, \qquad \forall F(W)\,.$$

Action in unfolded dynamics approach

$$S = \int L(W(x))\,, \qquad QL = 0 \qquad \textbf{(2005)}$$

$L = QM$: **total derivatives**

Actions and conserved charges: Q cohomology

for off-shell and on-shell unfolded systems, respectively

Nonlocality of HS Gauge Theory

Having infinitely many HS fields with higher derivatives in interactions, the HS Gauge Theory is nonlocal:

$$\lambda^{-1} D \sim 1$$

since

$$[\lambda^{-1} D\,, \lambda^{-1} D] \sim 1$$

A different mass scale parameter like α' is needed for a low-energy expansion

$4d$ massless fields in ten dimensions

To describe all $4d$ massless fields as two fields the Minkowski space-time M^4 has to be replaced by the ten dimensional space $\mathcal{M}_4$ of symmetric matrices $X^{\mu\nu} = X^{\nu\mu}$ Fronsdal 1985

$\mu, \nu = 1, 2, 3, 4$ Majorana (real) spinor indices $\mu = (\alpha, \dot{\alpha})$

$$X^{\mu\nu} = \left(x^{\alpha\dot{\beta}}, y^{\alpha\beta}, \bar{y}^{\dot{\alpha}\dot{\beta}}\right)$$

$x^{\alpha\dot{\beta}}$: Minkowski coordinates

$y^{\alpha\beta}, \bar{y}^{\dot{\alpha}\dot{\beta}}$: six spinning coordinates

From $d = 4$ to $d = 10$ via unfolded dynamics

Unfolded equations in the $4d$ flat Minkowski space

$$(d_x + dx^{\alpha\dot\beta}\frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot\alpha}})C(Y|x) = 0\,, \qquad d_x = dx^{\alpha\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}}$$

Extend $x^{\alpha\dot\alpha}$ to $X^{\mu\nu}$

$$(d_X + dX^{\mu\nu}\frac{\partial^2}{\partial Y^\mu \partial Y^\nu})C(Y|x) = 0\,, \qquad d_X = dX^{\mu\nu}\frac{\partial}{\partial X^{\mu\nu}}$$

There are only two dynamical fields in $\mathcal{M}_M$:

Scalar field $C(X)$ in the hyperspace $\mathcal{M}_4$ = all massless bosons in $4d$ Minkowski space.

Spinor field $C_\mu(X)$ in the hyperspace $\mathcal{M}_4$ = all massless fermions in $\mathbf{4d}$ Minkowski space.

Field Equations

(2001)

$$\textbf{bosons}: \quad \left(\frac{\partial^2}{\partial X^{\mu\nu}\partial X^{\rho\sigma}} - \frac{\partial^2}{\partial X^{\rho\nu}\partial X^{\mu\sigma}}\right) C(X) = 0$$

$$\textbf{fermions}: \quad \left(\frac{\partial}{\partial X^{\mu\nu}} C_\rho(X) - \frac{\partial}{\partial X^{\rho\nu}} C_\mu(X)\right) = 0$$

- No index contraction: no metric in ten dimensions
- The system is overdetermined
- Makes sense for $\mathcal{M}_M$ with $\mu, \nu = 1, 2 \ldots M$
- The field equations are $Sp(8)$ invariant

$Sp(8)$ is an extension of the $4d$ conformal group $SU(2,2)$.

$Sp(8)$ unifies all massless bosons and fermions into just two multiplets.

$X^{\mu\nu}$ are coordinates of the minimal $Sp(8)$ invariant space $\mathcal{M}_4$.

Fourier Transform

$C(X) = C_0 \exp ik_{\mu\nu}X^{\mu\nu}$

the field equation gives $k_{\mu\nu}k_{\rho\sigma} = k_{\mu\rho}k_{\nu\sigma}$, **i.e.**

$k_{\mu\nu} = k\xi_\mu\xi_\nu \,, \qquad k = \pm 1$

ξ_μ **is real.** $k = \pm 1$ **distinguishes between positive and negative energy branches:** **particles and antiparticles**

General solution

$$C(X) = \int d^M\xi \left(b^+(\xi) \exp i\xi_\mu\xi_\nu X^{\mu\nu} + b^-(\xi) \exp -i\xi_\mu\xi_\nu X^{\mu\nu} \right)$$

is parameterized by two functions of four real variables ξ_μ:

Initial data to be given on a M**-dimensional surface** E **in** $\mathcal{M}_M$.

E: **local Cauchy bundle**

For $M = 4$, $E = R^3 \times S^1$:

R^3 **is space in Minkowski space-time**, S^1**-modes describe helicity**

Time and Space

Let $T^{\mu\nu}$ be a positive definite matrix.

Space coordinates $x^{\mu\nu}$ are various $T-$ traceless matrices

$$X^{\mu\nu} \in \Sigma_t : \qquad X^{\mu\nu} = x^{\mu\nu} + tT^{\mu\nu}, \qquad x^{\mu\nu}T_{\mu\nu} = 0, \qquad T_{\mu\nu}T^{\nu\rho} = \delta^{\rho}_{\mu}.$$

$\mathcal{M}_M$ has one time parameter $t = \frac{1}{M}X^{\mu\nu}T_{\mu\nu}$. Using the ambiguity in $c^{\pm}(\xi)$ in the general solution

$$C(X) = \int d^M\xi \left(c^+(\xi) \exp i\xi_\mu\xi_\nu X^{\mu\nu} + c^-(\xi) \exp -i\xi_\mu\xi_\nu X^{\mu\nu}\right)$$

it is possible to localize solutions in M coordinates: physical events are M-dimensional.

Whether there exist some $d-1$ space-like coordinates x^n:

$$X^{\mu\nu} = \sigma^{\mu\nu}_n x^n$$

such that, using $c^{\pm}(\xi)$ it is possible to built solutions of the field equations proportional to (derivatives of)

$\delta^{d-1}(x-x_0)$ at any $x_0 \in R^{d-1}$?!

If yes, at given time we can switch on light at the point x_0 of our space $R^{d-1} \subset E$.

This happens if there exists a map $k_n = \sigma_n^{\mu\nu}\xi_\mu\xi_\nu$ onto R^{d-1}. By changing integration variables from ξ_μ to k_n plus some other variables in case $d-1 < M$, $\delta^{d-1}(x-x_0)$ can be obtained from the integration over k_n.

Usual space in $\mathcal{M}_M$ is realized in terms of Clifford algebra:

$$\gamma_n{}^{\mu}{}_{\nu} = \sigma_n^{\mu\rho} T_{\rho\nu}\,, \qquad \{\gamma_n, \gamma_m\} = 2\eta_{nm}\,,$$

$\mathcal{M}_M$ is visualized via Clifford algebras.

No metric tensor in the $sp(8)$ invariant dynamical equations in $\mathcal{M}_M$.

Space metric η_{nm} appears via Clifford algebra along with the concept of local event.

Different Physical Dimensions in $\mathcal{M}_4$

Different $sp(8)$ invariant equations visualize $\mathcal{M}_4$ as space-times of local events of different dimensions

Rank two equations

$$\frac{\partial^3}{\partial X^{[\mu_1\nu_1}\partial X^{\mu_2\nu_2}\partial X^{\mu_3]}{}_{\mu}{}^{\nu_3}}C(X) = 0$$

describe $6d$ **space-time with the** $SU(2)$ **spin variable**: $E = R^5 \times SU(2)$.

Rank four equations describe 10d space-time with the S^7 spin variable.

Delocalized branes of different dimensions in the same **10d** space-time $\mathcal{M}_4$?!

Rank two equations in $\mathcal{M}_M$ $\sim$ rank one in $\mathcal{M}_{2M}$ Gelfond, MV (2002)

$M = 2, 4, 8, 16$:

$d = 3, 4, 6, 10$ Bandos, Lukierski, Sorokin (1999)

Symmetries

Let some local Cauchy bundle $E = R^{d-1} \times S$ be chosen to visualize $\mathcal{M}_M$.

A transformation that maps Minkowski space-time to itself leaving the fibers intact is a usual conformal transformation.

A symmetry that does not shift points of the Minkowski space-time, acting on the coordinates of the fiber is the (generalized) electric-magnetic duality transformation that acts on all spins.

$Sp(2M)$ transformations that shift E in $\mathcal{M}_M$ look as nongeometric symmetries from the Minkowski space-time perspective, extending $su(2,2) \oplus u(1)$ to $sp(8)$ which mixes fields of different spins.

Riemann theta functions as solutions of massless field equations

A surprising property of the unfolded massless field equations formulated in $\mathcal{M}_M$

$$\left(\frac{\partial}{\partial Z^{\mu\nu}} + i\,h\frac{\partial^2}{\partial Y^\mu \partial Y^\nu}\right) C^+(Y|Z) = 0\,,$$

is that Riemann theta functions form their natural solutions

Gelfond, MV 2008

$$C^+(Y|Z) = \sum_{n^\mu \in \mathbb{Z}^M} c_n^+ \exp i(hZ^{\mu\nu}(2\pi n_\mu)(2\pi n_\nu) + 2\pi n_\rho Y^\rho)$$

$\mathcal{M}_M$ is a boundary of Siegel space

$c_n = 1$: $C^+(Y|Z)$ is Riemann theta function= D-function periodic in Y.

Space-time coordinates: period matrix?!

Conclusions

Nonlinear HS gauge theories do exist in various dimensions.

Unbroken HS gauge symmetries require

Infinite HS multiplets $+$ nonzero curvature$=$nonlocal theory

Free $4d$ HS theory admits concise formulation in the ten-dimensional space.

Metric tensor appears after coordinates of local events are defined.

Higher rank systems visualize physical space-times of different dimension as coexisting delocalized "branes" imbedded into $\mathcal{M}_M$

M: $M = 8 \to d = 6$, $M = 16 \to d = 10$, $M = 32 \to d = 11$**?!**

To do

Extend nonlinear HS theory to

Mixed symmetry fields

Matrix space-times $\mathcal{M}_M$

HS symmetry breaking mechanism

Low energy expansion parameter analogous to α'

Relation to String Theory

Exact solutions

So far very few exact solutions including

$m \neq 0$ **matter in** $3d$

selfdual in $4d$

Black hole in $4d$

Integrability?!

AdS/CFT!

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