Motivation

Theoretical description of SM process at the LHC

What is needed?

Fully differential cross-sections for the production of jets, heavy quarks and gauge bosons

I. Leading order calculations

- First estimates: leading order MC's based on Born amplitudes
- Multi-leg processes (up to 8 or more legs) are imporant at the LCH (see next slide)
- The use of standard Feynman-diagram approach already LO calculations are problematic:

  Stronger than factorial growth in number of external particles
  N-gluon scattering: CPU grows as $N^{(N-3)}$ (E-algorithm)

- Solution: use recursion relations (Berends, Giele; Britto, Cachazo, Feng,Witten) :
  CPU time has polynomial growth in the number of the external legs
  N^{α} (P- algorithm)
- Tree-level general purpose softwares: ALPGEN, HELAC (P), MADGRAPH (E)
- <u>More quantitative estimates require NLO (QCD and EW) corrections</u>

Status of Leading Order Calculations

fully automated, user friendly Leading Order generators:

Alpgen, CompHEP, CalcHEP, Comix, Helac, Madgraph, Sherpa, Whizard, ...
best implementations are working efficiently up to 11 legs

Tree amplitude:

- generate Feynman diagrams, evaluate helicity amplitudes numerically (Madgraph,Sherpa, CompHEP)
- use BG recursion relations (Alpgen, Helac,Comix) evaluate recursion fully numerically (color included, or stripped)

Phase space integral:

- small denominators of individual Feynman diagrams generate multi-chanel phase space integral according
- in case of recursion relations different recursion chains generate propagotor denominators,

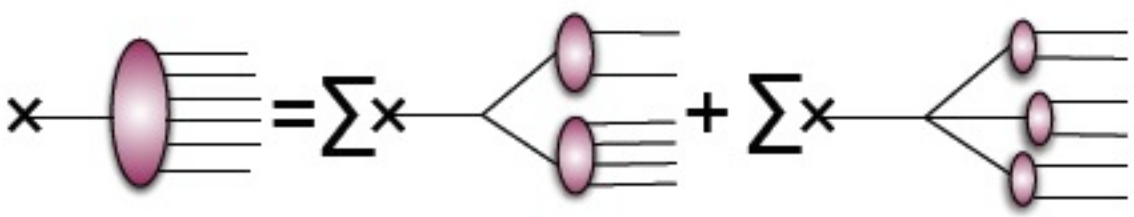
Difficulties:

quantitative description requires NLO accuracy

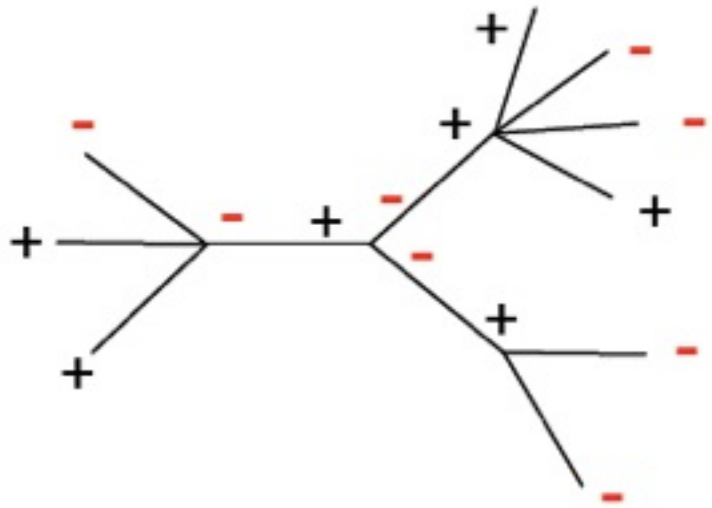
The efficient codes use recursion relations

Berends-Giele recursions:
color ordered amplitudes are constructed using off-shell currents
color can also be included (Helac, COMIX)

Berends, Giele '88

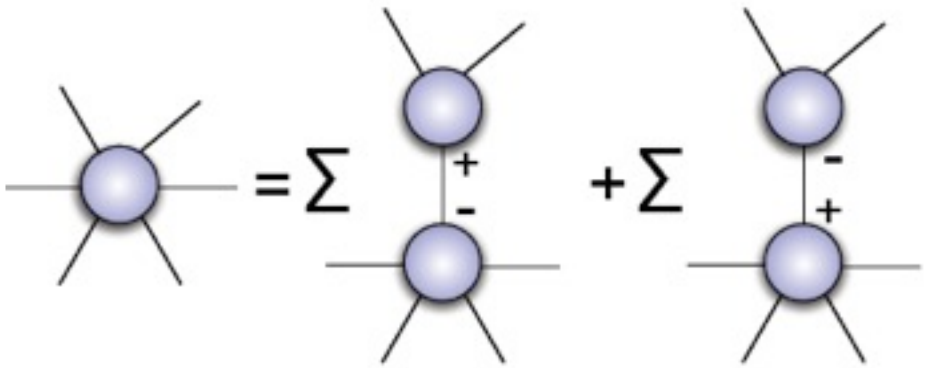


Cachazo, Svrcek, Witten '04



CFW recursions:
helicity amplitudes are calculated from MHV amplitudes

Britto, Cachazo, Feng '04



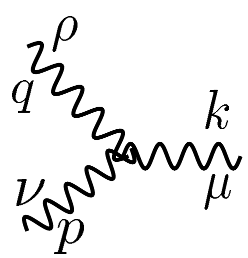
BCF recursions:
amplitudes via on-shell recursion
using complex shift of external momenta

In numerical implementations of BG is most efficient

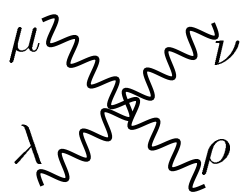
Colorless Feynman rules for color ordered amplitudes

$$(a_1, a_2, \ldots, a_n) = \mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n})$$

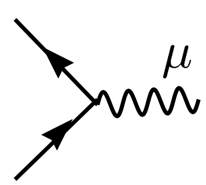
$$\mathcal{A}_n^{(0)}(1,2,3,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,3,\ldots,n)} (a_1 a_2 \ldots a_n)\, A_n^{(0)}(1,2,3,\ldots,n)$$



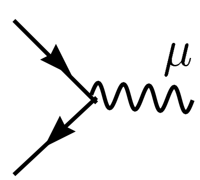
$$= \frac{i}{\sqrt{2}} \Big(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho \Big)$$



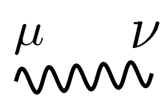
$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$



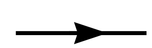
$$= \frac{i}{\sqrt{2}}\gamma_\mu$$



$$= -\frac{i}{\sqrt{2}}\gamma_\mu$$



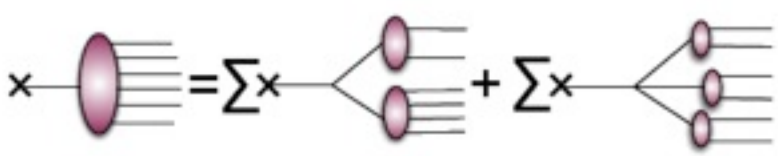
$$= -i\frac{\eta_{\mu\nu}}{p^2}$$



$$= \frac{i}{\not p}$$

Berends-Giele recursion relations for color ordered amplitudes

The color-ordered off-shell currents can be constructed recursively



$$J^{\mu}(1,2,\ldots,n) = \frac{-i}{P_{1,n}^2} \left\{ \sum_{k=1}^{n-1} V_3^{\mu\nu\rho}\left(P_{1,k}, P_{k+1,n}\right) J_{\nu}(1,\ldots,k) J_{\rho}(k+1,\ldots,n) + \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} V_4^{\mu\nu\rho\sigma} J_{\nu}(1,\ldots,j) J_{\rho}(j+1,\ldots,k) J_{\sigma}(k+1,\ldots,n) \right\},$$

$$P_{i,j} = p_i + p_{i+1} + \ldots + p_{j-1} + p_j,$$

The color-ordered n-point gluon off-shell current can also be defined as the sum of all color ordered Feynman-diagrams: n on-shell gluon, one off-shell gluon with polarization μ

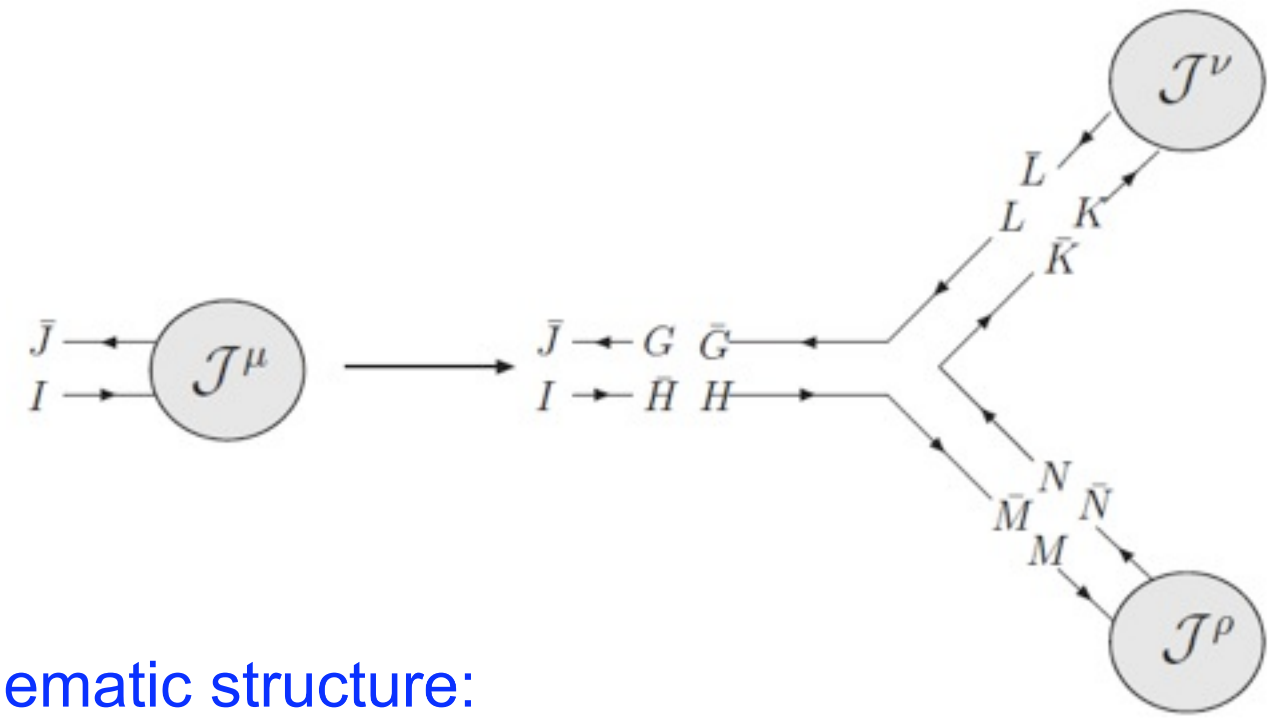
$V_3^{\mu\nu\rho}\left(P_{1,k}, P_{k+1,n}\right)$ and $V_4^{\mu\nu\rho\sigma}$ are color ordered vertices

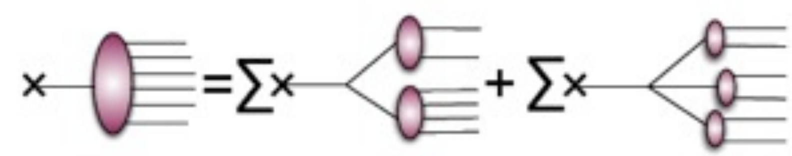
Color dressed Berends-Giele recursion relations

$$\mathcal{A}_n^{(0)}(1,2,3,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,3,\ldots,n)} \delta_{i_1 j_2}\delta_{i_2 j_3}\cdots\delta_{i_n j_1}\, A_n^{(0)}(1,2,3,\ldots,n)$$

In the color-flow decomposition a color-dressed gluon off-shell current is

$$\mathcal{J}^{\mu}_{I\bar{J}}(1,2,\ldots,n) = \sum_{\sigma\in S} \delta^{\bar{J}}_{i_{\sigma_1}}\delta^{\bar{j}_{\sigma_1}}_{i_{\sigma_2}}\ldots\delta^{\bar{j}_{\sigma_n}}_{I}\, J^{\mu}(\sigma_1,\sigma_2,\ldots,\sigma_n),$$





unordered in on-shell gluons

Schematic structure:

$$J_n(\pi) = P_n(\pi) \sum_{N=1}^{n} \sum_{\mathcal{P}_N(\pi)} V_N(\pi_1,\ldots,\pi_N)\, J_{i_1}(\pi_1)\ldots J_{i_N}(\pi_N)\ .$$

Instead of ordering we have partitioning

II. Ingredients of traditional NLO calculations

tree-level corrections from N+1 parton processes

- local subtraction terms
- recursion relations
- divergences analytical: from phase space integration over the subtraction terms

virtual corrections to N parton processes

- Feynman diagram evaluation, automated tools are used
- Passarino-Veltman or vanNeerven-Vermaseren type of reduction of tensor integrals to scalar integrals
- divergent terms: come from divergences of the scalar integrals

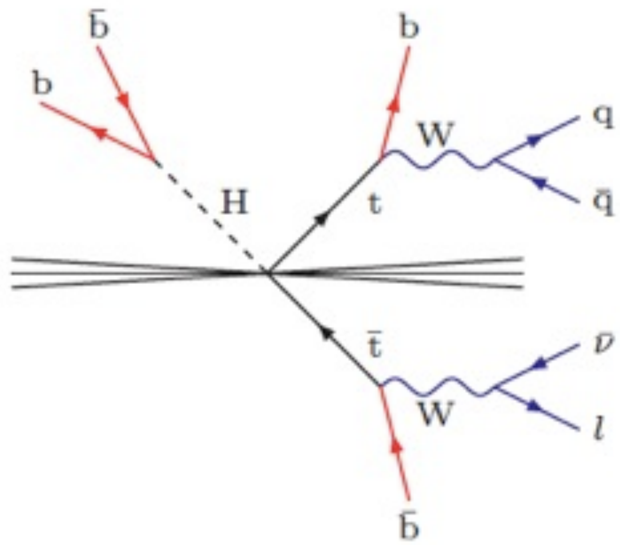
Straightforward calculations based on Feynman diagrams plagued by worse than factorial growth of the computer time. Difficult to push beyond N=6.

Bottleneck: virtual corrections.

N>5 leg processes will be important at the LHC

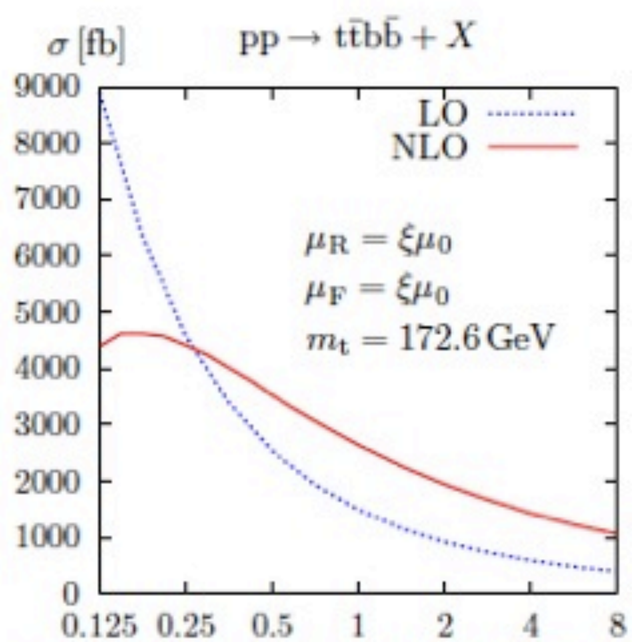
NLO calculation for six-leg process based on Feynman diagrams

NLO corrections for $pp \to b\bar{b}t\bar{t}$

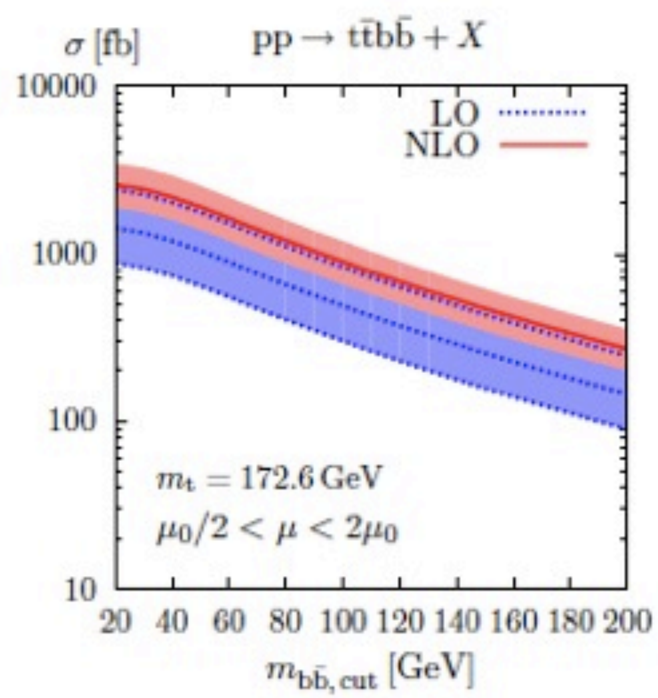


- $pp \to b\bar{b}t\bar{t}$ important background process for $pp \to t\bar{t}H$ at LHC
- form factor approach, numerical tensor reduction
- very efficient Feynman diagrammatic computation !
  $gg \to b\bar{b}t\bar{t}$ 160 ms on 3 GHz Intel Xeon processor

Bredenstein, Denner, Dittmaier, Pozzorini 2009



Bredenstein et.al. hep-ph:0905.0110



Generalized Unitarity Method

1) P-algorithm (exponential) for NLO virtual calculations
2) More suitable for automated implementations

III. The Unitarity Method: successful and promsing approach for calculating NLO corrections

Bern, Dixon, Kosower (1994-…)

gauge theory one-loop amplitudes from tree amplitudes

i) BDK theorem: SUSY gauge theories have no rational parts
applications to N=1,N=4 SYM also multi-loops
ii) Impressive QCD results: e.g. $e^+ + e^-$ annihilation to four jets in NLO (1998)

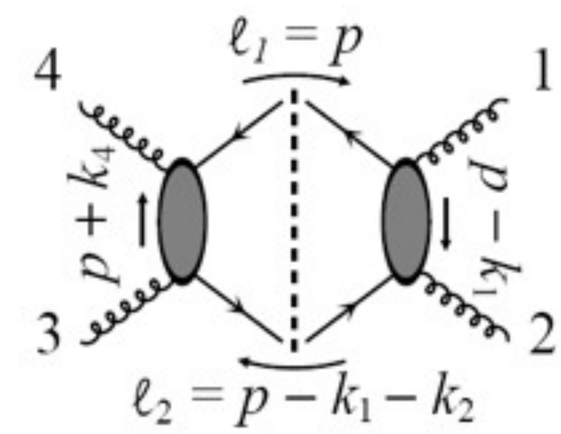
series of nifty tricks: analytic results, only four-dimensional states on cut lines, spinor helicity formalism, rational part is obtained from soft and collinear limits, triple cuts, SUSY identities etc.

DIFFICULTIES in QCD applications

i) Reduction of cut tensor integrals (Passarino-Veltman,Neerven-Vermaseren)
ii) The cut lines are treated in four dimensions (no rational parts)
iii) Only double cuts have been applied. Usefulness of triple cut.

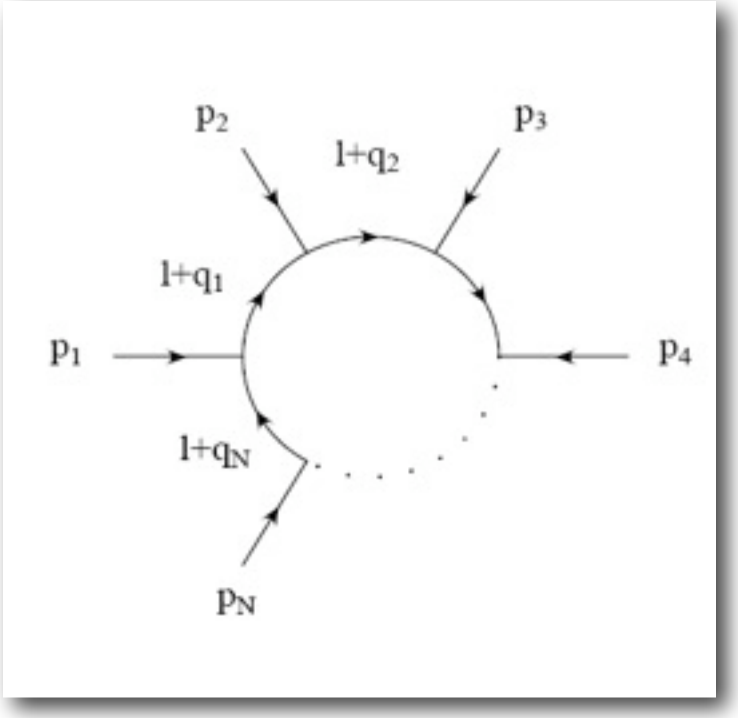
Constraints from Unitarity: $M^\dagger - M = -iM^\dagger M$

Imaginary part from tree amplitudes, iterative in coupling



$$-i\,\mathrm{Disc}\, A_4(1,2,3,4)\Big|_{s-\mathrm{cut}} = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\delta^{(+)}(\ell_1^2 - m^2)\, 2\pi\delta^{(+)}(\ell_2^2 - m^2) \times A_4^{\mathrm{tree}}(-\ell_1, 1, 2, \ell_2)\, A_4^{\mathrm{tree}}(-\ell_2, 3, 4, \ell_1)\,,$$

* factorization structure on the cuts (only double cuts)
* discontinuity given by tree amplitudes
* how to get the real part?
* how to get the coefficients efficiently



Insipiration from twistor formulations

i) Witten: Tree amplitudes with on-shell complex momenta

ii) Britto, Cachazo, Feng: Generalized unitarity , complex four momenta

OPP reduction and unitarity cut in D-dimensions

i) OPP: Ossola, Papadopoulos, Pittau ,2006
an alternative to Passarino-Veltman (1979) reduction

ii) Unitarity in D-dimension: Giele ZK Melnikov (2008)
full reconstruction of loop amplitudes from on-shell tree amplitudes
but complex momenta and in D=8,6 dimensions

Three-point amplitudes

On shell massless three point function is not well defined for real kinematics: all kinematical invariants s_{ij} vanish

Use spinor variables to take the kinematics complex

$$(\lambda_i)_\alpha \equiv [u_+(k_i)]_\alpha\,, \qquad (\tilde{\lambda}_i)_{\dot\alpha} \equiv [u_-(k_i)]_{\dot\alpha}\,, \qquad i = 1, 2, \ldots, n.$$

we introduce also bra and ket notations

$$\lambda_i \;=\; |i^+\rangle \;=\; \langle i^-|\,, \qquad \tilde{\lambda}_i \;=\; |i^-\rangle \;=\; \langle i^+|\,. \qquad k_i^\mu(\sigma_\mu)_{\alpha\dot\alpha} = (\lambda_i)_\alpha(\tilde{\lambda}_i)_{\dot\alpha}\,.$$

$$\langle j\, l\rangle = \varepsilon^{\alpha\beta}(\lambda_j)_\alpha(\lambda_l)_\beta = \bar{u}_-(k_j)u_+(k_l)\,,$$

$$[j\, l] = \varepsilon^{\dot\alpha\dot\beta}(\tilde{\lambda}_j)_{\dot\alpha}(\tilde{\lambda}_l)_{\dot\beta} = \bar{u}_+(k_j)u_-(k_l)\,.$$

$$\langle l\, j\rangle\,[j\, l] = 2k_j \cdot k_l = s_{jl}\,.$$

For real momenta, λ_i and $\tilde{\lambda}_i$ are complex conjugates of each other. Therefore the spinor products are complex square roots of the Lorentz products,

$$\langle j\, l \rangle = \sqrt{s_{jl}} e^{i\phi_{jl}}\,, \qquad [j\, l] = \pm\sqrt{s_{jl}} e^{-i\phi_{jl}}\,.$$

if all the s_{jl} vanish, then so do all the spinor products.

for complex momenta it is possible to choose all three left-handed spinors to be proportional,

$$\tilde{\lambda}_1 = c_1 \tilde{\lambda}_3 \quad \tilde{\lambda}_2 = c_2 \tilde{\lambda}_3$$

while the right-handed spinors are not proportional, but because of momentum conservation, $k_1 + k_2 + k_3 = 0$, they obey the relation,

$$c_1 \lambda_1 + c_2 \lambda_2 + \lambda_3 = 0$$

$$[1\, 2] = [2\, 3] = [3\, 1] = 0,$$

$\langle 1\, 2 \rangle$, $\langle 2\, 3 \rangle$ and $\langle 3\, 1 \rangle$ are all nonvanishing

$$A_3^{\text{tree}}(1^-,2^-,3^+) = \frac{i}{\sqrt{2}}\Big[\varepsilon_1^-\cdot\varepsilon_2^-\,\varepsilon_3^+\cdot(k_1-k_2)+\varepsilon_2^-\cdot\varepsilon_3^+\,\varepsilon_1^-\cdot(k_2-k_3)+\varepsilon_3^+\cdot\varepsilon_1^-\,\varepsilon_2^-\cdot(k_3-k_1)\Big]$$

where $$\varepsilon_i^{\pm,\mu} = \varepsilon^{\pm,\mu}(k_i,q_i) = \pm\frac{\langle q_i^\mp|\gamma^\mu|k_i^\mp\rangle}{\sqrt{2}\langle q_i^\mp|k_i^\pm\rangle}$$

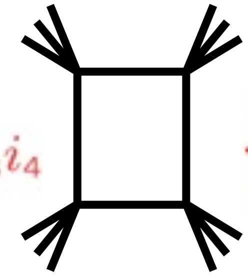
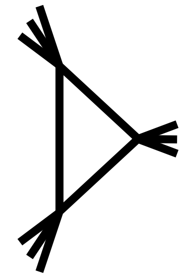
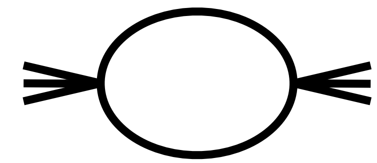
choose $q_2 = q_1$ and $q_3 = k_1$, then $\varepsilon_1^-\cdot\varepsilon_2^- = \varepsilon_3^+\cdot\varepsilon_1^- = 0$

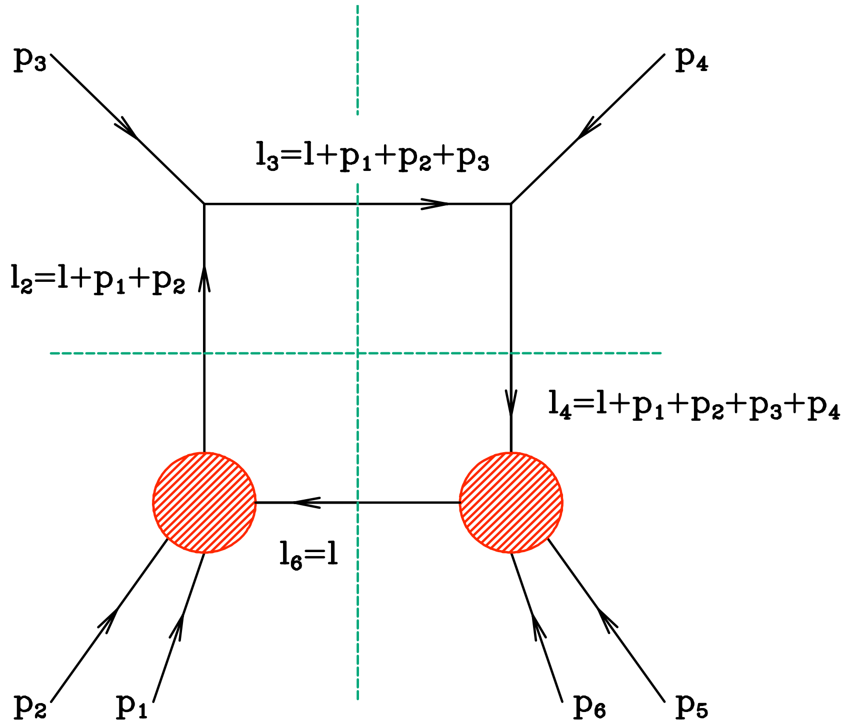
$$A_3^{\text{tree}}(1^-,2^-,3^+) = i\sqrt{2}\,\varepsilon_2^-\cdot\varepsilon_3^+\,\varepsilon_1^-\cdot k_2 = i\,\frac{[q_1\,3]\,\langle 1\,2\rangle}{[q_1\,2]\,\langle 1\,3\rangle}\,\frac{[q_1\,2]\,\langle 2\,1\rangle}{[q_1\,1]} = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

Park-Taylor formula for LHC amplitudes

$$A_n^{\text{tree MHV},\,jk} \equiv A_n^{\text{tree}}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

Generalized unitarity

$\mathcal{A}_N(\{p_i\}) = \sum d_{i_1i_2i_3i_4}$  $+ \sum c_{i_1i_2i_3}$  $+ \sum b_{i_1i_2}$  $+\mathcal{R}$



Quadrupole cut **$d_i=d_j=d_k=d_l=0$ (two solutions)**

$$l^\mu = V_4^\mu + \alpha_1\, n_1^\mu$$

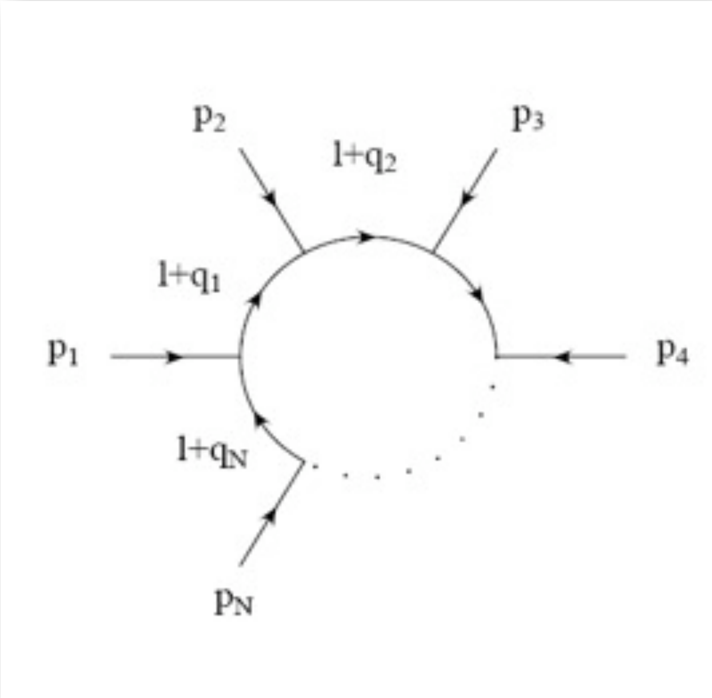
$$l_\pm^\mu = V_4^\mu \pm i\sqrt{V_4^2 - m_l^2} \times n_1^\mu$$

Complex valued loop momenta

OPP method to determine the coefficient of scalar integrals in D=4 dimension in terms of tree amplitudes

The unintegrated one-loop amplitude is linear combination of quadro-, triple-,double-,single-pole and polynomial terms

partial decomposition for the integrand



$$\mathcal{A}_N(p_1, p_2, \ldots, p_N; l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N} =$$

$$\sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1 \le i_1 < i_2 < i_3 \le N} \frac{\bar{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \le i_1 < i_2 \le N} \frac{\bar{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \le i_1 \le N} \frac{\bar{a}_{i_1}(l)}{d_{i_1}}$$

Parametrization of the numerators

$$\mathcal{A}_N(l) = \sum_{1\le i_1<i_2<i_3<i_4\le N} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1\le i_1<i_2<i_3\le N} \frac{\bar{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1\le i_1<i_2\le N} \frac{\bar{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1\le i_1\le N} \frac{\bar{a}_{i_1}(l)}{d_{i_1}}$$

parametric integral over the loop momentum

18 structures but only 3 non-vanishing integrals

$$\bar{d}_{ijkl}(l) \equiv \bar{d}_{ijkl}(n_1 \cdot l) = d_{ijkl} + \tilde{d}_{ijkl}\, s_1 \;, \quad s_i = n_i \cdot l$$

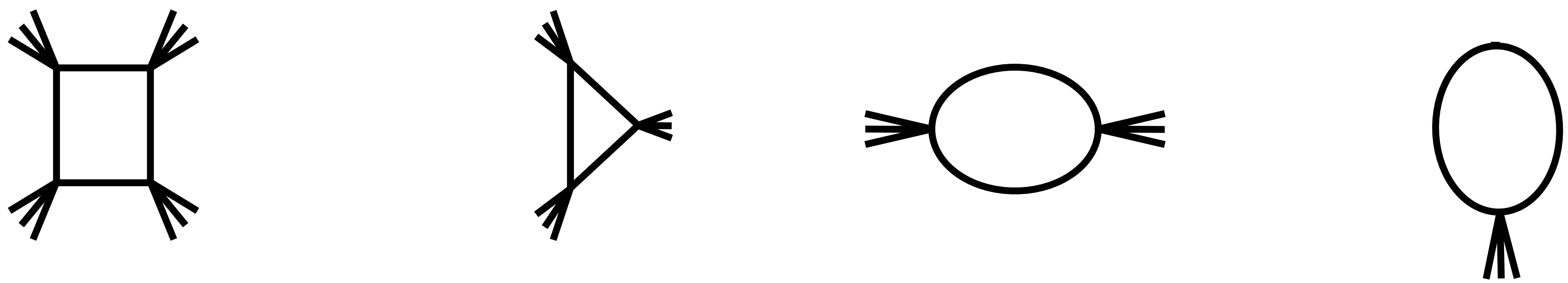
$$\bar{c}_{ijk}(l) = c^{(0)}_{ijk} + c^{(1)}_{ijk} s_1 + c^{(2)}_{ijk} s_2 + c^{(3)}_{ijk}(s_1^2 - s_2^2) + s_1 s_2 (c^{(4)}_{ijk} + c^{(5)}_{ijk} s_1 + c^{(6)}_{ijk} s_2)$$

$$\bar{b}_{ij}(l) = b^{(0)}_{ij} + b^{(1)}_{ij} s_1 + b^{(2)}_{ij} s_2 + b^{(3)}_{ij} s_3 + b^{(4)}_{ij}(s_1^2 - s_3^2) + b^{(5)}_{ij}(s_2^2 - s_3^2) + b^{(6)}_{ij} s_1 s_2 + b^{(7)}_{ij} s_1 s_3 + b^{(8)}_{ij} s_2 s_3$$

Carry out the integral over the loop momentum

$$\int [d\,l]\,\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} = \int [d\,l]\,\frac{d_{ijkl} + \tilde{d}_{ijkl}\,n_1\cdot l}{d_i d_j d_k d_l} = d_{ijkl}\int [d\,l]\,\frac{1}{d_i d_j d_k d_l} = d_{ijkl} I_{ijkl}\ ,$$

Scalar integrals



loop momenta on the cut $d_j = 0$

1. Quadrupole cut d_i=d_j=d_k=d_l=0 (two solutions)

$$l^\mu = V_4^\mu + \alpha_1 n_1^\mu$$

$$l_\pm^\mu = V_4^\mu \pm i\sqrt{V_4^2 - m_l^2} \times n_1^\mu$$

Complex valued loop momenta

2. Triple cut, infinite number of solutions (on a circle circle)

$$l^\mu = V_3^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu$$

$$l_{\alpha_1\alpha_2}^\mu = V_3^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu; \;\; \alpha_1^2 + \alpha_2^2 = -(V_3^2 - m_k^2)$$

3. Double cut, infinite number of solutions (on a “sphere”)

$$l^\mu = V_2^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu$$

$$l_{\alpha_1\alpha_2\alpha_3}^\mu = V_2^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu; \;\; \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = -(V_2^2 - m_j^2) \; .$$

The parameters are fixed by linear algebraic equations in terms of products of loop amplitudes

generalized unitarity: the residues are taken with (complex) "cut loop momenta"

$$\text{Res}_{ij\cdots k}\left[F(l)\right] \equiv \left[d_i(l)d_j(l)\cdots d_k(l)F(l)\right]\Big|_{l=l_{ij\cdots k}} \ .$$

$$\bar{d}_{ijkl}(l) = \text{Res}_{ijkl}\left(\mathcal{A}_N(l)\right)$$ $d_i=d_j=d_k=d_l=0$ two solutions

$$\bar{c}_{ijk}(l) = \text{Res}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k}\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$ $d_i=d_j=d_k=0$ infinite # of solutions

$$\bar{b}_{ij}(l) = \text{Res}_{ij}\left(\mathcal{A}_N(l) - \sum_{k\neq i,j}\frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{2!}\sum_{k,l\neq i,j}\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$ $d_i=d_j=0$ infinite # of solutions

unitarity: the residues factorize into the products of tree amplitudes

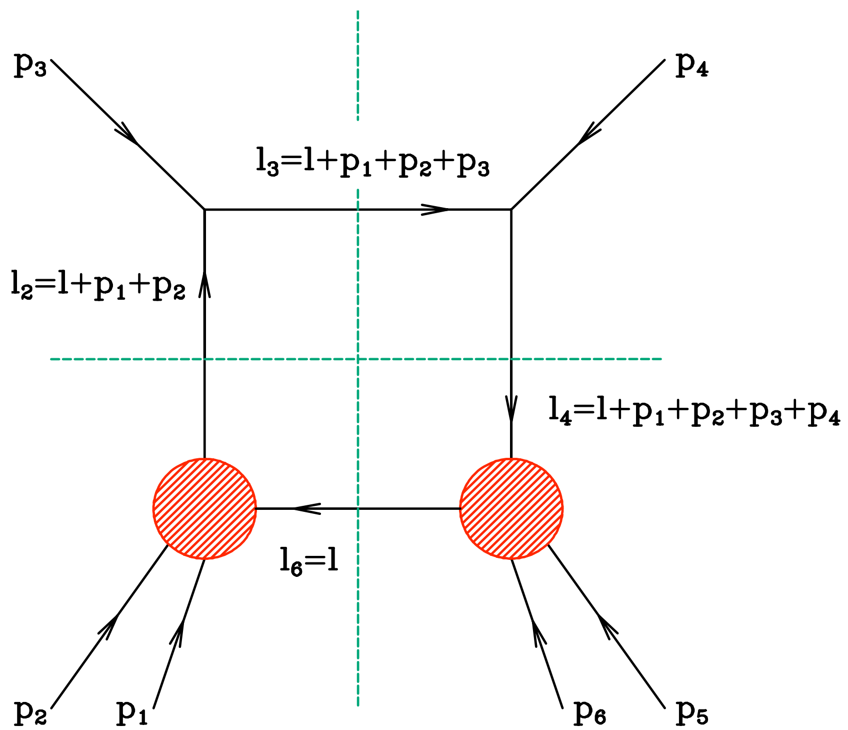
we fully reconstruct the integrand in terms of product of tree amplitudes

in combination with the s_j factors and denominator factors, no Feynman diagrams

The box residue

$$\text{Res}_{2346}\left(\mathcal{A}_6(l^{\pm})\right) = \mathcal{M}_4^{(0)}(l_6^{\pm}; p_1, p_2; -l_2^{\pm}) \times \mathcal{M}_3^{(0)}(l_2^{\pm}; p_3; -l_3^{\pm})\mathcal{M}_3^{(0)}(l_3^{\pm}; p_4; -l_4^{\pm})$$

$$\times\mathcal{M}_4^{(0)}(l_4^{\pm}; p_5, p_6; -l_6^{\pm}) \quad = \quad \overline{d}_{ijkl}(l) = d_{ijkl} + \tilde{d}_{ijkl}\, l \cdot n_1$$



$$d_{ijkl} = \frac{\text{Res}_{ijkl}\left(\mathcal{A}_N(l^+)\right) + \text{Res}_{ijkl}\left(\mathcal{A}_N(l^-)\right)}{2}$$

$$\tilde{d}_{ijkl} = \frac{\text{Res}_{ijkl}\left(\mathcal{A}_N(l^+)\right) - \text{Res}_{ijkl}\left(\mathcal{A}_N(l^-)\right)}{2i\sqrt{V_4^2 - m_l^2}}$$

Unitarity in D-dimension: uniform treatment of the cut constructible and rational parts (GKM)

Two sources of D-dependence

i) spin-polarization states live in D_s .

ii) loop momentum component live in D. (D_s>D)

$$\mathcal{A}_{(D,D_s)}(\{p_i\},\{J_i\}) = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{\mathcal{N}^{(D_s)}(\{p_i\},\{J_i\};l)}{d_1 d_2 \cdots d_N}.$$

We can calculate the D_s dependence before carrying out the integral over the loop momentum

$$\sum_{i=1}^{D_s-2} e_\mu^{(i)}(l) e_\nu^{(i)}(l) = -g_{\mu\nu}^{(D_s)} + \frac{l_\mu b_\nu + b_\mu l_\nu}{l \cdot b},$$

$$l^2 = \bar{l}^2 - \tilde{l}^2 = l_1^2 - l_2^2 - l_3^2 - l_4^2 - \sum_{i=5}^{D} l_i^2$$

Dirac spinors in 6 dimensions

gamma-matrices in $D_s = 4$ $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^5\}$

gamma-matrices in $D_s = 6$ $\Gamma^0 = \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix}, \quad \Gamma^{i=1,2,3} = \begin{pmatrix} \gamma^i & 0 \\ 0 & \gamma^i \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & \gamma^5 \\ -\gamma^5 & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix}$

$$u^{(s)}(l,m) = \frac{(l_\mu \Gamma^\mu + m)}{\sqrt{l_0 + m}} \eta^{(s)}_{D_s}, \qquad s = 1, \ldots, 2^{D_s/2-1} .$$

$$\eta_4^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_4^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\eta_6^{(1)} = \begin{pmatrix} \eta_4^{(1)} \\ \mathbf{0} \end{pmatrix}, \quad \eta_6^{(2)} = \begin{pmatrix} \eta_4^{(2)} \\ \mathbf{0} \end{pmatrix}, \quad \eta_6^{(3)} = \begin{pmatrix} \mathbf{0} \\ \eta_4^{(1)} \end{pmatrix}, \quad \eta_6^{(4)} = \begin{pmatrix} \mathbf{0} \\ \eta_4^{(2)} \end{pmatrix} .$$

conjugate spinors:

$$\bar{u}^{(s)}(l,m) = \bar{\eta}^{(s)}_{D_s} \frac{(l_\mu \Gamma^\mu + m)}{\sqrt{l_0 + m}}$$

l_μ is not conjugated

Two key features

Dependence on D_s is linear

$$\mathcal{N}^{(D_s)}(l) = \mathcal{N}_0(l) + (D_s - 4)\mathcal{N}_1(l)$$

full D_s dependence

- Choose two integer values $D_s = D_1$ and $D_s = D_2$ to reconstruct the full D_s dependence.
- Suitable for numerical implementation
- $D_s = 4 - 2\varepsilon$ 't Hooft Veltman scheme, $D_s = 4$ FDHS (Bern, Koswer)
- for closed fermion loops $\mathcal{N}^{D_s}(l) = 2^{(D_s-4)/2}\mathcal{N}_0(l)$

The loop momentum effectively has only 4+1 component

$$\mathcal{N}(l) = \mathcal{N}(\tilde{l}, \mu), \qquad l^2 = \tilde{l}^2 - \mu^2$$

maximum 5 unitarity constraints: pentagon cuts

Loop integrals are in $D < D_s$ dimensions $D = 4 - 2\epsilon$

OPP reduction is well defined for any integer D_s and D dimensions

- **We need to carry out the analytic continuation to $D = 4 - 2\epsilon$ only at the evaluation of the scalar integral functions.**

- **In D dimensions the loop momenta allow for**

  i) penta poles,
  ii) new structures in the numerators
  iii) four new non-vanishing integrals

$$\frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} = \sum_{[i_1|i_5]} \frac{\bar{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\bar{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}$$
$$+ \sum_{[i_1|i_3]} \frac{\bar{c}^{(D_s)}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\bar{b}^{(D_s)}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\bar{a}^{(D_s)}_{i_1}(l)}{d_{i_1}} .$$

New structures and new integrals

$$\bar{e}_{ijkmn}^{(D_s)}(l) = e_{ijkmn}^{(D_s,(0))}$$

no new scalar integrals

$$\bar{d}_{ijkn}^{\mathrm{FDH}}(l) = d_{ijkn}^{(0)} + d_{ijkn}^{(1)} s_1 + (d_{ijkn}^{(2)} + d_{ijkn}^{(3)} s_1) s_e^2 + d_{ijkn}^{(4)} s_e^4,$$

two new scalar integrals

$$\bar{c}_{ijk}^{\mathrm{FDH}}(l) = \ldots + c_{ijk}^{(7)} s_1 s_e^2 + c_{ijk}^{(8)} s_2 s_e^2 + c_{ijk}^{(9)} s_e^2,$$

one new scalar integrals

$$\bar{b}_{ij}^{\mathrm{FDH}}(l) = \ldots + b_{ij}^{(9)} s_e^2$$

one new scalar integrals

dependence

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}.$$

ϵ

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3 i_4} = 0,$$

$$\lim_{D\to 4} \frac{(D-4)(D-2)}{4} I^{(D+4)}_{i_1 i_2 i_3 i_4} = -\frac{1}{3},$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2 i_3} = \frac{1}{2},$$

$$\lim_{D\to 4} \frac{(D-4)}{2} I^{(D+2)}_{i_1 i_2} = -\frac{m_{i_1}^2 + m_{i_2}^2}{2} + \frac{1}{6} (q_{i_1} - q_{i_2})^2 .$$

One-loop amplitudes up to terms of order ε

One loop amplitudes as sum of cut-constructible and rational parts:

$$\mathcal{A}_N = \mathcal{A}_N^{CC} + R_N.$$

The cut constructible part is as before (EGK):

$$\mathcal{A}_N^{CC} = \sum_{[i_1|i_4]} \tilde{d}^{(0)}_{i_1 i_2 i_3 i_4}\, I^{(4-2\epsilon)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c^{(0)}_{i_1 i_2 i_3}\, I^{(4-2\epsilon)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b^{(0)}_{i_1 i_2}\, I^{(4-2\epsilon)}_{i_1 i_2} + \sum_{i_1=1}^{N} a^{(0)}_{i_1}\, I^{(4-2\epsilon)}_{i_1},$$

The rational part is new (GKM):

$$R_N = -\sum_{[i_1|i_4]} \frac{d^{(4)}_{i_1 i_2 i_3 i_4}}{6} + \sum_{[i_1|i_3]} \frac{c^{(7)}_{i_1 i_2 i_3}}{2} - \sum_{[i_1|i_2]} \left(\frac{(q_{i_1} - q_{i_2})^2}{6} - \frac{m^2_{i_1} + m^2_{i_2}}{2} \right) b^{(9)}_{i_1 i_2},$$

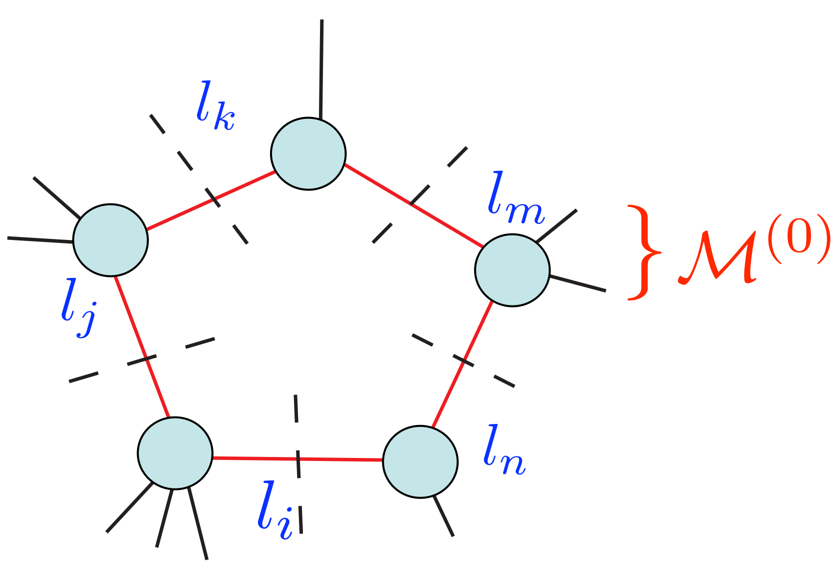
The residues are sum over the products of tree amplitudes in D=6 and 8 dimensions

$$\bar{e}^{(D_s)}_{i_1\cdots i_5}(\ell) = \mathrm{Res}_{i_1\cdots i_5}\left[\mathcal{A}_N^{(D_s)}(\ell)\right] \equiv d_{i_1}(\ell)\cdots d_{i_5}(\ell)\,\mathcal{A}_N^{(D_s)}(\ell)\Big|_{d_{i_1}(\ell)=\cdots=d_{i_5}(\ell)=0}$$

the residues are products of tree amplitudes of D_s dimensions with complex on-shell D=5 loop momenta l summed over helicities

$$\mathrm{Res}_{i_1\cdots i_M}\left[\mathcal{A}_N^{(D_s)}(\ell)\right] = \sum_{\{\lambda_1,\ldots,\lambda_M\}=1}^{D_s-2}\left\{\prod_{k=1}^{M}\mathcal{M}^{(0)}\left(\ell_{i_k}^{(\lambda_k)}; p_{i_k+1},\ldots,p_{i_{k+1}}; -\ell_{i_{k+1}}^{(\lambda_{k+1})}\right)\right\}$$

sum is over internal polarization states



$$\ell_{i_k} = \ell + q_{i_k} - q_{i_M}$$

Example for OPP reduction: the tensor bubble

$$\mathcal{A}_{(\mathcal{D})}(q,\mu,\nu) = I_2^{D,\mu\nu} = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{l^\mu l^\nu}{d_1 d_2}, \quad \text{where} \quad d_1 = l^2, \quad d_2 = (l+q)^2$$

the integrand

$$A_{(D)}(q,\mu,\nu;\, l) = \frac{l^\mu l^\nu}{d_1 d_2} = \frac{b(l)}{d_1 d_2}$$

general parameterization of a double cut is given by

$$\begin{aligned} \bar{b}^{\mu\nu}(l) &= b_0^{\mu\nu} + b_1^{\mu\nu} s_1 + b_2^{\mu\nu} s_2 + b_3^{\mu\nu} s_3 + b_4^{\mu\nu}(s_1^2 - s_3^2) + b_5^{\mu\nu}(s_2^2 - s_3^2) + b_8^{\mu\nu} s_2 s_3 \\ &+ b_6^{\mu\nu} s_1 s_2 + b_7^{\mu\nu} s_1 s_3 + b_9^{\mu\nu} s_e^2, \end{aligned}$$

$s_1 = l \cdot n_1$, $s_2 = l \cdot n_2$ and $s_3 = l \cdot n_3$

$$l^\mu = -\frac{1}{2}q^\mu + \sum_{i=1}^{3} s_i n_i^\mu + s_e n_e^\mu$$

note that with this parametrization if $l^2 = (l+q)^2 = 0$

the sum of the diagonal terms is not independent

$$s_1^2 + s_2^2 + s_3^2 = w = -s_e^2 - \frac{1}{4}q^2$$

$$l^\mu l^\nu = \frac{1}{4}q^\mu q^\nu - \frac{1}{2}\sum_{i=1}^{3} s_i(q^\mu n_i^\nu + q^\nu n_i^\mu) + \ldots + \sum_{i=1}^{3} n_i^\mu n_i^\nu s_i^2 + n_e^\mu n_e^\nu s_e^2$$

We can read out analytically

$$b_0^{\mu\nu} = \frac{1}{4} q^\mu q^\nu - \frac{q^2}{12} W_4^{\mu\nu} \qquad b_9^{\mu\nu} = -\frac{1}{3} W_4^{\mu\nu}.$$

OPP reduction gives

$$\int \frac{d^D l}{i(\pi)^{D/2}} \frac{l^\mu l^\nu}{d_1 d_2} = b_0^{\mu\nu} I_2^{(D)} + b_9^{\mu\nu} \frac{q^2}{6}$$

Passarino Veltman reduction gives

$$\int \frac{d^D l}{i(\pi)^{D/2}} \frac{l^\mu l^\nu}{d_1 d_2} = \frac{1}{4} q^\mu q^\nu I_2^{(D)} - \frac{1}{4} \frac{q^2}{D-1} W^{\mu\nu} I_2^{(D)}$$

We can read out the coefficients also numerically by solving the linear equations