Lattice QCD and Flavour

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Indirect Searches for New Physics at the time of the LHC
GGI, Florence, March 23rd 2010
1. Introduction

- There has been a huge improvement in the precision of lattice calculations in the last 3 years or so.
- There are a number of groups focussing on different aspects on flavour physics.
- I will talk about progress in kaon physics, particularly from the RBC-UKQCD collaboration using Domain Wall Fermions (set in context).
  - RBC=RIKEN, Brookhaven National Laboratory, Columbia University.
  - UKQCD in this project = Edinburgh and Southampton Universities.
  - We coordinate the generation of (expensive) ensembles and work in subgroups on a wide variety of physics topics.
  - The 2008 paper describing our old ensembles had 33 authors and we are preparing the analogous paper for our new ensembles.
- A set of references is found at the end of the talk.
Plan of the Talk

1. Introduction
2. Determination of $V_{us}$
   2.i $f_K/f_\pi$.
   2.ii $K_{\ell 3}$ decays.
3. $B_K$
4. $\eta$ and $\eta'$ mesons and mixing.
5. $K \rightarrow \pi\pi$ Decays
6. Conclusions and Prospects
We use two datasets of DWF with the Iwasaki Gauge Action with a lattice spacing of about 0.114 fm:

- $24^3 \times 64 \times 16$ ($L \simeq 2.74$ fm)
- $(16^3 \times 32 \times 16)$ ($L \simeq 1.83$ fm)

On the $24^3$ lattice measurements have been made with 4 values of the light-quark mass:

- $m_\alpha = 0.03$ ($m_\pi \simeq 670$ MeV);
- $m_\alpha = 0.02$ ($m_\pi \simeq 555$ MeV);
- $m_\alpha = 0.01$ ($m_\pi \simeq 415$ MeV);
- $m_\alpha = 0.005$ ($m_\pi \simeq 330$ MeV).

(Using partial quenching the lightest pion in our analysis has a mass of about 240 MeV.)

On the $16^3$ lattice results were obtained with $m_\alpha = 0.03$, 0.02 and 0.01.

For the (sea) strange quark we take $m_\alpha = 0.04$, although a posteriori we see that this is a little too large.

We are completing the analysis of an ensemble on a $32^3 \times 64 \times 16$ lattice with $a \simeq 0.081$ fm ($L \simeq 2.6$ fm) with three dynamical masses ($m_\pi \simeq 310$, 365 and 420 MeV).

This will enable us to reduce the discretization errors significantly.

Some preliminary results were presented at Lattice 2008, 2009 and elsewhere.
Imagine an idealized situation where simulations are possible at all quark masses for a variety of $\beta$s ($\beta = \beta_i, i = 1, 2, \cdots, N$). We can choose to fix $m_{ud}(\beta_i), m_s(\beta_i)$ and $a(\beta_i)$ by requiring that 3 physical quantities take their physical values. This defines a *Scaling Trajectory*.

- We use $m_\pi, m_K$ and $m_\Omega$.

We can then calculate other physical quantities ($f_\pi(\beta_i), B_K(\beta_i), \cdots$). These will have lattice artefacts of $O(a_i^2\Lambda_{\text{QCD}}^2)$ and we imagine extrapolating the results to the continuum limit.

At present however, we have to extrapolate to the physical values of $m_{ud}$ (and interpolate to $m_s$). We have invested considerable effort in defining and performing global fits in which we keep physical Low Energy Constants at all (both) $\beta_i$ and yet treat the artefacts consistently. ALMOST DONE.

$$O(m_\pi^2/\Lambda_{\chi}^2), O(a^2\Lambda_{\text{QCD}}^2) \sqrt{\cdot}, O((m_\pi/\Lambda_{\chi})^4), O(a^2m_\pi^2), O((a\Lambda_{\text{QCD}})^4) \cdots \times.$$  

- We use other ansatz also.
Lattice Issues

- Topology Changing.
  - Although the algorithms used in the generation of field ensembles are formally ergodic, in a finite simulation it may be that the space of field configurations has not been fully sampled.
  - Procedures for calculating autocorrelations exist, but can not be 100% reliable.
  - It has recently been stressed that for fine lattices \((a \lesssim 0.04 \text{ fm})\), the topological charge does not change (for the actions generally used).

There is a large amount of algorithmic work being devoted to overcome this problem.

Step Scaling

- Although the idea of step-scaling and the *femto universe* have been advocated for a long time by the Alpha collaborations, up to recently they have only been used by a small number of groups.
- Improved precision in the calculation of physical quantities \(\Rightarrow\) this is becoming a more widely used technique (*B*-physics, Non-perturbative renormalization etc.)
- Match lattices at different \(\beta\) until we end up with a very fine, but small, lattice where connection with continuum QCD can be made reliably.

Zeuthen and CERN groups, \(\cdots\).

Alpha Collaboration.
- **Reweighting**

  Although we can simulate at $m_s^{\text{phys}}$, we only know its value a posteriori. We therefore have to estimate what $m_s$ is before performing the simulations.

  Imagine that we wish to compute (Dirac operator $D_q = D[U,m_q]$)

  \[
  \langle O \rangle_2 = \frac{\int d[U] e^{-S_g} \sqrt{\det(D_2^\dagger D_2)} O(U)}{\int d[U] e^{-S_g} \sqrt{\det(D_2^\dagger D_2)}}
  \]

  Imagine also that we performed the simulation with mass $m_1$. Now

  \[
  \langle O \rangle_2 = \frac{\int d[U] e^{-S_g} \sqrt{\det(D_1^\dagger D_1)} O(U) w(U)}{\int d[U] e^{-S_g} \sqrt{\det(D_1^\dagger D_1)} w(U)}
  \]

  where

  \[
  w[U] = \det \left( \frac{D_2^\dagger[U]D_2[U]}{D_1^\dagger[U]D_1[U]} \right)^{1/2} \equiv \det^{-1/2}(\Omega) = \left( \frac{\int D\xi e^{-\xi^\dagger \sqrt{\Omega[U]\xi}}}{\int D\xi e^{-\xi^\dagger \xi}} \right).
  \]

  Jointly sampling $U$ and $\xi$ fields $\Rightarrow \langle O \rangle_2$.

  One (small) systematic error removed.
2. $V_{us} - f_K/f_\pi$ FLAG Compendium – Preliminary

- All groups calculate $f_K/f_\pi$.

![Graph showing $f_K/f_\pi$ values for different groups with their respective $N_f$ values.]

- Flag Compendium – Preliminary:
  - $f_K/f_\pi = 1.190(2)(10)$ – Direct $N_f = 2 + 1$;
The calculation requires a reliable chiral extrapolation.

\[ \Rightarrow \quad \text{SU}(2) \text{ ChPT.} \]

Is the chiral extrapolation as well under control for all quantities as we think?

Very soon, as the simulated masses $\rightarrow m_{\pi}^{\text{phys}}$ the chiral extrapolation will be a smaller concern.
Comparison of Results obtained using SU(2) and SU(3) ChPT

Study is performed at NLO in the chiral expansion.

- black points - partially quenched results with $am_l = 0.01$ ($m_{\pi}^{\text{unitary}} \simeq 420\text{ MeV}$).
- red points - partially quenched results with $am_l = 0.005$ ($m_{\pi}^{\text{unitary}} \simeq 330\text{ MeV}$).

We find:

$$f_{\pi}/f \simeq 1.08, \quad f/f_0 = 1.23(6).$$

The corresponding results from the MILC collaboration, who do an NNLO analysis (partly in staggered chiral perturbation theory), with NNNLO analytic terms:

$$f_{\pi}/f = 1.052(2) \left(\frac{+6}{-3}\right), \quad f/f_0 \text{ MILC} = 1.15(5) \left(\frac{+13}{-3}\right),$$

The large value of $f_{\pi}/f_0$ (and even larger values of $f_{PS}/f_0$ of $\sim 1.6$ where we have data) lead RBC/UKQCD (and ETMC) to present results based on SU(2)×SU(2) ChPT.
$K\ell_3$ Decays

\[ \langle \pi(p_\pi) | \bar{s} \gamma_\mu u | K(p_K) \rangle = f_0(q^2) \frac{M_K^2 - M_\pi^2}{q^2} q_\mu + f_+(q^2) \left[ (p_\pi + p_K)_\mu - \frac{M_K^2 - M_\pi^2}{q^2} q_\mu \right] \]

where $q \equiv p_K - p_\pi$.

To be useful in extracting $V_{us}$ we require $f_0(0) = f_+(0)$ to better than about 1% precision.

\[ \chi\text{PT} \Rightarrow f_+(0) = 1 + f_2 + f_4 + \cdots \quad \text{where} \quad f_n = O(M_K^n, \pi, \eta). \]

Reference value $f_+(0) = 0.961 \pm 0.008$ where $f_2 = -0.023$ is relatively well known from $\chi\text{PT}$ and $f_4, f_6, \cdots$ are obtained from models.  

Leutwyler & Roos (1984)
Our final result from the $K_{\ell 3}$ project is

\[ f^+_{K\pi}(0) = 0.964(5). \]

P.A. Boyle et al. [RBC&UKQCD Collaborations – arXiv:0710.5136 [hep-lat]]
\[ V_{ud} = 0.97372(10)(15)(19) \]

W. Marciano, Kaon2007

\[ V_{ud} = 0.97424(23) \]

I. Towner and J. Hardy, CKM(2008)

The uncertainties on \(|V_{ud}|^2\) and \(|V_{us}|^2\) are comparable!
\[ f_+(0) \]

- \( N_f = 2+1 \) RBC/UKQCD 08
- \( N_f = 2 \) ETM 08
- \( N_f = 2 \) QCDSF 07
- \( N_f = 2 \) RBC 06
- \( N_f = 2 \) JLQCD 05
- \( N_f = 0 \) SPQCD 05

- our estimate

- nuclear \( \beta \) decay
- semi-inclusive \( \tau \) decay

- \( N_f = 2 \) RBC-UKQCD and ETM to lighter masses.

Chris Sachrajda (UKQCD/RBC Collaboration)
Florence, 23/3/2010
We are now able to calculate the form-factor directly at \( q^2 = 0 \) (using twisted boundary conditions).

For example for the 330 MeV pion:

\[
\begin{align*}
    f_{K\pi}^\text{pole}(0) &= 0.9774(35); \\
    f_{K\pi}^\text{polynomial}(0) &= 0.9749(59); \\
    f_{K\pi}(0)^\text{TBC} &= 0.9757(44).
\end{align*}
\]

An important source of systematic error has been eliminated.
Where we have data the results are robust.

The principal uncertainty is in the chiral extrapolation.

For example, what value should we take for $f$ in

$$f_2 = \frac{3}{2} H_{\pi K} + \frac{3}{2} H_{\eta K}; \quad H_{PQ} = - \frac{1}{64 \pi^2 f^2} \left[ M_P^2 + M_Q^2 + \frac{2M_P^2 M_Q^2}{M_P^2 - M_Q^2} \log \frac{M_Q^2}{M_P^2} \right]$$

Examples (all of which fit the lattice data well):

$$f = 100, 115, 131.5 \text{ MeV} \Rightarrow f^K_+(0) = 0.9556, 0.9599, 0.9631 \text{ respectively}.$$
The emphasis must now be to reduce the error due to the chiral extrapolation.
Lattice simulations are being performed at lighter masses.
Need theoretical guidance in optimizing the chiral extrapolation.

**Hard Pion SU(2) Chiral Perturbation Theory:**

\[ f_0(0) = f_+(0) = F_+ \left( 1 - \frac{3}{4} \frac{m_\pi^2}{16\pi^2 f^2} \log \left( \frac{m_\pi^2}{\mu^2} \right) + c_+ m_\pi^2 \right) \]

\[ f_-(0) = F_- \left( 1 - \frac{3}{4} \frac{m_\pi^2}{16\pi^2 f^2} \log \left( \frac{m_\pi^2}{\mu^2} \right) + c_- m_\pi^2 \right). \]

It would be useful to know the result at NNLO.
3. $B_K$

- Flavour and Chiral symmetry properties of DWF well suited to this calculation.
- $\Delta S = 2$ operator renormalizes multiplicatively and is renormalized nonperturbatively.

Our published results are

$$B_K^{\overline{MS}}(2 \text{ GeV}) = 0.524(10)(28) \quad (\hat{B}_K = 0.720(13)(37)).$$

The largest component of the uncertainty is due to the single lattice spacing.

- Analysis with a second $a$ and continuum extrapolation almost ready (v18 of draft).
- Aubin, Laiho, Van de Water, $\hat{B}_K = 0.724(8)(28)$, (DWF/Staggered Mixed Action) arXiv:0905.3947

- Other groups have preliminary results.
We are almost completed the full analysis of $B_K$ on our finer lattice and hence to be able to compute the continuum extrapolation.

We are currently repeating the procedure for all the possible dimension 6 $\Delta S = 2$ operators which contribute in extensions of the standard model.

We have been generalizing the Rome-Southampton Non-Perturbative Renormalization method (RI-MOM) to non-exceptional momenta.


\[
p_1^2 = p_2^2 = (p_1 - p_2)^2
\]
Evidence for small chiral symmetry breaking

\[ \Lambda_A - \Lambda_V. \]

\[ \Lambda_S \text{ and } \Lambda_P. \]

We have also renormalized \( O^{\Delta S=2} \) using non-exceptional momentum configurations.

Y. Aoki arXiv:0901.2595 [hep-lat]
To study $\eta$ and $\eta'$ we need to evaluate disconnected diagrams.

Here $l$ represents the $u$ or $d$ quark ($m_u = m_d$) and $s$ the strange quark.

For disconnected diagrams the needed exponential decrease in $t$ comes from increasingly large statistical cancelations implying a rapidly vanishing signal-to-noise ratio.
\( \eta \) and \( \eta' \) Mesons

- Let

\[
O_l = \frac{\bar{u} \gamma_5 u + \bar{d} \gamma_5 d}{\sqrt{2}} \quad \text{and} \quad O_s = \bar{s} \gamma_5 s.
\]

- We calculate the correlation functions

\[
X_{\alpha\beta}(t) = \frac{1}{32} \sum_{t'=0}^{3} 1 \left\langle O_{\alpha}(t+t') O_{\beta}(t') \right\rangle \quad \text{where} \quad \alpha, \beta = l, s.
\]

- Sources are generated for each time slice (T=32).
- \( X_{ls} \neq 0 \) because of the \( D_{ls} = D_{sl} \) diagrams.

- The four correlation functions correspond to the diagrams as follows:

\[
\begin{pmatrix}
X_{ll} & X_{ls} \\
X_{sl} & X_{ss}
\end{pmatrix}
= \begin{pmatrix}
C_{ll} - 2D_{ll} & -\sqrt{2}D_{ls} \\
-\sqrt{2}D_{sl} & C_{ss} - D_{ss}
\end{pmatrix}.
\]

- The usual expectation that disconnected diagrams and the resulting mixing are small does not apply here.
We diagonalize $X(t)$ at each $t$:

$$X(t) = A^T \begin{pmatrix} e^{-m_\eta t} & 0 \\ 0 & e^{-m_{\eta'} t} \end{pmatrix} A,$$

where

$$A = \begin{pmatrix} \langle \eta | O_l | 0 \rangle & \langle \eta | O_s | 0 \rangle \\ \langle \eta' | O_l | 0 \rangle & \langle \eta' | O_s | 0 \rangle \end{pmatrix}.$$ 

To be more precise we diagonalize $X(t_0)^{-1} X(t)$. 

Lüscher and Wolff (1990)
\( \eta - \eta' \text{ mixing} \)

- In the standard phenomenological treatment of \( \eta - \eta' \) mixing

\[
\begin{pmatrix}
|\eta\rangle \\
|\eta'\rangle
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
|8\rangle_{\text{sym}} \\
|1\rangle_{\text{sym}}
\end{pmatrix}
\]

- In the \( O_8 \) and \( O_1 \) basis

\[
A = \begin{pmatrix}
\sqrt{Z_8} \cos \theta & -\sqrt{Z_1} \sin \theta \\
\sqrt{Z_8} \sin \theta & \sqrt{Z_1} \cos \theta
\end{pmatrix}
\]

where \( \text{sym} \langle a | O_b | 0 \rangle = \sqrt{Z_a} \delta_{ab} \).

- If this model is correct then the columns of \( A \) are orthogonal. We find for the dot product \(-0.009(49)\) for \( m_l = 0.01 \) and \( 0.008(24) \) for \( m_l = 0.02 \).

- The mixing angle can be determined from

\[
\frac{A_{\eta 1}A_{\eta'8}}{A_{\eta 8}A_{\eta'1}} = -\tan^2 \theta.
\]
We find $m_\eta = 583(15)$ MeV and $m_{\eta'} = 853(123)$ MeV and $\theta = -9.2(4.7)^\circ$. (Statistical errors only.)

To our accuracy, our calculation demonstrates that QCD can explain the relatively large mass of the ninth pseudoscalar meson and its small mixing with the SU(3) octet state.

There is plenty more to do!
5. $K \rightarrow \pi\pi$ decay amplitudes from $K \rightarrow \pi$ Matrix Elements

- At lowest order in the SU(3) chiral expansion one can obtain the $K \rightarrow \pi\pi$ decay amplitude by calculating $K \rightarrow \pi$ and $K \rightarrow \text{vacuum}$ matrix elements.
- In 2001, two collaborations published some very interesting (quenched) results on non-leptonic kaon decays in general and on the $\Delta I = 1/2$ rule and $\varepsilon'/\varepsilon$ in particular:

<table>
<thead>
<tr>
<th>Collaboration(s)</th>
<th>$\text{Re } A_0/\text{Re } A_2$</th>
<th>$\varepsilon'/\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RBC</td>
<td>$25.3 \pm 1.8$</td>
<td>$-(4.0 \pm 2.3) \times 10^{-4}$</td>
</tr>
<tr>
<td>CP-PACS</td>
<td>$9 \div 12$</td>
<td>$-(-7 \div -2) \times 10^{-4}$</td>
</tr>
<tr>
<td>Experiments</td>
<td>$22.2$</td>
<td>$(17.2 \pm 1.8) \times 10^{-4}$</td>
</tr>
</tbody>
</table>

- This required the control of the ultraviolet problem, the subtraction of power divergences and renormalization of the operators – highly non-trivial.
  - Four-quark operators mix, for example, with two quark operators $\Rightarrow$ power divergences:
Re $A_0/Re A_2$ as a function of the meson mass.

$\omega^{-1}$ = $Re A_0/Re A_2$

$\epsilon'/\epsilon$ as a function of the meson mass.

The RBC and CP-PACS simulations were quenched, and relied on the validity of lowest order $\chi$PT in the region of approximately 400-800 MeV.

Given the cancellations between different matrix elements (particularly $O_6$ and $O_8$) the negative value of $\epsilon'/\epsilon$ is not such an embarrassment but

Must do better!
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**Must do better!**
RBC/(UKQCD) have repeated the calculation with the $24^3$ DWF ensembles in the pion-mass range 240-415 MeV.

For illustration consider the determination of $\alpha_{27}$, the LO LEC for the $(27,1)$ operator. Satisfactory fits were obtained, but again the corrections were found to be huge, casting serious doubt on the approach.

Soft pion theorems are not sufficiently reliable $\Rightarrow$ need to compute $K \to \pi\pi$ matrix elements.

To arrive at this important conclusion required a major effort.

$$O^{3/2}_{(27,1)} = (\bar{s}d)_L \left\{ (\bar{u}u)_L - (\bar{d}d)_L \right\} + (\bar{s}u)_L (\bar{u}d)_L$$
To make progress we need to be able to calculate \( K \to \pi\pi \) matrix elements directly and the RBC/UKQCD Collaboration is undertaking a major study. T. Blum, P. Boyle, D. Broemmel, J. Flynn, E. Goode, T. Izubuchi, C. Kim, M. Lightman, Qi Liu, R. Mawhinney, N. Christ, C. Sachrajda, A. Soni.

The main theoretical ingredients of the \emph{infrared} problem with two-pions in the s-wave are now understood.

Two-pion quantization condition in a finite-volume

\[
\delta(q^*) + \phi^P(q^*) = n\pi,
\]

where \( E^2 = 4(m^2_\pi + q^*2) \), \( \delta \) is the s-wave \( \pi\pi \) phase shift and \( \phi^P \) is a kinematic function. M. Lüscher, 1986, 1991, \ldots.

The relation between the physical \( K \to \pi\pi \) amplitude \( A \) and the finite-volume matrix element \( M \)

\[
|A|^2 = 8\pi V^2 \frac{m_K E^2}{q^*2} \left\{ \delta'(q^*) + \phi'^P(q^*) \right\} |M|^2,
\]

where \( \prime \) denotes differentiation w.r.t. \( q^* \).

Use the Wigner-Eckart Theorem to relate the physical $K \to \pi^+ \pi^0$ matrix element to that for $K \to \pi^+ \pi^+$

$$I=2 \langle \pi^+(p_1)\pi^0(p_2) | O^{3/2} | K^+ \rangle = \frac{3}{2} \langle \pi^+(p_1)\pi^+(p_2) | O^{3/2} | K^+ \rangle,$$

Calculate the $K \to \pi^+ \pi^+$ matrix element with the $u$-quark with twisted boundary conditions with twisting angle $\theta$.

Perform a Fourier transform of one of the pion interpolating operators with additional momentum $-2\pi/L$.

The ground state now corresponds to one pion with momentum $\theta/L$ and the other with momentum $(\theta - 2\pi)/L$.

The corresponding $\pi\pi$ s-wave phase-shift can then be obtained by the Lüscher formula as a function of $\theta \Rightarrow$ this allows for the derivative of the phase-shift to be evaluated directly at the masses being simulated.

We have carried this procedure out in an exploratory calculation. Fig

Unfortunately this technique does not work for $K \to (\pi\pi)_{I=0}$ decays.
Exploratory Evaluation of the Lellouch-Lüscher Factor

C.h. Kim and CTS, arXiv:1003.3191

LL factor

\[ a(q^*, \phi^+) \]

\[ q^* \]

\[ q^* \]

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Unfortunately this technique does not work for $K \rightarrow (\pi\pi)_{I=0}$ decays.
We are starting a major project to calculate the $\Delta I = 3/2$ $K \to \pi\pi$ Decay Amplitudes. There are no significant obstacles to completing this.

- An exploratory quenched study with improved Wilson fermions was completed in 2004 but at the time we did not understand the Finite-Volume corrections at non-zero total momentum.

  P. Boucaud, V. Gimenez, C. J. D. Lin, V. Lubicz, G. Martinelli, M. Papinutto and C. T. Sachrajda,

- The first results of an exploratory quenched study with Domain Wall Fermions were presented at Lattice 2009.

  M.Lightman and E.J.Goode, arXiv:0912.1667

Novel features included:

- using the Wigner-Eckart Theorem:

  \[ I=2 \langle \pi^+(p_1)\pi^0(p_2) | O^{3/2} | K^+ \rangle = \frac{3}{2} \langle \pi^+(p_1)\pi^+(p_2) | O'^{3/2} | K^+ \rangle , \]

  where $O'^{3/2}$ has the flavour structure $\bar{s}d\bar{u}d$.

- using antiperiodic boundary conditions so that the final state is

  \[ \langle \pi^+(\pi/L)\pi^+( -\pi/L) | . \]

  C-h Kim, Ph.D. Thesis
We have been using an exploratory quenched study to learn about suitable parameters for the main simulation.

The plots show the matrix elements as a function of the $t$ for the insertion of the operator. $t_{\pi\pi} = 0$, $t_K = 24$.
For $I=2$ $\pi\pi$ states the correlation function is proportional to $D-C$.

We are also exploring whether it will be feasible to compute the $\Delta I = 1/2\ K \rightarrow \pi\pi$ Decay Amplitudes.

For $I=0$ $\pi\pi$ states the correlation function is proportional to $2D+C-6R+3V$.

The major practical difficulty is to subtract the vacuum contribution with sufficient precision.
Two-pion Correlation Functions (Cont.)

- **RBC/UKQCD, Preliminary, Qi Liu et al. arXiv:0910.2658**

- **$I = 2$ (Correlator and Effective Mass)**

- **$I = 0$ (Correlator and Effective Mass)**
6. Conclusions and Prospects

- Huge recent improvement in reliability and precision of lattice computations of quantities relevant for flavour physics.
- As $m_\pi \rightarrow m_\pi^{\text{phys}}$, the chiral extrapolation becomes less of a problem.
  - LECs of Chiral Pert. Th. being computed with unprecedented precision.
  - (I am not convinced that the current representation of lattice data by NNLO/models is fully under control yet!)
- Future:
  - Improve precision still further.
  - Extend the physics reach of the computations.
  
  Discussion with wider flavour community needed here.
- Other speakers would have focussed on different important topics, e.g.:
  - Alpha Collaboration: HQET at $O(1/m)$ using NPR and step-scaling.
  - HPQCD: Large range of B-physics with NRQCD and charm physics using highly improved actions.
  - FNAL, CP-PACS, RBC-UKQCD - Symanzik-improvement based approach.
  - However, I am very much of the opinion that power divergences must be subtracted non-perturbatively.
  - We still don’t know how to study $B \rightarrow M_1M_2$ decays, even in principle.

Maiani, Martinelli, CTS (1992)
The precision of lattice calculations is now reaching the point where we need good interactions with the $N^n LO$ QCD perturbation theory community.

The traditional way of dividing responsibilities is:

\[
\text{Physics} = C \times \langle f \mid O \mid i \rangle \\
\uparrow \quad \uparrow \\
\text{Perturbative} \quad \text{Lattice} \\
\text{QCD} \quad \text{QCD}
\]

The two factors have to be calculated in the same scheme.

Can we meet half way?

bare operators

lattice → ? ← renormalized operators

in $\overline{\text{MS}}$ scheme

What is the best scheme for ? (RI-SMOM, Schrödinger Functional, ...)?
1. Physical Results from 2+1 Flavor Domain Wall QCD and SU(2) Chiral Perturbation Theory,
   C. Allton et al., (32 Authors, 133 pages)

2. $K_\ell 3$ semileptonic form factor from 2+1 flavour lattice QCD,
   P. A. Boyle, A. Jüttner, R.D. Kenway, C.T. Sachrajda, S. Sasaki, A. Soni,
   R.J. Tweedie and J.M. Zanotti,

3. Hadronic form factors in lattice QCD at small and vanishing momentum transfer,
   P. A. Boyle, J. M. Flynn, A. Juttner, C. T. Sachrajda and J. M. Zanotti,

4. The pion’s electromagnetic form factor at small momentum transfer in full lattice QCD,
   P.A. Boyle, J.M. Flynn, A. Jüttner, C. Kelly, H. Pedroso de Lima, C.M. Maynard,
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We have already seen the two precise results:

\[ \left| \frac{V_{us} f_K}{V_{ud} f_\pi} \right| = 0.27599(59) \quad \text{and} \quad |V_{us} f_+(0)| = 0.21661(47) \]

We can view these as two equation for the four unknowns \( f_K / f_\pi, f_+(0), V_{us} \) and \( V_{ud} \).

Within the Standard Model we also have the unitarity constraint:

\[ |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \]

Thus we now have 3 equations for four unknowns.

There has been considerable work recently in updating the determination of \( V_{ud} \) based on 20 different superallowed transitions.

\[ |V_{ud}| = 0.97425(22) \]

If we accept this value then we are able to determine the remaining 3 unknowns:

\[ |V_{us}| = 0.22544(95), \quad f_+(0) = 0.9608(46), \quad \frac{f_K}{f_\pi} = 1.1927(59). \]