QCD and statistical physics

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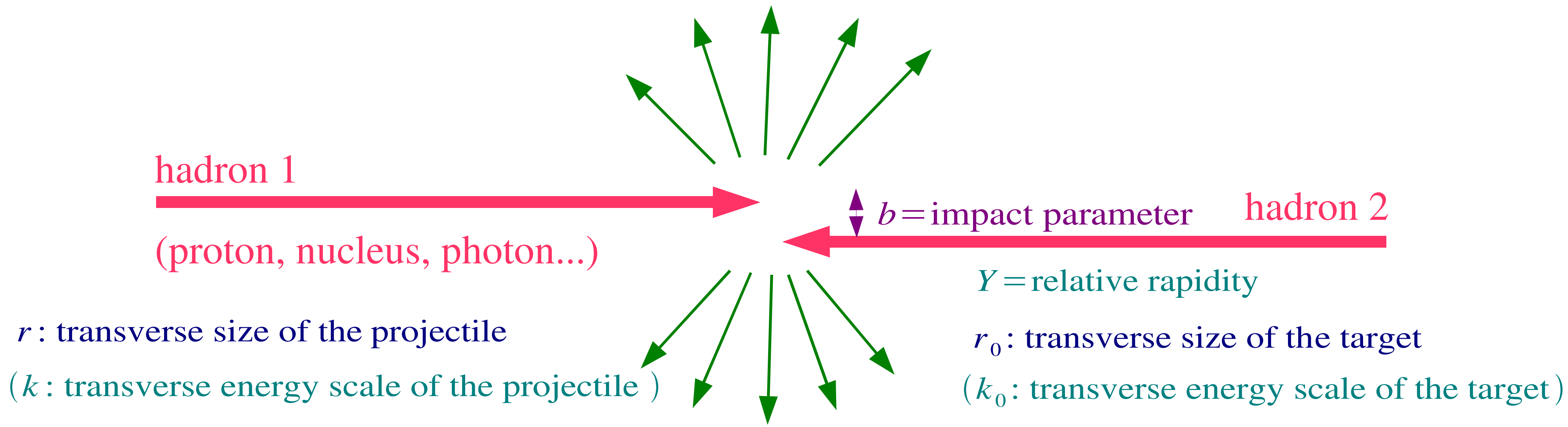
High energy QCD



$$A(Y, \overset{k}{r}) = \int d^2 b\, A(b, Y, \overset{k}{r}) = \text{elastic amplitude}$$

$$A(b, Y, \overset{k}{r}) = \text{fixed impact parameter amplitude} \quad \leq 1$$

(High) energy dependence of QCD amplitudes?

The Balitsky equation

Balitsky (1996)

Rapidity evolution of the scattering amplitude:

$\overline{\alpha}=\frac{\alpha_s N_c}{\pi}$ BFKL kernel; acts on transverse coordinates

$T=\frac{1}{N_c}\mathrm{Tr}(U\,\overline{U}),\langle T\rangle=A$

Infinite hierarchy, more complex operators at each step

$$\partial_{\overline{\alpha}Y} A=\chi*(A-\langle T\,T\rangle)$$

$$\partial_{\overline{\alpha}Y}\langle T\,T\rangle=\chi*(\langle T\,T\rangle-\langle T\,T\,T\rangle)+\chi_2*\langle\mathrm{Tr}(U\,\overline{U}\,U\,\overline{U}\,U\,\overline{U})\rangle$$
$$+\text{source terms}$$

$$\dots$$

A "mean field" approximation gives the Balitsky-Kovchegov (simpler) equation:

$$\langle T\,T\rangle=\langle T\rangle\langle T\rangle=A\cdot A \quad\Rightarrow\quad \partial_{\overline{\alpha}Y} A=\chi*(A-A\cdot A)$$

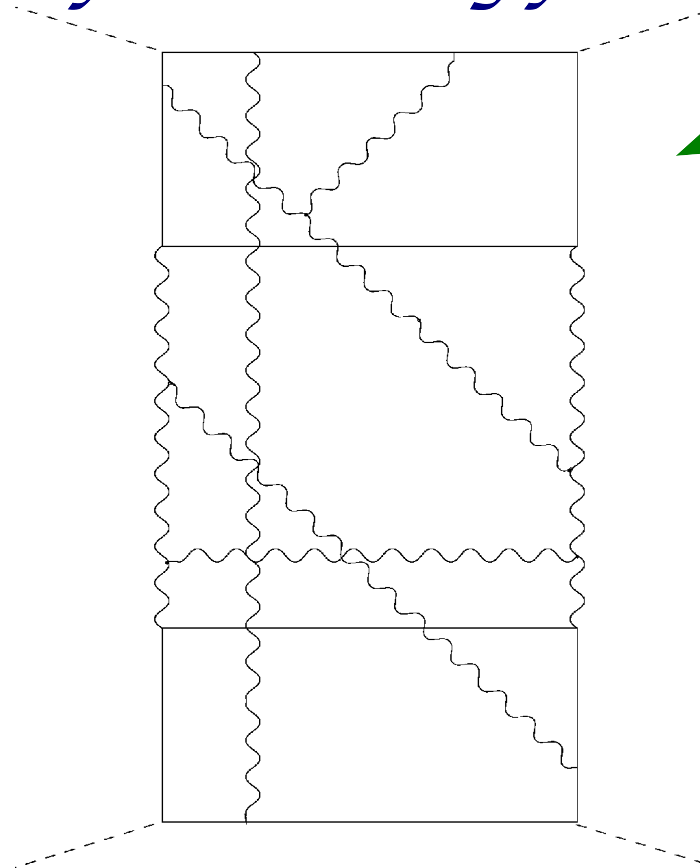
Balitsky (1996);
Kovchegov (1999)

Understand and solve the full high energy evolution equations!

See also JIMWLK and further developments
Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

High energy QCD in the field-theory formulation

Inside the Balitsky equation:

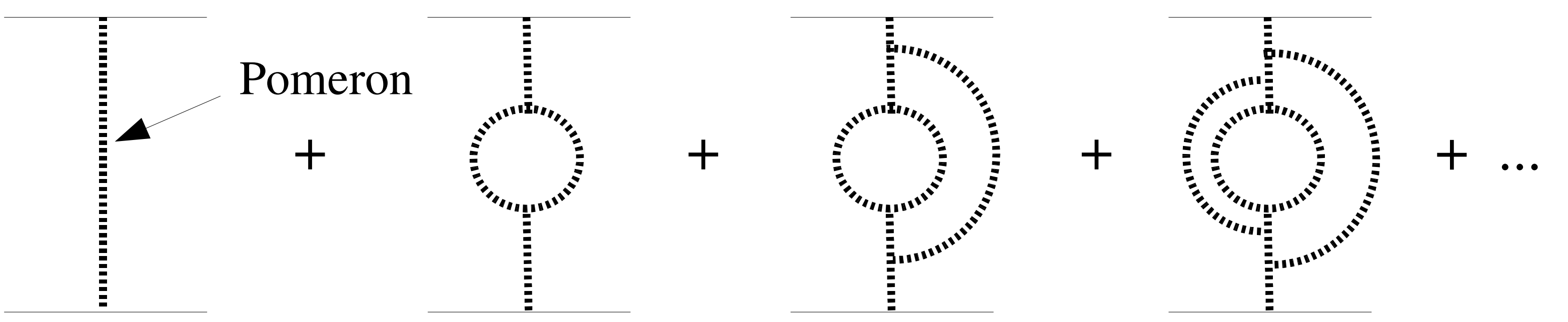


Balitsky (1996)

+ ...

Effective formulation:
"Pomeron" diagrams

$A =$



$=$

$\partial_{\bar{\alpha} Y} A = \chi * A$

?

Alternative philosophy

Breakthrough by Mueller and Shoshi, 3 years ago:

"Small x physics beyond the Kovchegov equation"

This talk:

Subsequent interpretation of their calculation in the light of some models well-known in statistical mechanics (namely **reaction-diffusion processes**).

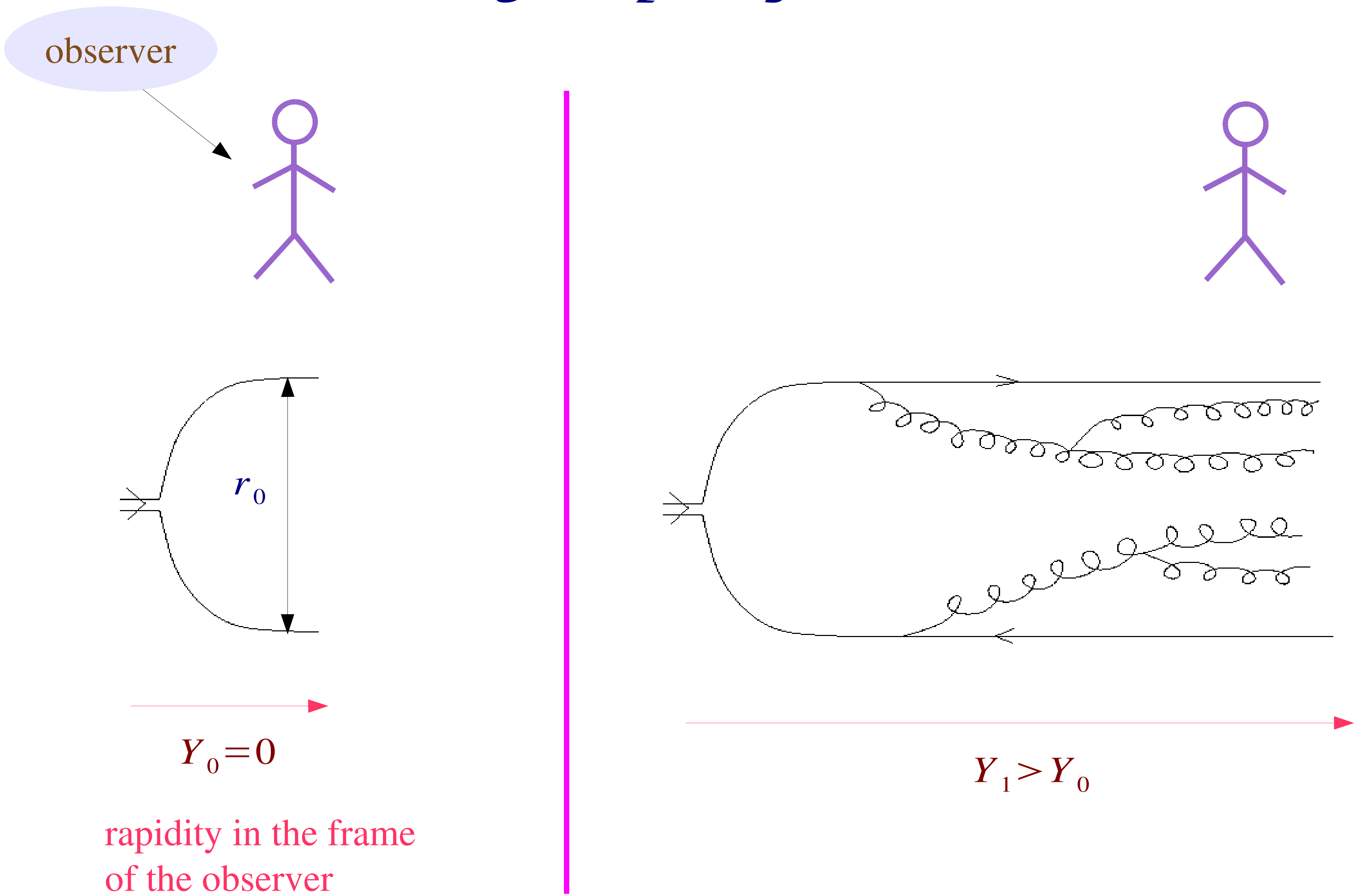
- go beyond the Mueller-Shoshi results
- simple picture, based on the parton model
- connects the QCD problem to more general physics and mathematics

Instead of a direct approach, identify the universality class from the physics of the parton model, then apply general results!

Outline

- ✯ High energy QCD and reaction-diffusion
- ✯ Field theory versus statistical methods for a simple particle model
- ✯ Statistical methods and application to QCD

How a high rapidity hadron looks



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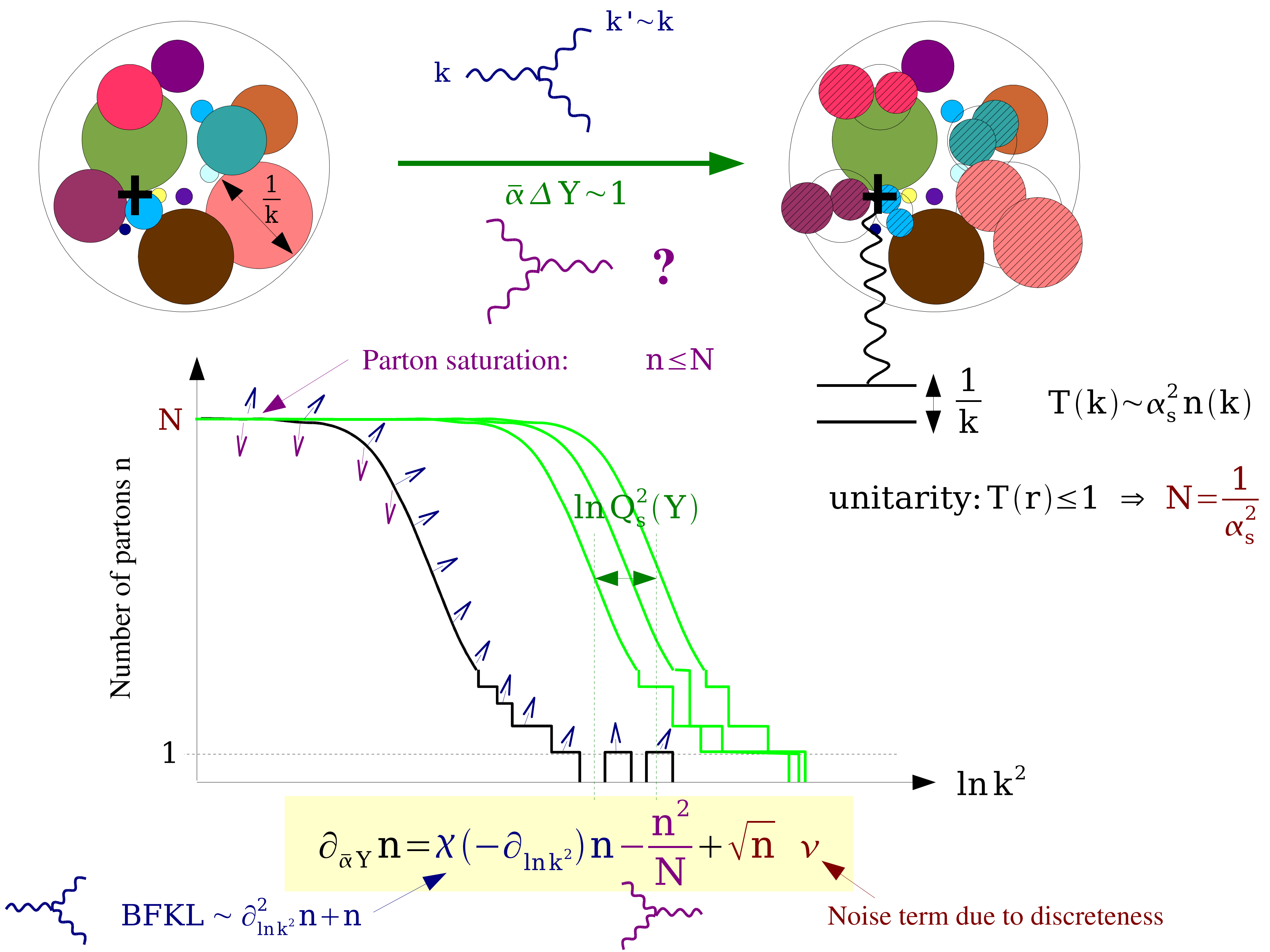

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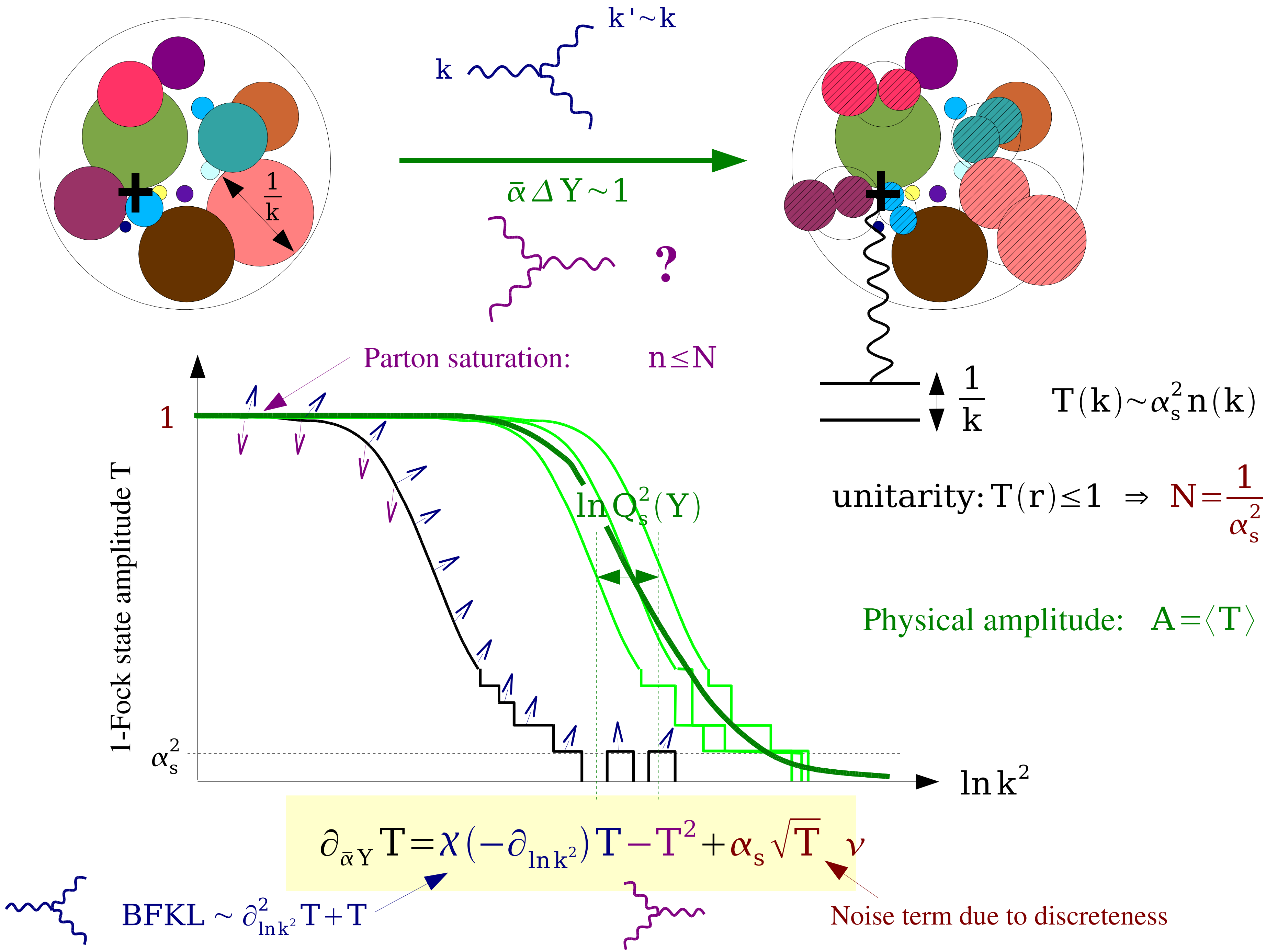
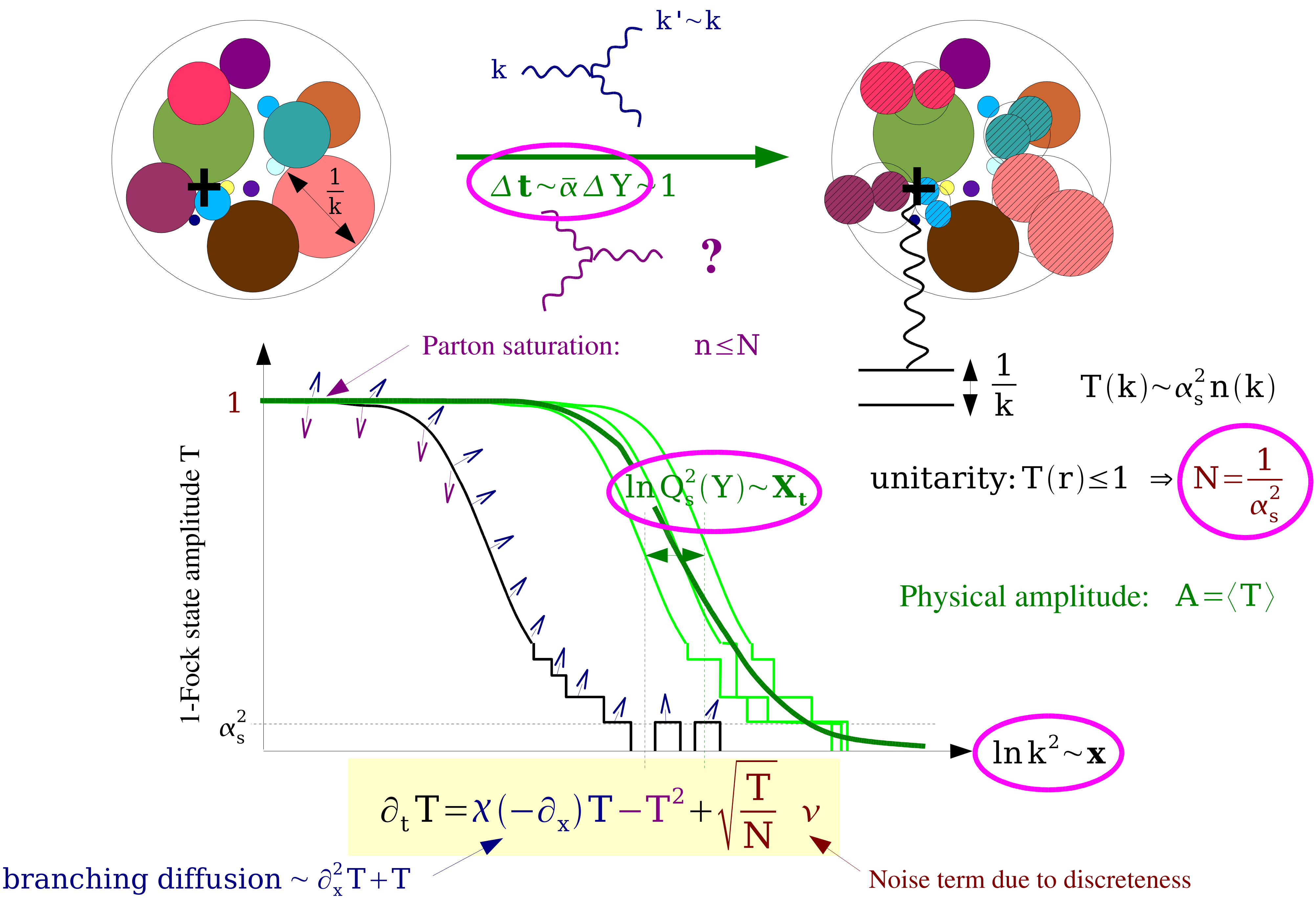
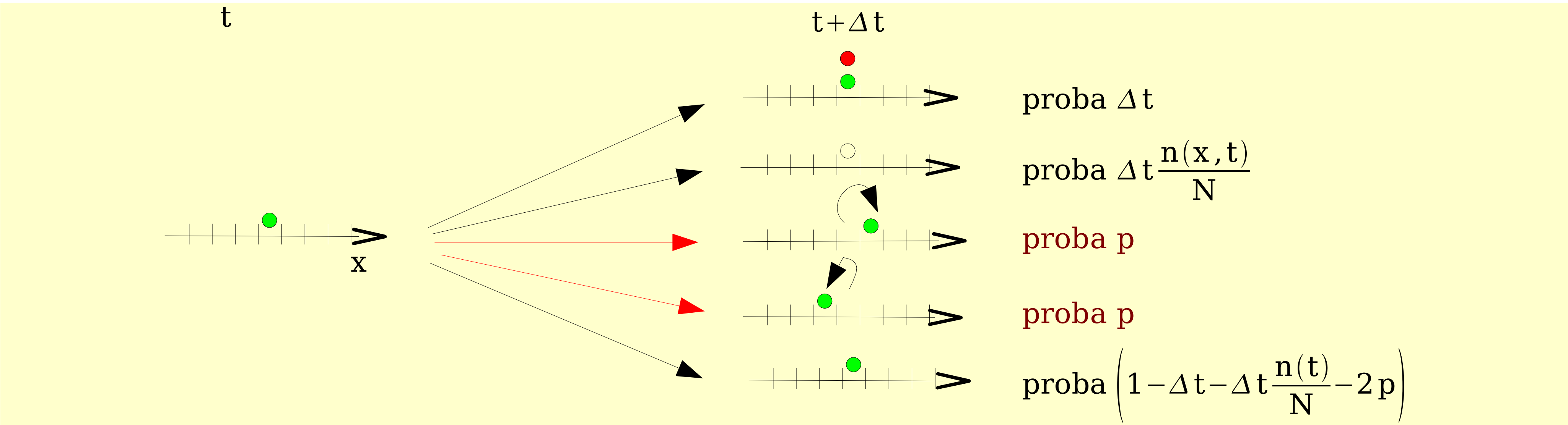
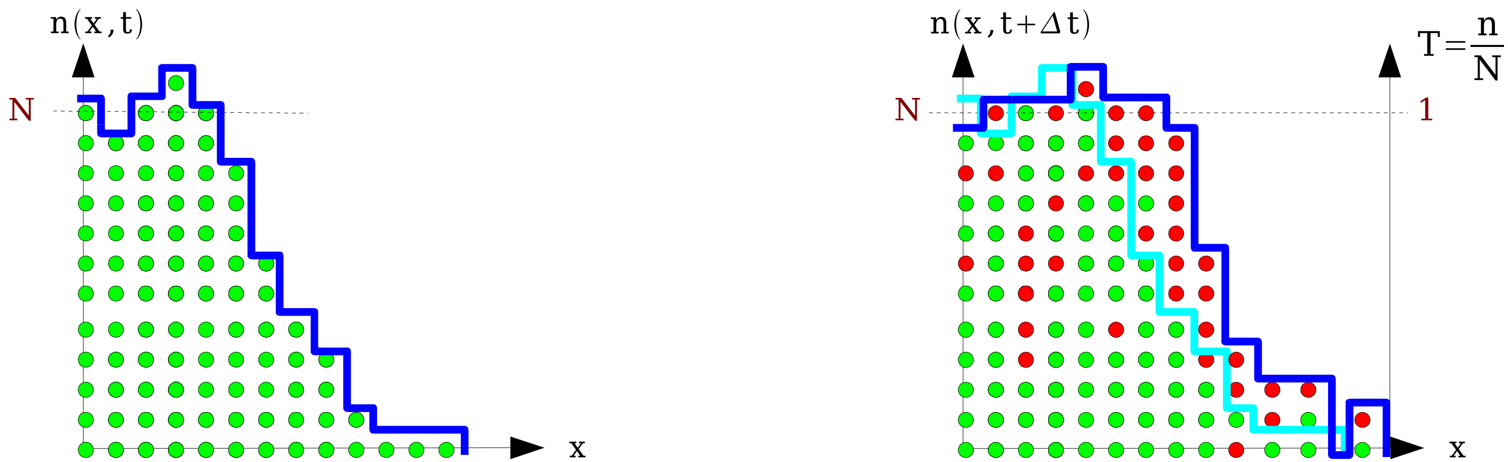


How a high rapidity hadron looks



Reaction-diffusion





$$T(x,t+\Delta t)=T(x,t)+p(T(x+\Delta x,t)+T(x-\Delta x,t)-2T(x,t))+\Delta t\,T(x,t)-\Delta t\,T^2(x,t)+\Delta t\sqrt{\frac{T}{N}}\,\nu(x,t+\Delta t)$$

$$\partial_t T = \qquad \chi(-\partial_x)T \qquad -T^2 \qquad +\sqrt{\frac{T}{N}}\;\nu$$

Prototype equation: sFKPP equation $\quad \partial_t T=\partial_x^2 T+T-T^2+\sqrt{\frac{2}{N}T(1-T)}\,\nu$

Fisher; Kolmogorov, Petrovsky, Piscunov (1937)

Dictionary

Reaction-diffusion		High energy QCD
Position x		$\ln(k^2/k_0^2)$
Time t		$\bar{\alpha} Y$
Particle density T	$\longleftrightarrow$	Partonic amplitude T
Maximum/equilibrium number of particles N		$\frac{1}{\alpha_s^2}$
Position of the wave front X		Saturation scale $\ln(Q_s^2/k_0^2)$

sFKPP equation

$$\partial_t T = \partial_x^2 T + T - T^2 + \sqrt{\frac{2}{N} T(1-T)}\, \nu$$

QCD evolution in the parton model

$$\partial_{\bar{\alpha} Y} T = \chi(-\partial_{\ln k^2}) T - T^2 + \alpha_s \sqrt{T}\ \nu$$

Outline

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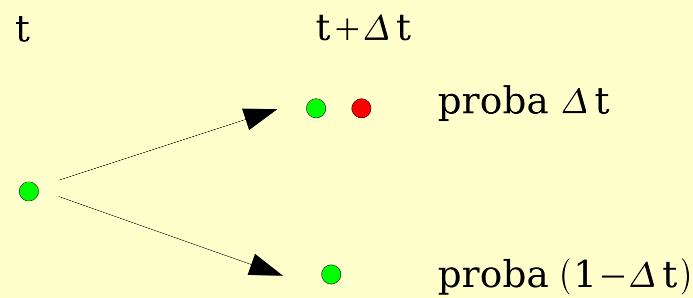
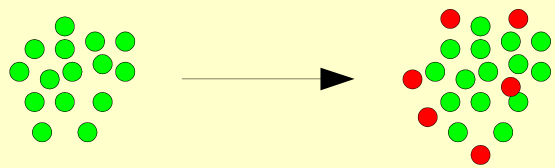
- Field theory versus statistical methods for a simple particle model
- Statistical methods and application to QCD

Simple particle model



t   t+Δt

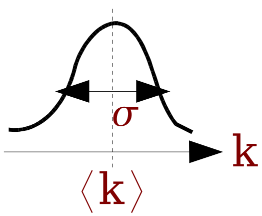
k particles added: k particles split, n-k do not split



$n(t)$ $n(t+\Delta t)=n(t)+k(t+\Delta t)$

$$\text{proba } P_n(k)=\binom{n}{k}(\Delta t)^k(1-\Delta t)^{n-k}$$

$$\begin{cases}\langle k\rangle = n\Delta t\\ \sigma^2=\langle (k-\langle k\rangle)^2\rangle = n\Delta t\end{cases}$$

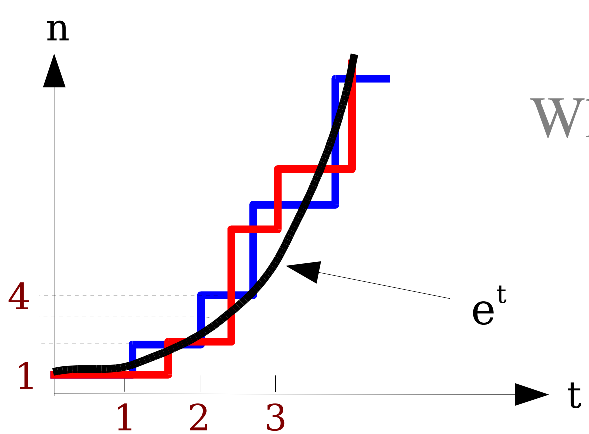


define $\nu=\dfrac{k-\langle k\rangle}{\sigma}\dfrac{1}{\sqrt{\Delta t}}$

such that $\sum_t^{t+1}\nu\sim\pm 1$

$$\begin{cases}\langle \nu\rangle = 0\\ \langle \nu^2\rangle = \dfrac{1}{\Delta t}\end{cases}$$

$$n(t+\Delta t)=n(t)+\Delta t\left(n(t)+\sqrt{n(t)}\,\nu(t+\Delta t)\right)\quad\xrightarrow[\Delta t\to 0]{}\quad \frac{dn}{dt}=n+\sqrt{n}\,\nu$$

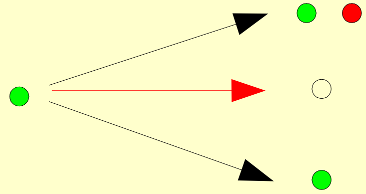


What is, *in average*, the number of particles at time t?

$\langle n(t)\rangle$ obtained by solving the trivial equation $\dfrac{d\langle n\rangle}{dt}=\langle n\rangle$

Simple particle model

t $\qquad$ $t+\Delta t$



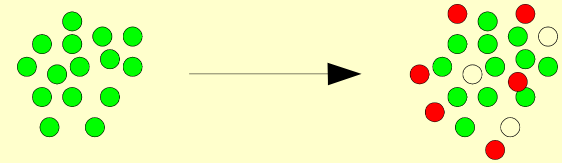
proba Δt

proba $\Delta t \frac{n(t)}{N}$

proba $\left(1-\Delta t-\Delta t\frac{n(t)}{N}\right)$

t $\qquad$ $t+\Delta t$

k_1 particles added, k_2 particles removed

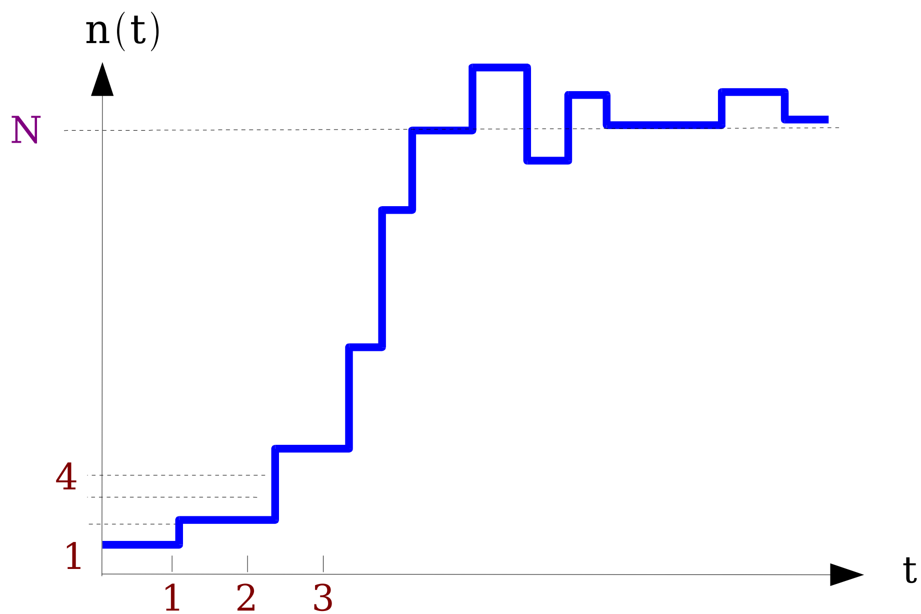


$n(t)$ $\qquad$ $n(t+\Delta t)=n(t)+k_1(t+\Delta t)-k_2(t+\Delta t)$

$$\text{proba } P_n(k_1,k_2)=\binom{n}{k_1\,k_2}(\Delta t)^{k_1}\left(\Delta t\frac{n(t)}{N}\right)^{k_2}\left(1-\Delta t-\Delta t\frac{n(t)}{N}\right)^{n-k_1-k_2}$$

$$\frac{dn}{dt}=n-\frac{n^2}{N}+\sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$

$$\begin{cases}\langle\nu\rangle=0\\ \langle\nu^2\rangle=\frac{1}{dt}\end{cases}$$



$\langle n(t)\rangle$ ***is not*** obtained by solving a trivial equation!

$$\begin{cases}\frac{d\langle n\rangle}{dt}=\langle n\rangle-\frac{1}{N}\langle n^2\rangle\\ \frac{d\langle n^2\rangle}{dt}=\ldots\end{cases}$$

...infinite hierarchy!

similar to the Balitsky equation in 0D

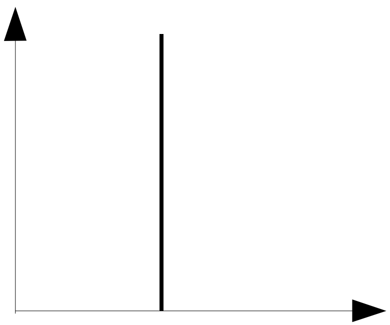
Mean field approximation: $\frac{d\langle n\rangle}{dt}=\langle n\rangle-\frac{\langle n\rangle^2}{N}$

similar to the Balitsky-Kovchegov equation

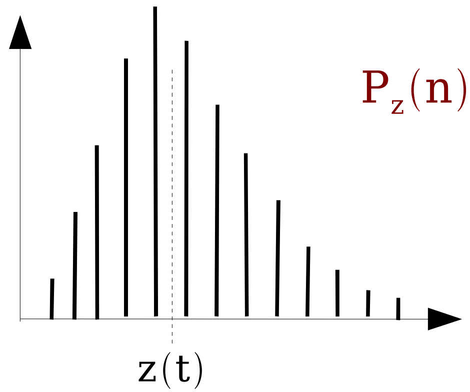
Field-theoretical formulation

Doi (1975)
Mueller (1995)
Shoshi, Xiao (2005)

Statistical formulation:
evolution of fixed particle number states



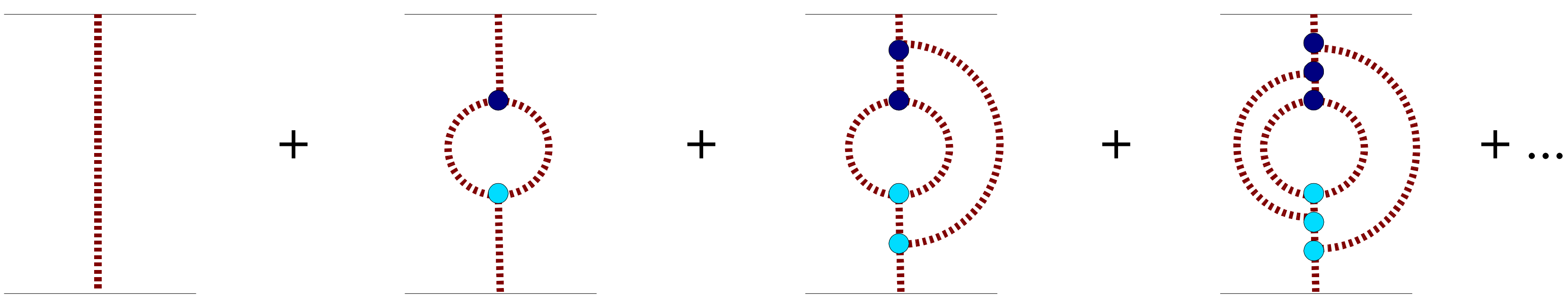
Evolution of Poissonian states



$$P_z(n) = \frac{z^n}{n!} e^{-z}$$

$\langle n(t) \rangle = \langle z(t) \rangle$ ← Path integral average, with weight
$$\exp\left(-\int dt \left[\bar{z}\left(\frac{d}{dt} - 1\right) z - \bar{z}\bar{z}z + \frac{1}{N}(\bar{z}zz + \bar{z}\bar{z}zz)\right]\right)$$

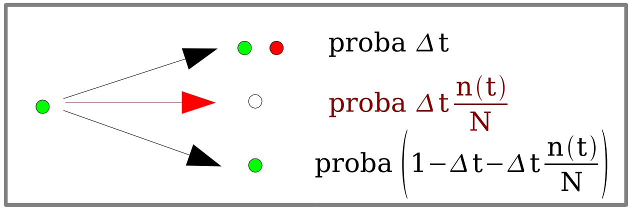
$\langle n(t) \rangle =$



$$e^t - \frac{2}{N} e^{2t} + \frac{6}{N^2} e^{3t} - \frac{24}{N^3} e^{4t} + \ldots$$

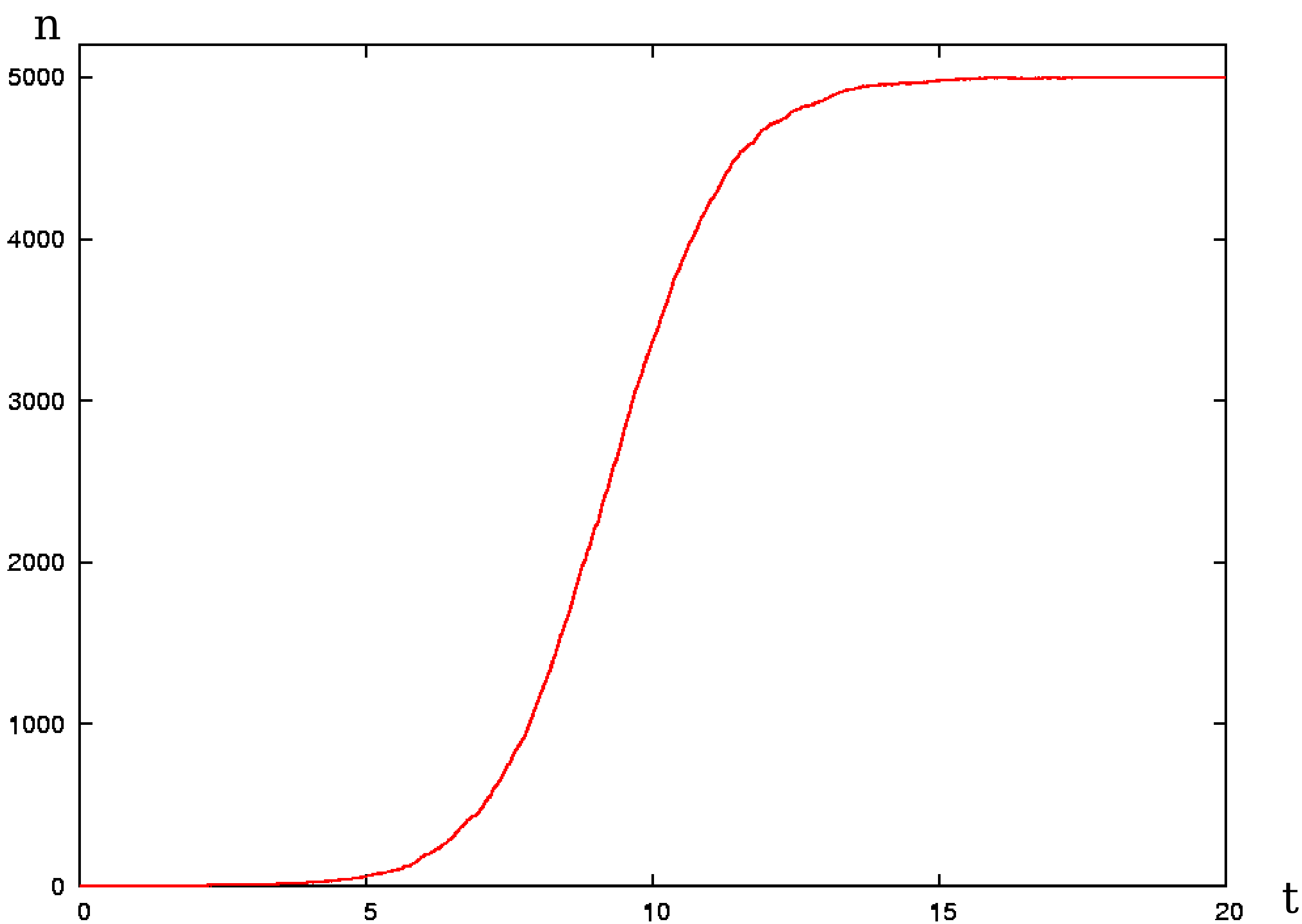
After Borel resummation:
$$\langle n(t) \rangle = N\left(1 - N e^{-t} \int_0^\infty \frac{db}{1+b} e^{-N\exp(-t) b}\right)$$

Statistical method

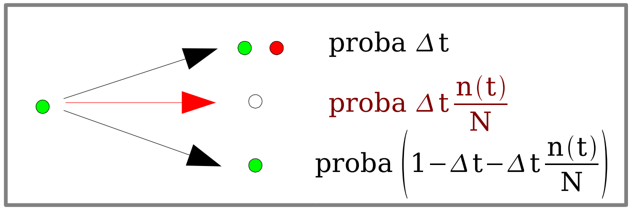


$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$

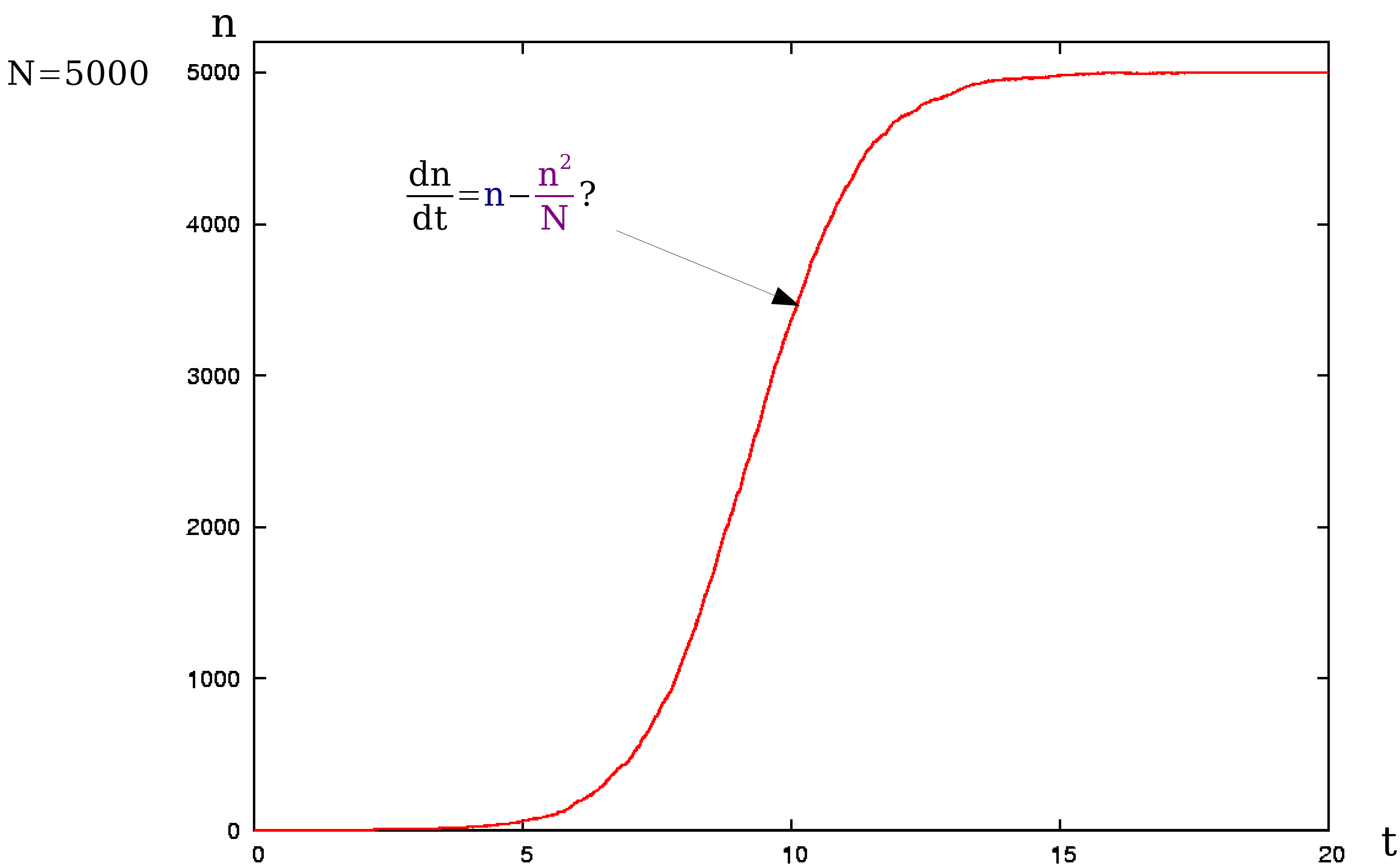
N=5000



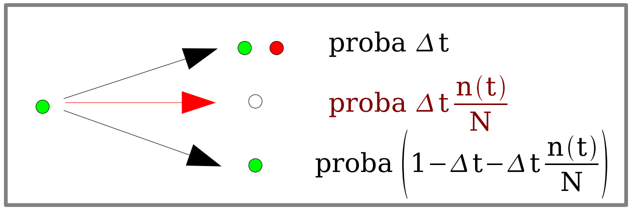
Statistical method



$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$

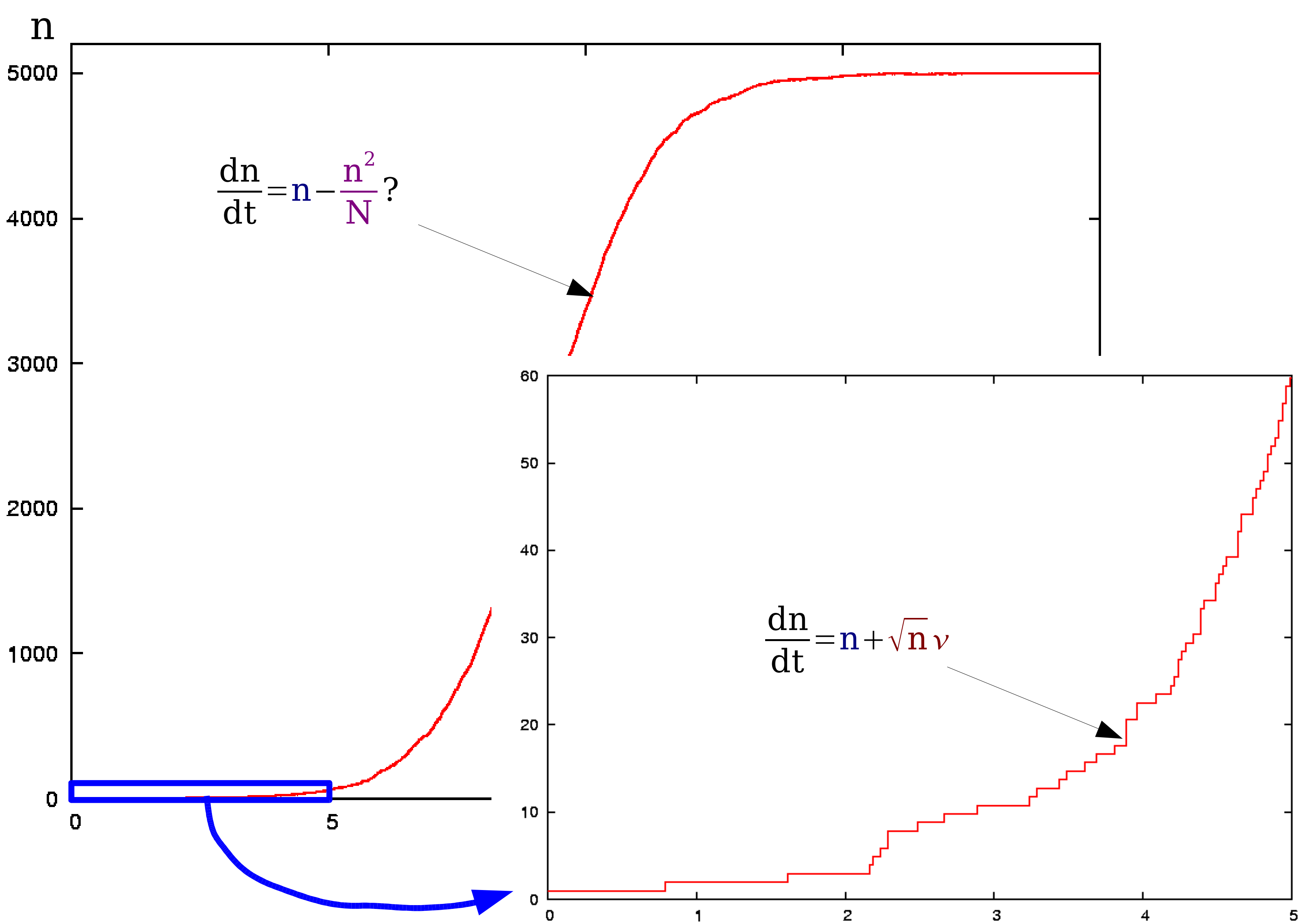


Statistical method

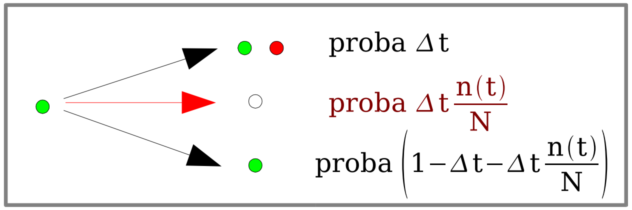


$$\frac{dn}{dt}=n-\frac{n^2}{N}+\sqrt{n\left(1+\frac{n}{N}\right)}\nu$$

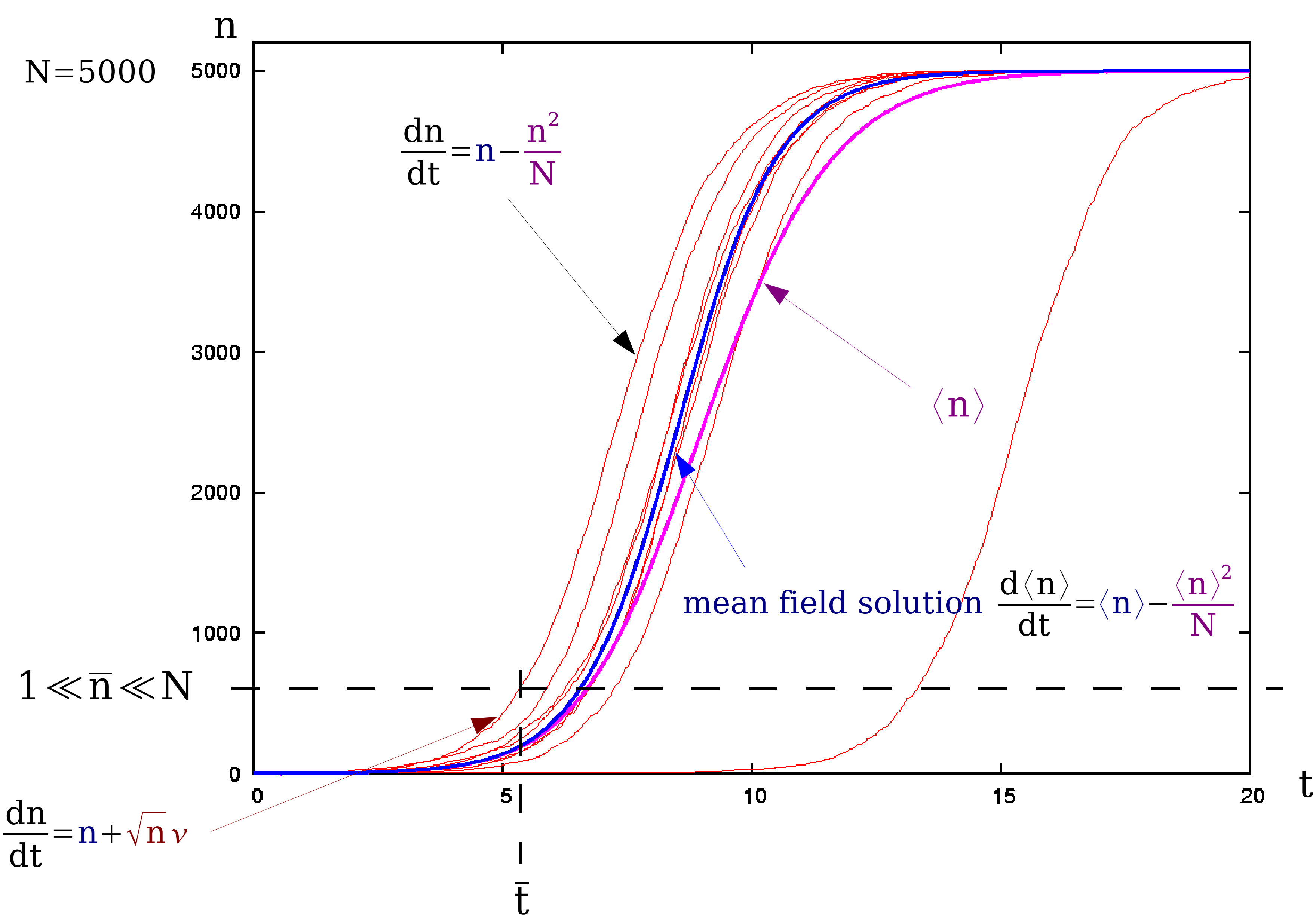
N=5000



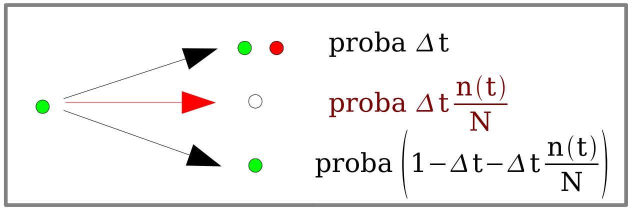
Statistical method



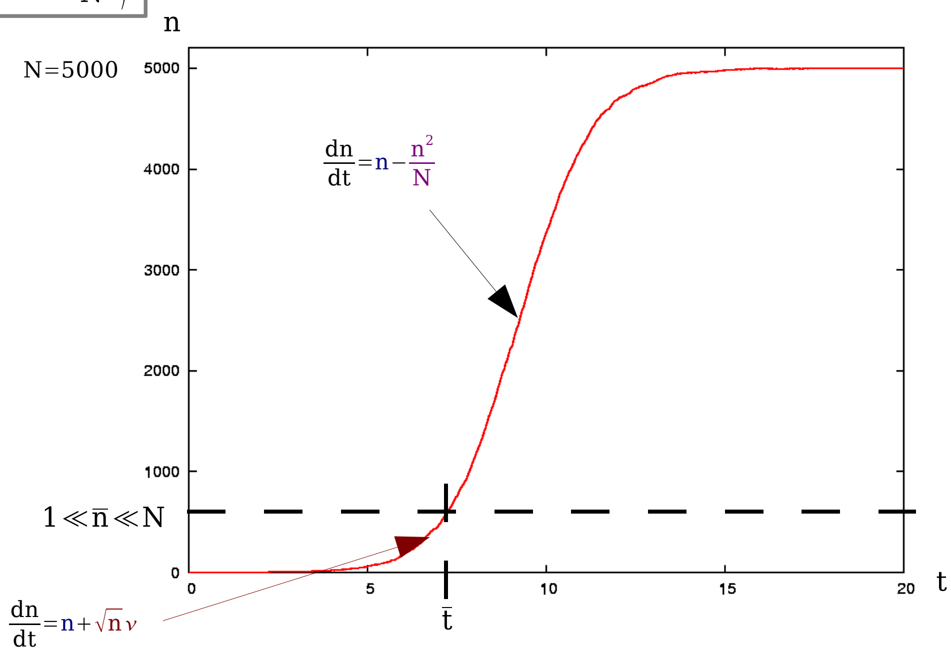
$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$



Statistical method



$$\frac{dn}{dt}=n-\frac{n^2}{N}+\sqrt{n\left(1+\frac{n}{N}\right)}\nu$$



Solution of $\frac{dn}{dt}=n+\sqrt{n}\,\nu$ for $\bar{t}$

Solution of the mean-field equation $\frac{dn}{dt}=n-\frac{n^2}{N}$ with the initial condition $n(\bar{t})=\bar{n}$

$$\langle n(t)\rangle=\int_0^\infty d\bar{t}\;\bar{n}\,e^{-\bar{t}-\bar{n}\exp(-\bar{t})}\;\frac{N}{1+\frac{N}{\bar{n}}e^{-(t-\bar{t})}}$$

$\Longleftrightarrow$

Field-theoretical result:

$$\langle n(t)\rangle=N\left(1-N e^{-t}\int_0^\infty \frac{db}{1+b}e^{-N\exp(-t)b}\right)$$

+ Well-established systematics
− Complex, abstract

+ Simple, intuitive
− No systematics

Summary of the part on simple particle models

We have considered a model that evolve according to nonlinear stochastic differential equations of the form

$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1 + \frac{n}{N}\right)}\,\nu$$

For the nonlinearity, $\langle n \rangle$ does not obey a closed equation, but an infinite hierarchy of equations of the Balitsky type. A field-theoretical resolution is difficult, on the other hand, the simple mean field solution completely fails!

However, there is a simple factorization at the level of individual realizations:

If N is large enough, realizations evolve first through the ***stochastic <u>but</u> linear equation***

$$\frac{dn}{dt} = n + \sqrt{n}\,\nu$$

until n is large enough for the noise term to be small, and continues evolving through the ***nonlinear <u>but</u> deterministic equation***

$$\frac{dn}{dt} = n - \frac{n^2}{N} \qquad \text{when } n \gg 1 .$$

Then, $\langle n \rangle$ is obtained from the averaging of many such realizations

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- ✯ Field theory versus statistical methods for a simple particle model
- ✯ Statistical methods and application to QCD

QCD as a reaction-diffusion process

Reaction-diffusion		*High energy QCD*
Position x		$\ln(k^2/k_0^2)$
Time t		$\bar{\alpha} Y$
Particle density T	⟷	Partonic amplitude T
Maximum/equilibrium number of particles N		$\frac{1}{\alpha_s^2}$
Position of the wave front X		Saturation scale $\ln(Q_s^2/k_0^2)$

sFKPP equation

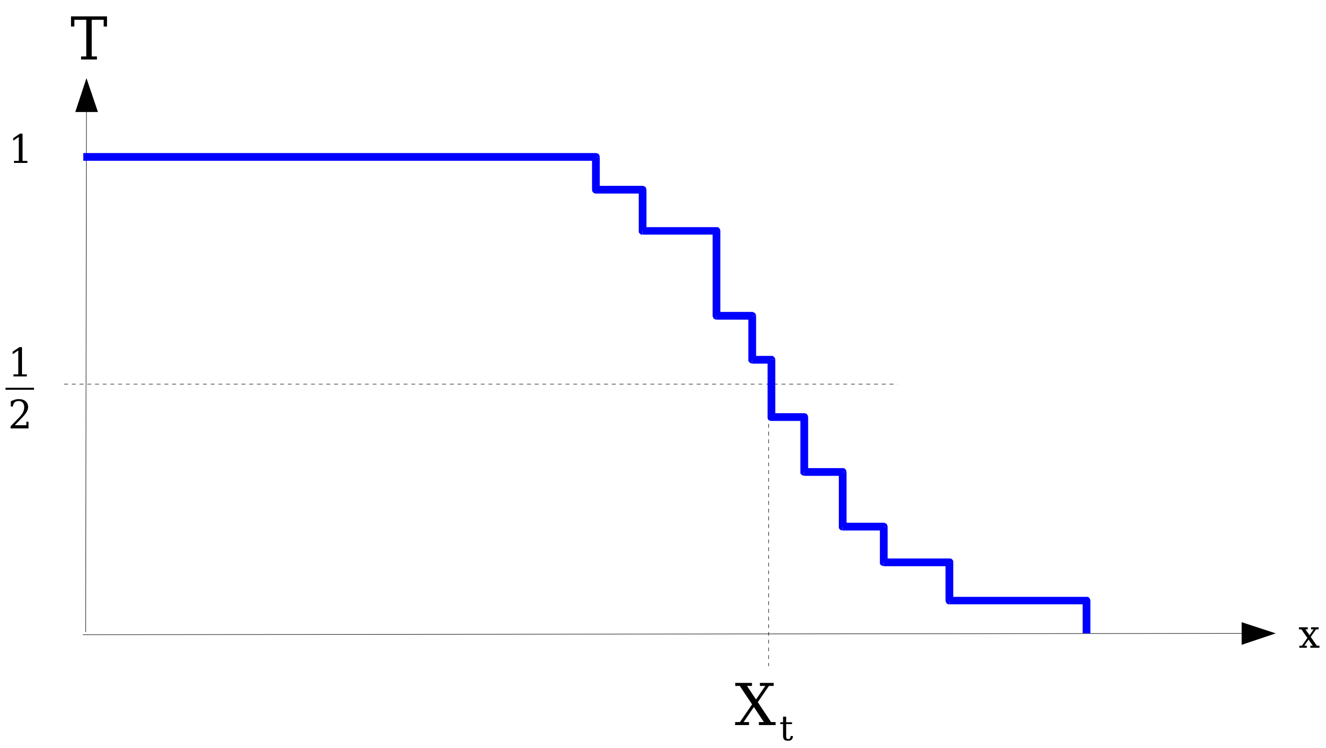
$$\partial_t T = \partial_x^2 T + T - T^2 + \sqrt{\frac{2}{N} T(1-T)}\, \nu$$

QCD evolution in the parton model

$$\partial_{\bar{\alpha} Y} T = \chi(-\partial_{\ln k^2}) T - T^2 + \alpha_s \sqrt{T}\ \ \nu$$

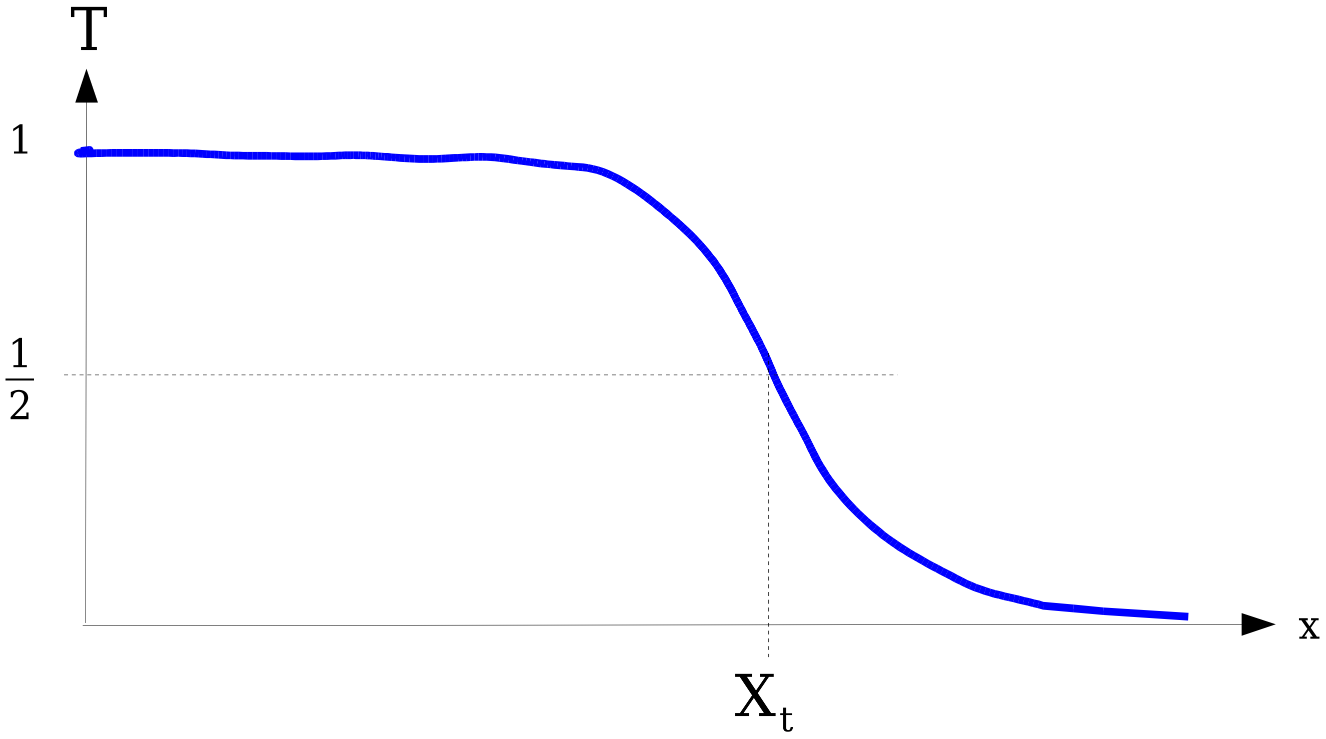
The infinite particle number limit

$$\partial_t T = \partial_x^2 T + T - T^2 + \sqrt{\frac{2}{N} T(1-T)}\,\nu$$



The infinite particle number limit

$$\partial_t T = \partial_x^2 T + T - T^2 + \sqrt{\frac{2}{N} T(1-T)}\,\nu$$

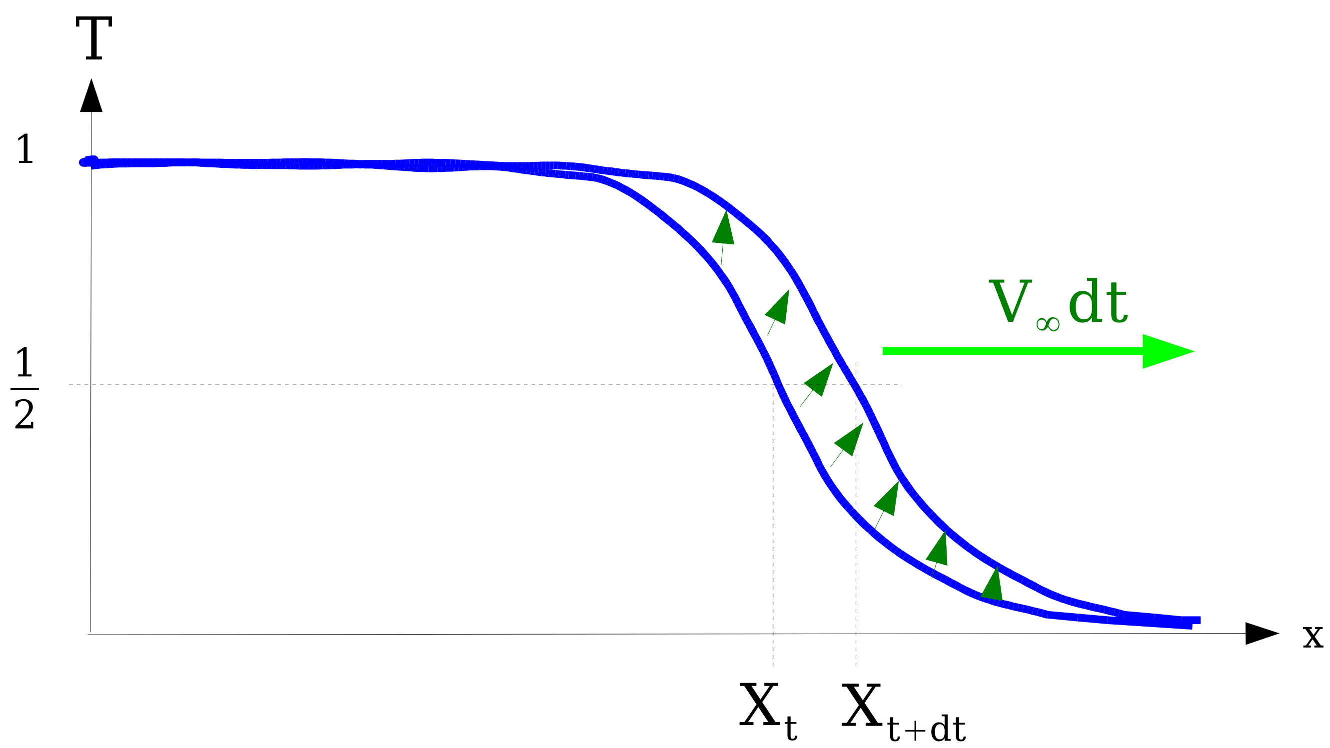


The infinite particle number limit

$$\partial_t T = \partial_x^2 T + T - T^2 + \sqrt{\frac{2}{N} T(1-T)}\,\nu$$

The large time asymptotics are exact traveling waves. Mathematical result by Bramson (1984)

The evolution of T is driven by the (linear) branching diffusion part.
The nonlinearity only tames the growth when $T \sim 1$



The infinite particle number limit

$$\partial_t T = \underbrace{\partial_x^2 T + T}_{\chi(-\partial_x)T} - T^2 + \sqrt{\frac{2}{N} T(1-T)}\,\nu$$

$\chi(\gamma) = \gamma^2 + 1$ characteristic function of the diffusion kernel

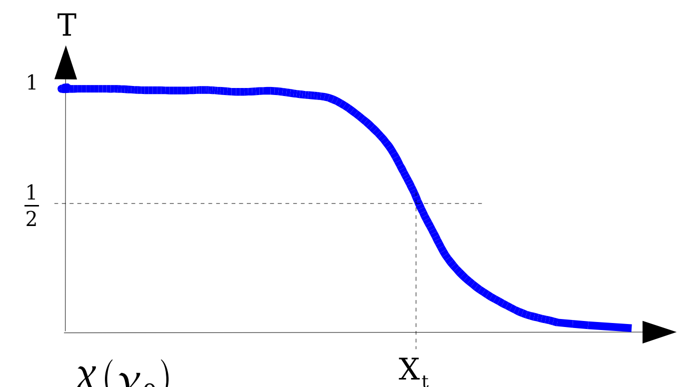
Look for solutions of the form $T_\gamma = \exp\left(-\gamma\left(x - v(\gamma)t\right)\right)$

Solution: $v(\gamma) = \dfrac{\chi(\gamma)}{\gamma}$

$v(\gamma) = \gamma + \dfrac{1}{\gamma}$ in the F-KPP case

General solution: arbitrary superposition of different wave numbers

$$T = \int d\gamma\, f(\gamma) T_\gamma = \int d\gamma\, f(\gamma) \exp\left(-\gamma\left(x - v(\gamma)t\right)\right)$$



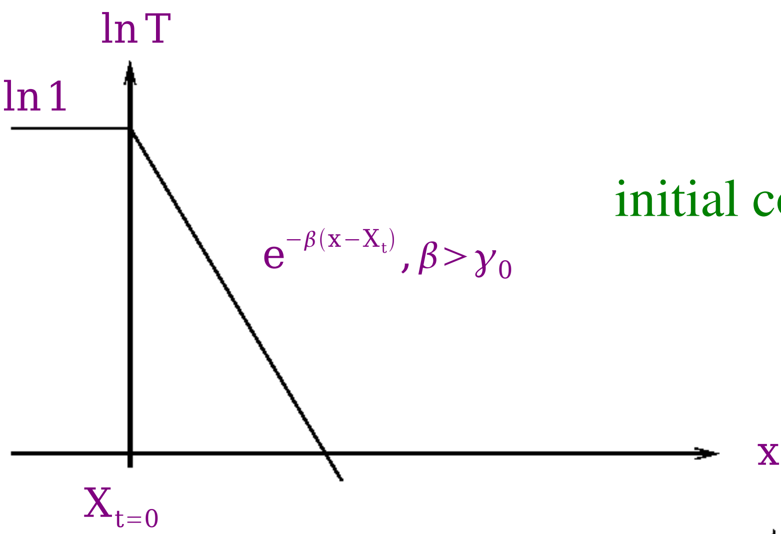
Large times (saddle point ***at constant*** T), select the wave that travels with minimum velocity:

$$v'(\gamma_0) = 0 \;\Rightarrow\; \chi'(\gamma_0) = \frac{\chi(\gamma_0)}{\gamma_0}$$

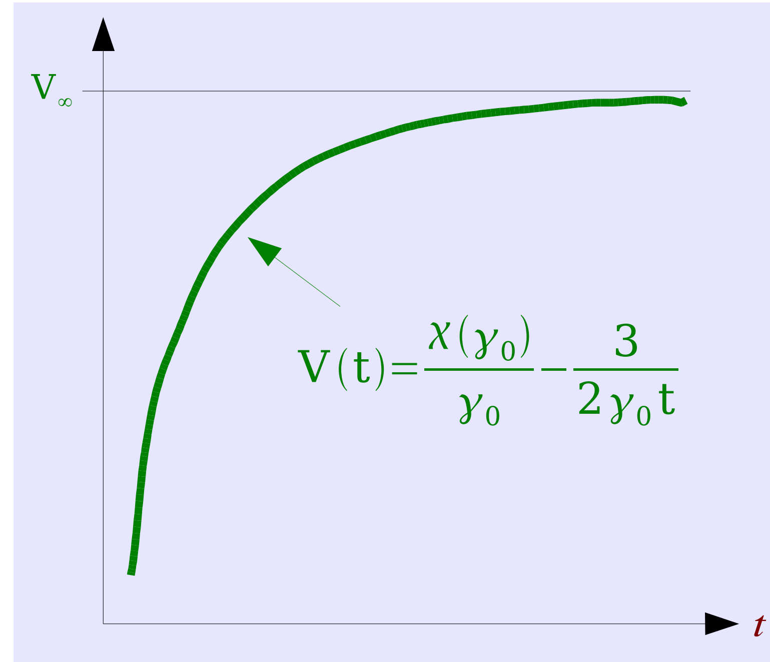
$$\begin{cases} V_\infty = \dfrac{dX_t}{dt} = v(\gamma_0) = \dfrac{\chi(\gamma_0)}{\gamma_0} \\ T(x,t) \sim e^{-\gamma_0(x - X_t)} \end{cases}$$

$\gamma_0 = 1, V_\infty = 2$ in the F-KPP case

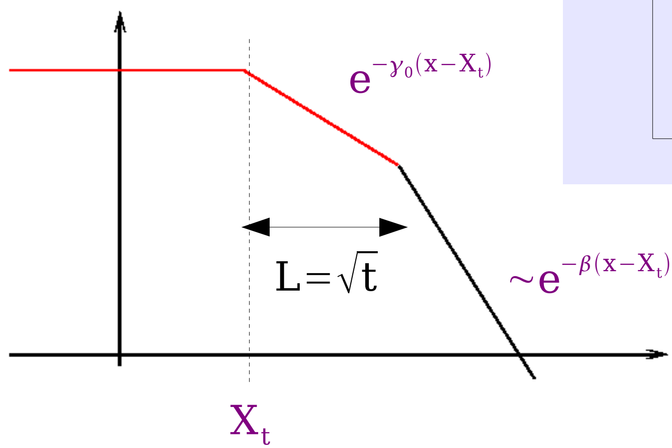
Transition to the asymptotics



initial condition

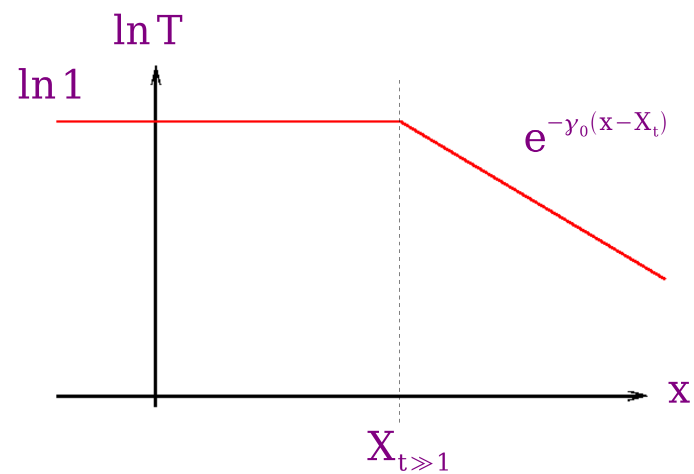


transients:

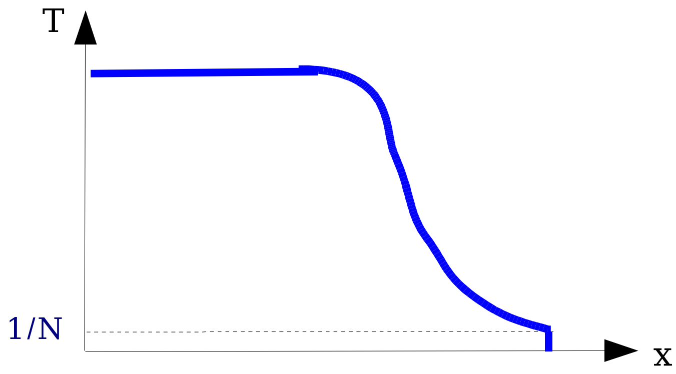


traveling wave, asymptotic velocity:

$$V_\infty = \frac{\chi(\gamma_0)}{\gamma_0}$$



Accounting for discreteness



Observation: T is either 0 or larger than $1/N$

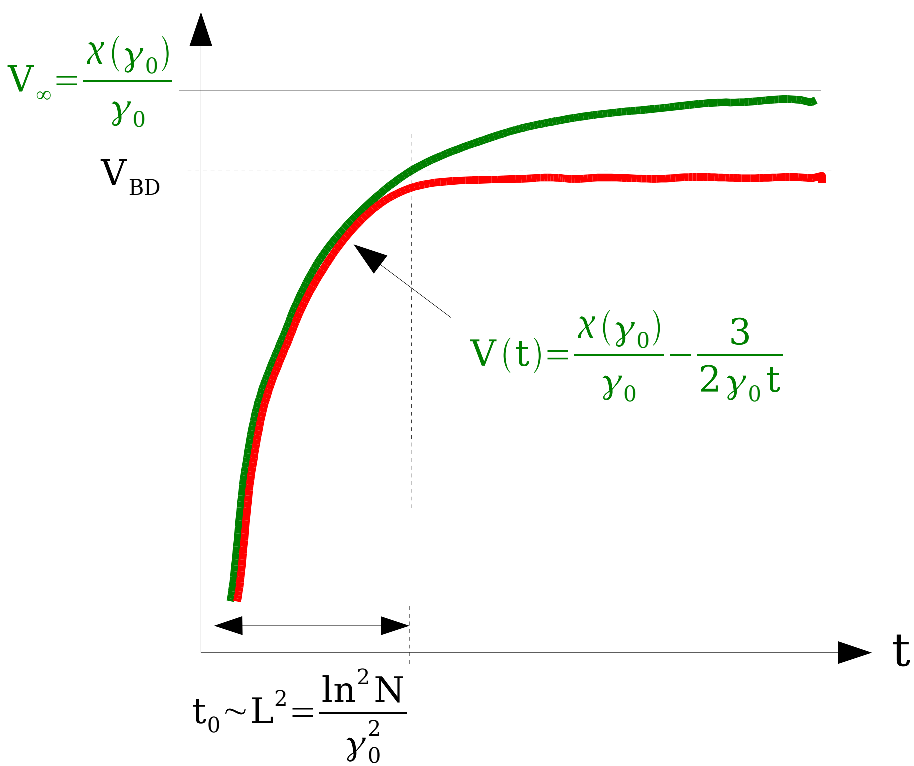
Recipe: Whenever there is more than 1 particle on a site apply the mean field evolution

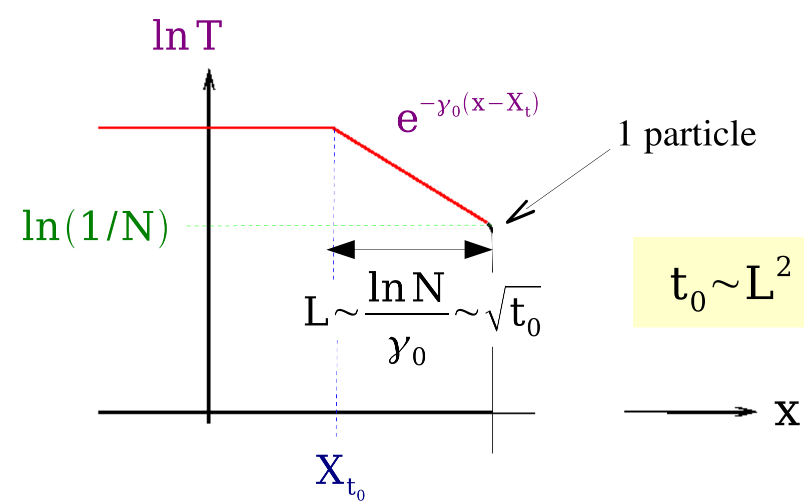
Brunet, Derrida (1997)

Infinite N equation + cut-off

(still deterministic)

$$\partial_t T = (\partial_x^2 T + T - T^2)\,\Theta(T - 1/N)$$





$t_0 \sim L^2$

$$V_{BD} = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2\ln^2 N}$$

Velocity of a front of size $L = \frac{\ln N}{\gamma_0}$

Summary of the mean field approach

The FKPP equation $\partial_t T = \partial_x^2 T + T - T^2$

admits asymptotic traveling wave solutions, of shape $e^{-\gamma_0 (x - X_t)}$

and velocity $V_\infty = \frac{dX_t}{dt} = \frac{\chi(\gamma_0)}{\gamma_0}$ where $\chi(\gamma) = \gamma^2 + 1$ and γ_0 minimizes $v(\gamma) = \frac{\chi(\gamma)}{\gamma}$

in the F-KPP case

Gribov, Levin, Ryskin (1980)

The traveling wave builds up diffusively from a given initial condition

and its velocity during that phase reads $V(t) = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{3}{2\gamma_0 t}$

Bramson (1984)
Mueller, Triantafyllopoulos (2002)

The FKPP equation may be modified to take into account the fact that in real particle models, occupation numbers are discrete, 0,1,2... : $\partial_t T = (\partial_x^2 T + T - T^2)\Theta(T - 1/N)$

The front reaches its asymptotic shape of width $L = \frac{\ln N}{\gamma_0}$

after a time L^2 and the corresponding velocity is $V_{BD} = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2 \ln^2 N}$

Brunet, Derrida (1997)
Mueller, Shoshi (2004)

Confirmed to be the right average front velocity in numerical simulations of fully stochastic models!

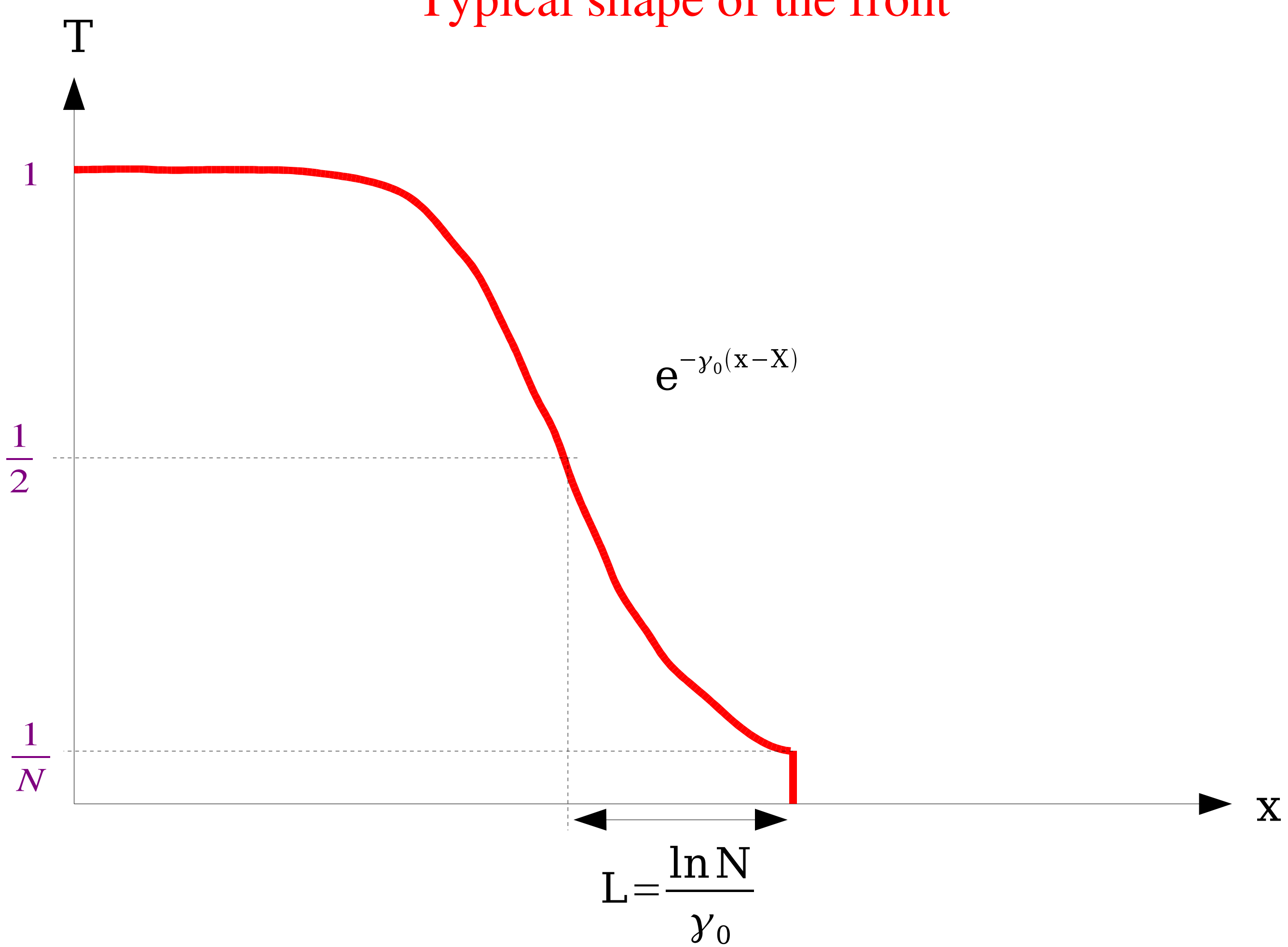
Brunet, Derrida; Moro;
Pechenik, Levine; Panja...

Accounting for fluctuations

Brunet, Derrida, Mueller, SM (2005)

Assumption #1: the evolution of the stochastic front is essentially deterministic

Typical shape of the front



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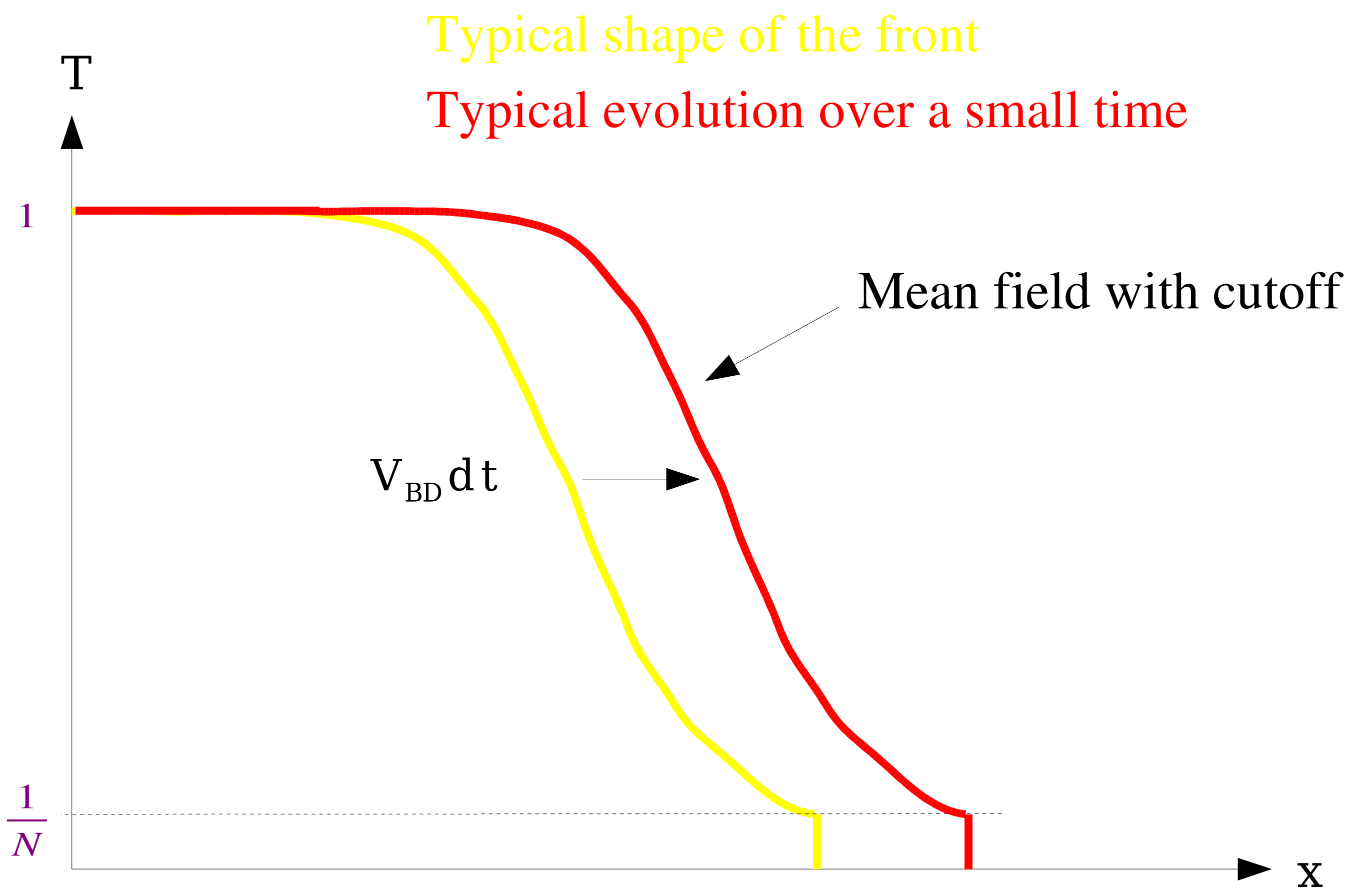
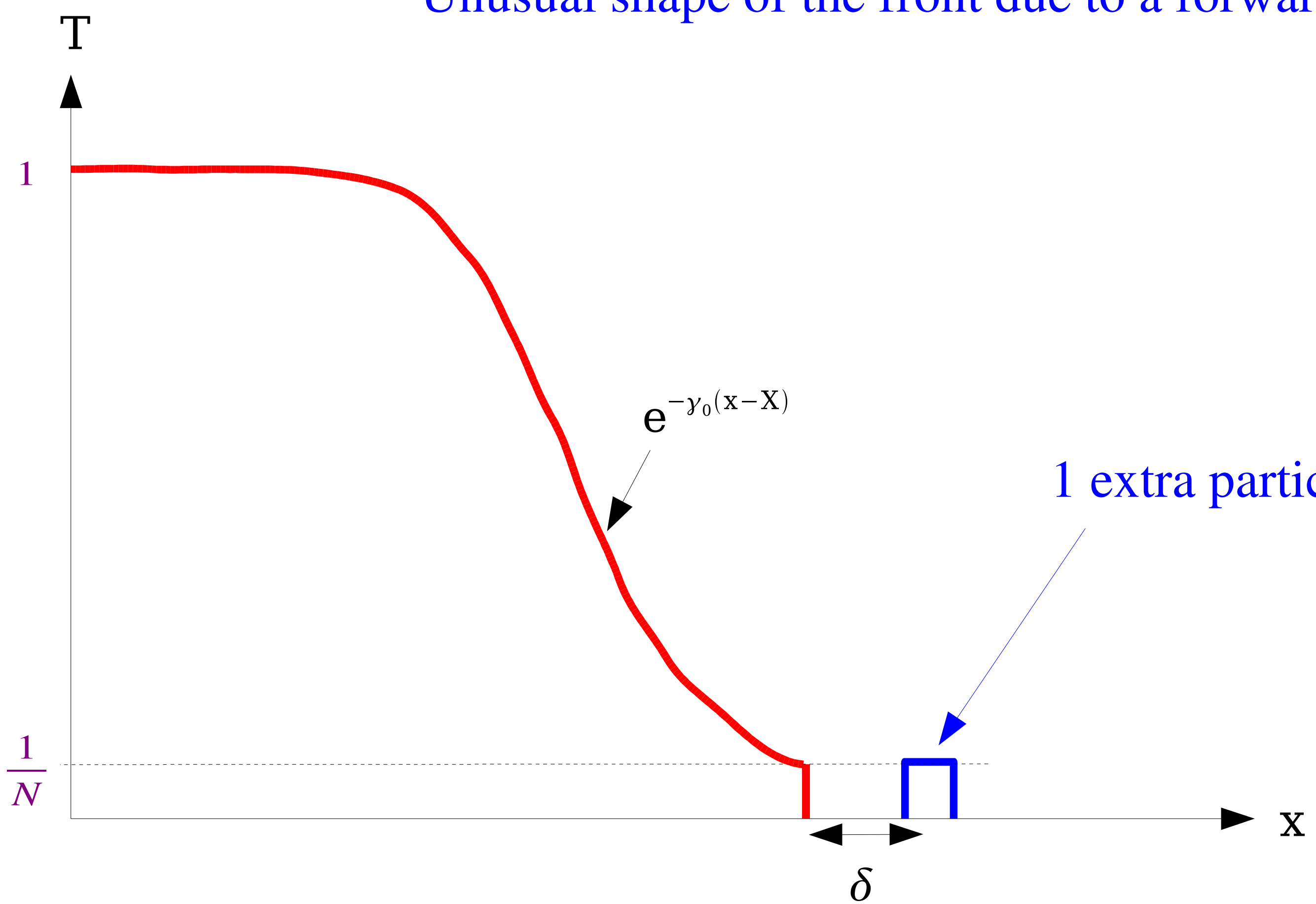


Accounting for fluctuations

Brunet, Derrida, Mueller, SM (2005)

Assumption #1: the evolution of the stochastic front is essentially deterministic, ***except for some occasional extra-particles in the tail***

Unusual shape of the front due to a forward extra particle



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Assumption #2: the probability for such extra-particles is $p(\delta)\,d\delta\,dt = C_1 e^{-\gamma_0 \delta}\,d\delta\,dt$

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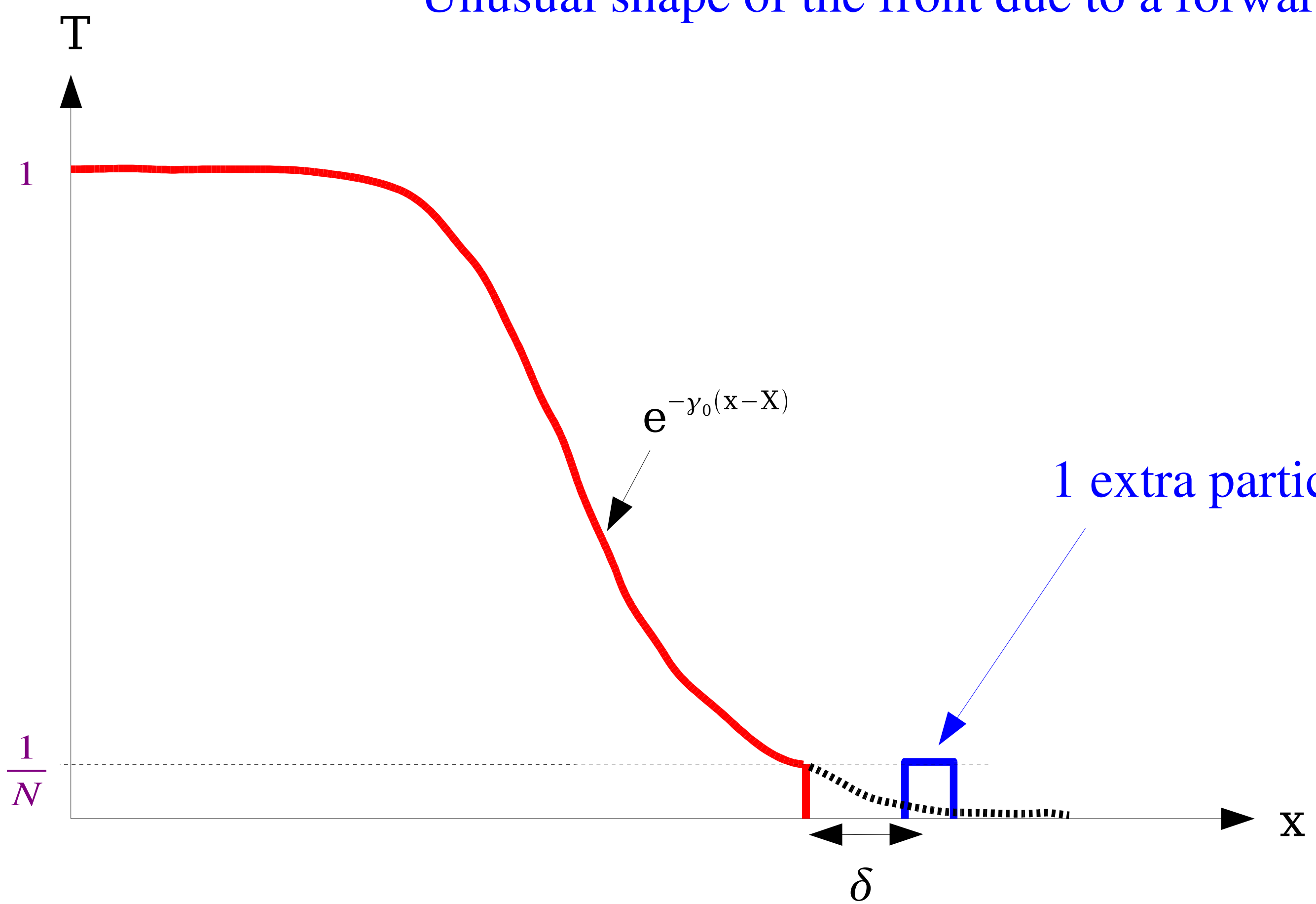

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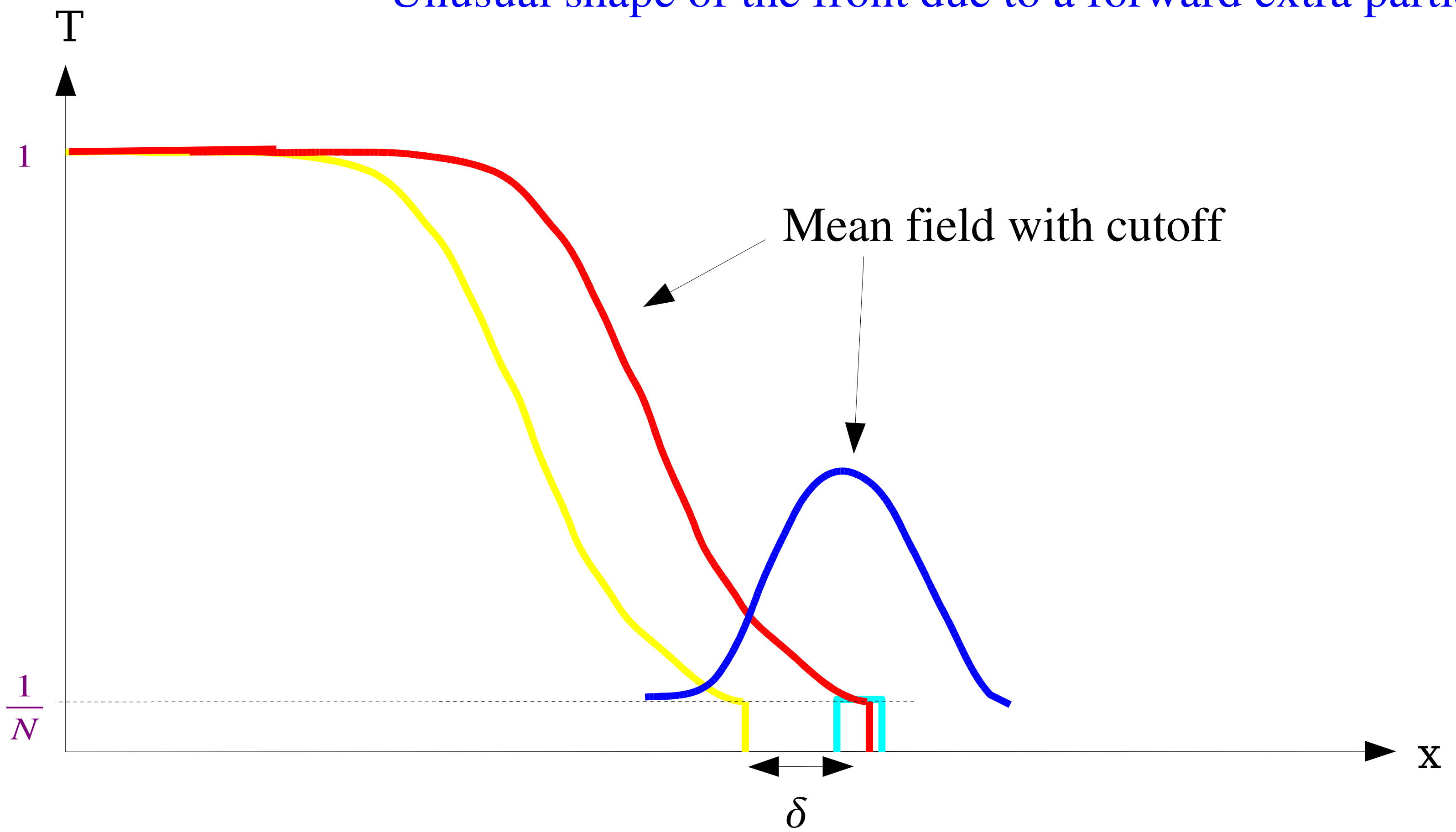

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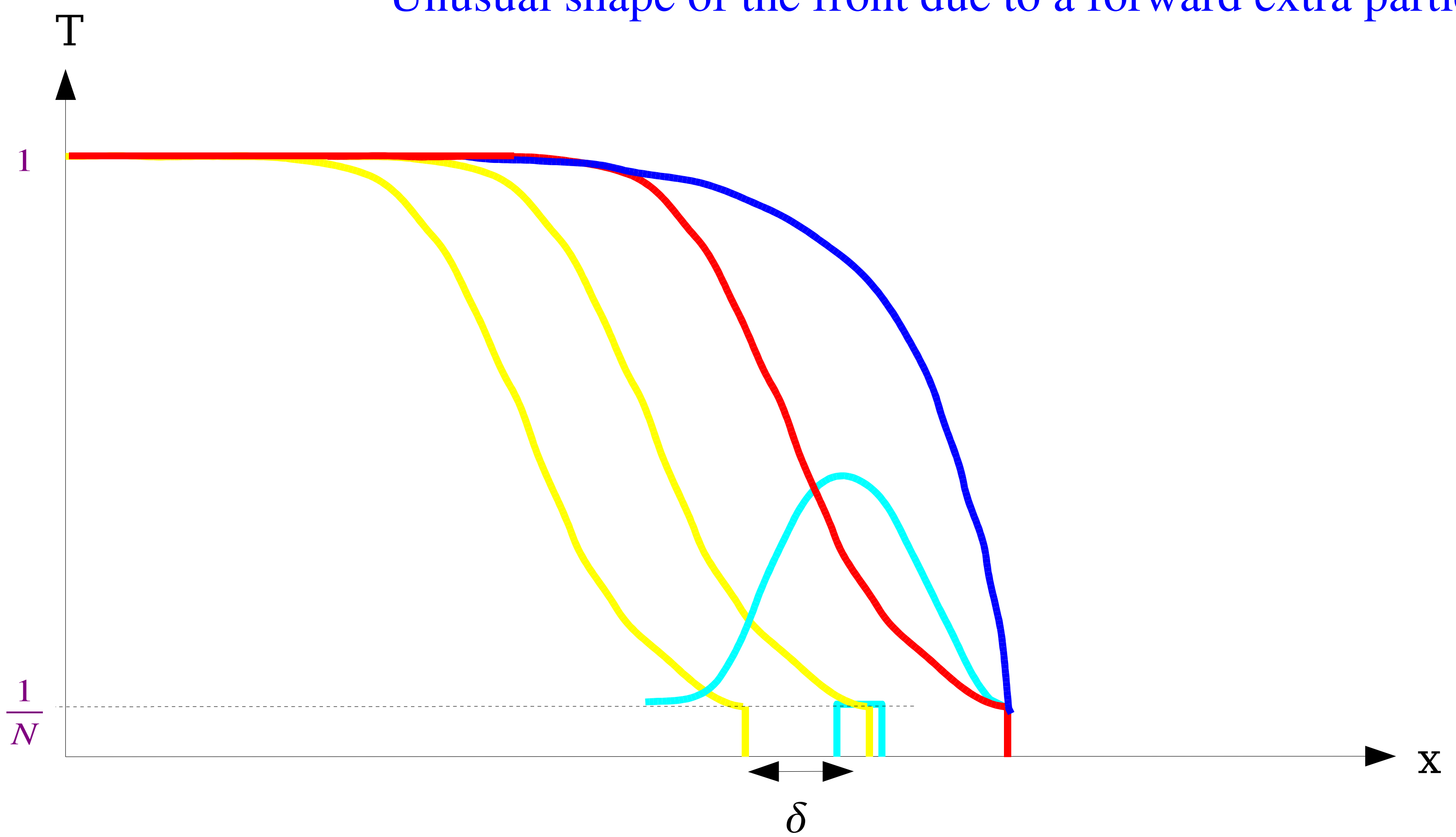

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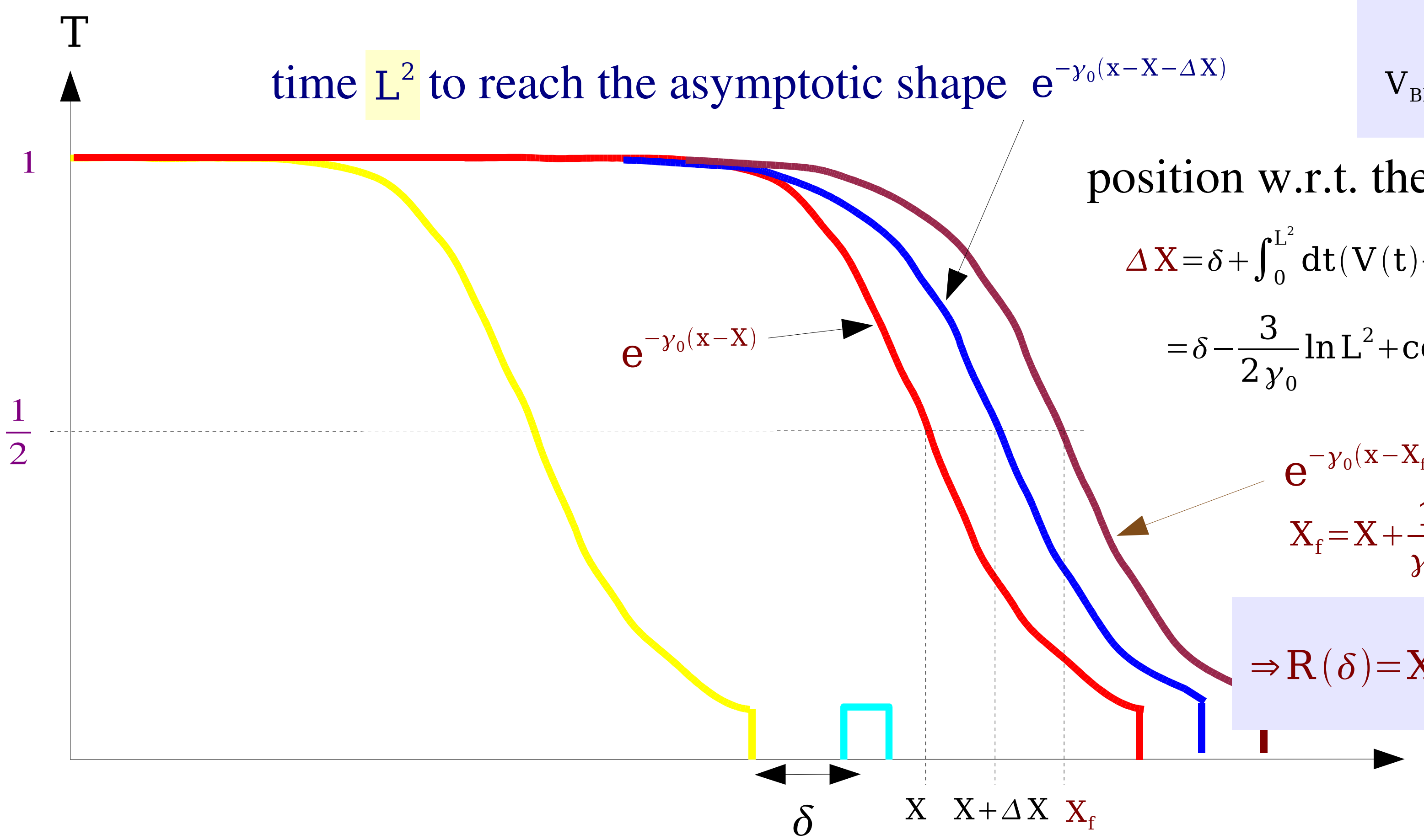

position w.r.t. the deterministic front:

$$\Delta X = \delta + \int_0^{L^2} dt\,(V(t) - V_{BD}) = \delta - \int^{L^2} dt\,\frac{3}{2\gamma_0 t} + \text{const}$$

$$= \delta - \frac{3}{2\gamma_0} \ln L^2 + \text{const} = \frac{1}{\gamma_0}(\gamma_0 \delta - \ln L^3) + \text{const}$$

$$e^{-\gamma_0(x-X_f)} = e^{-\gamma_0(x-X)} + e^{-\gamma_0(x-X-\Delta X)}$$

$$X_f = X + \frac{1}{\gamma_0} \ln\left(1 + e^{\gamma_0 \Delta X}\right)$$

$$\Rightarrow R(\delta) = X_f - X = \frac{1}{\gamma_0} \ln\left(1 + C_2 \frac{e^{\gamma_0 \delta}}{L^3}\right)$$

Accounting for fluctuations

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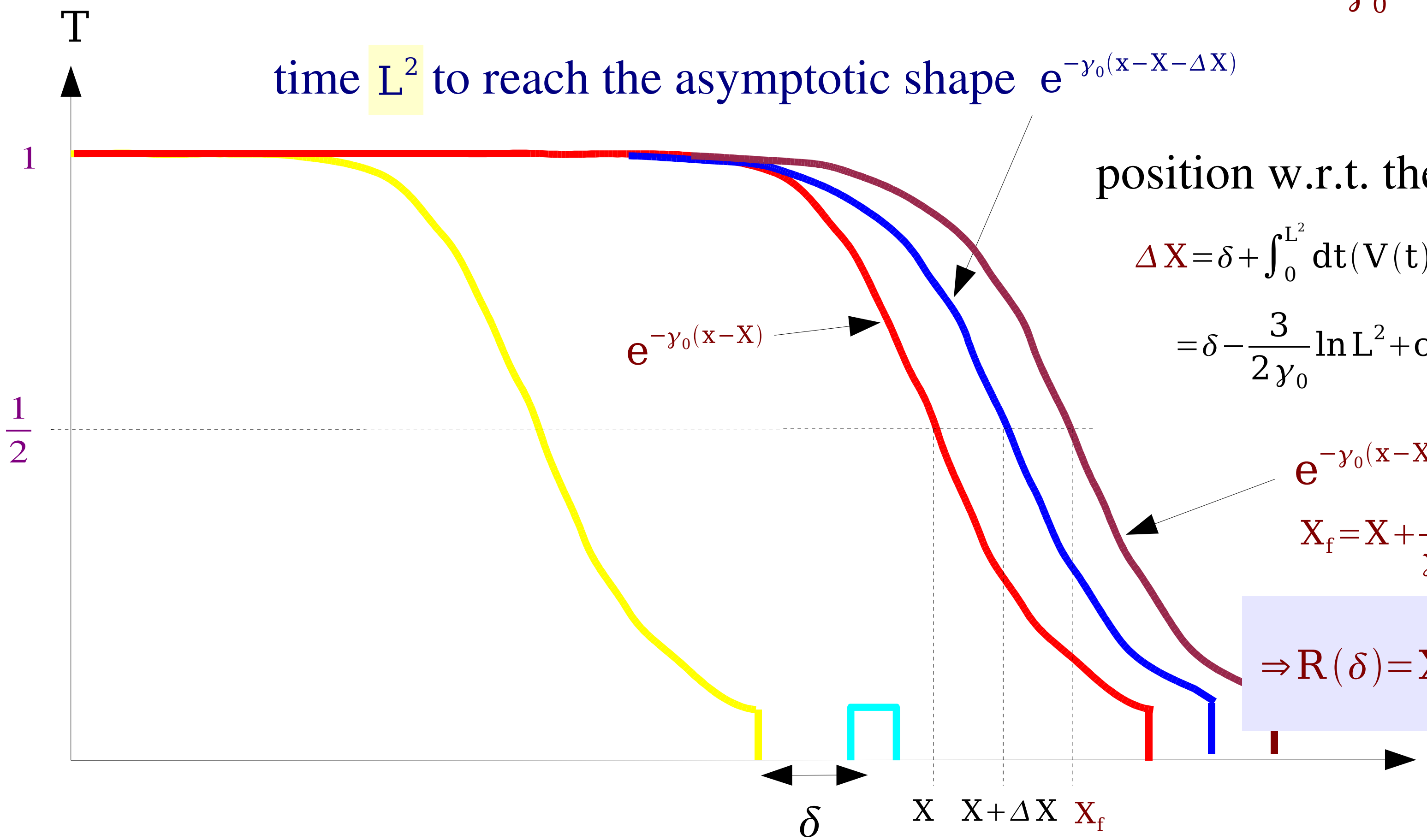
Assumption #3: their effect on the front position is $R(\delta) = X_f - X = \frac{1}{\gamma_0} \ln\left(1 + C_2 \frac{e^{\gamma_0 \delta}}{L^3}\right)$



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Stochastic rules for the effective evolution of the position of the front:

$$X_{t+dt} = \begin{cases} X_t + V_{BD}\,dt, & \text{if no fluctuation occurs} \\ X_t + V_{BD}\,dt + R(\delta), & \text{with proba } p(\delta)\,d\delta\,dt \end{cases}$$

$$V - V_{BD} = \int d\delta\, p(\delta) R(\delta) \qquad = \frac{C_1 C_2}{\gamma_0} \frac{3 \ln L}{\gamma_0 L^3}$$

$$\frac{[n-\text{th cumulant}]}{t} = \int d\delta\, p(\delta) R^n(\delta) \qquad = \frac{C_1 C_2}{\gamma_0} \frac{n!\,\zeta(n)}{\gamma_0^n L^3}$$

$$L = \frac{\ln N}{\gamma_0}$$

Summary of the effect of fluctuations

Brunet, Derrida, Mueller, SM (2005)

We proposed a phenomenological model for the propagation stochastic fronts, that we expect to be valid in the weak noise limit (for a large enough number of particles). This model is summarized in the following assumptions:

Assumption #1: the evolution of the stochastic front is essentially deterministic, except for some occasional extra-particles in the tail

Assumption #2: the probability for such extra-particles is $p(\delta)\,d\delta\,dt = C_1 e^{-\gamma_0\delta}\,d\delta\,dt$

Assumption #3: their effect on the front position is $R(\delta) = \frac{1}{\gamma_0}\ln\left(1 + C_2\frac{e^{\gamma_0\delta}}{L^3}\right)$

(**Assumption #4**: needed to get the constant $C_1 C_2$)

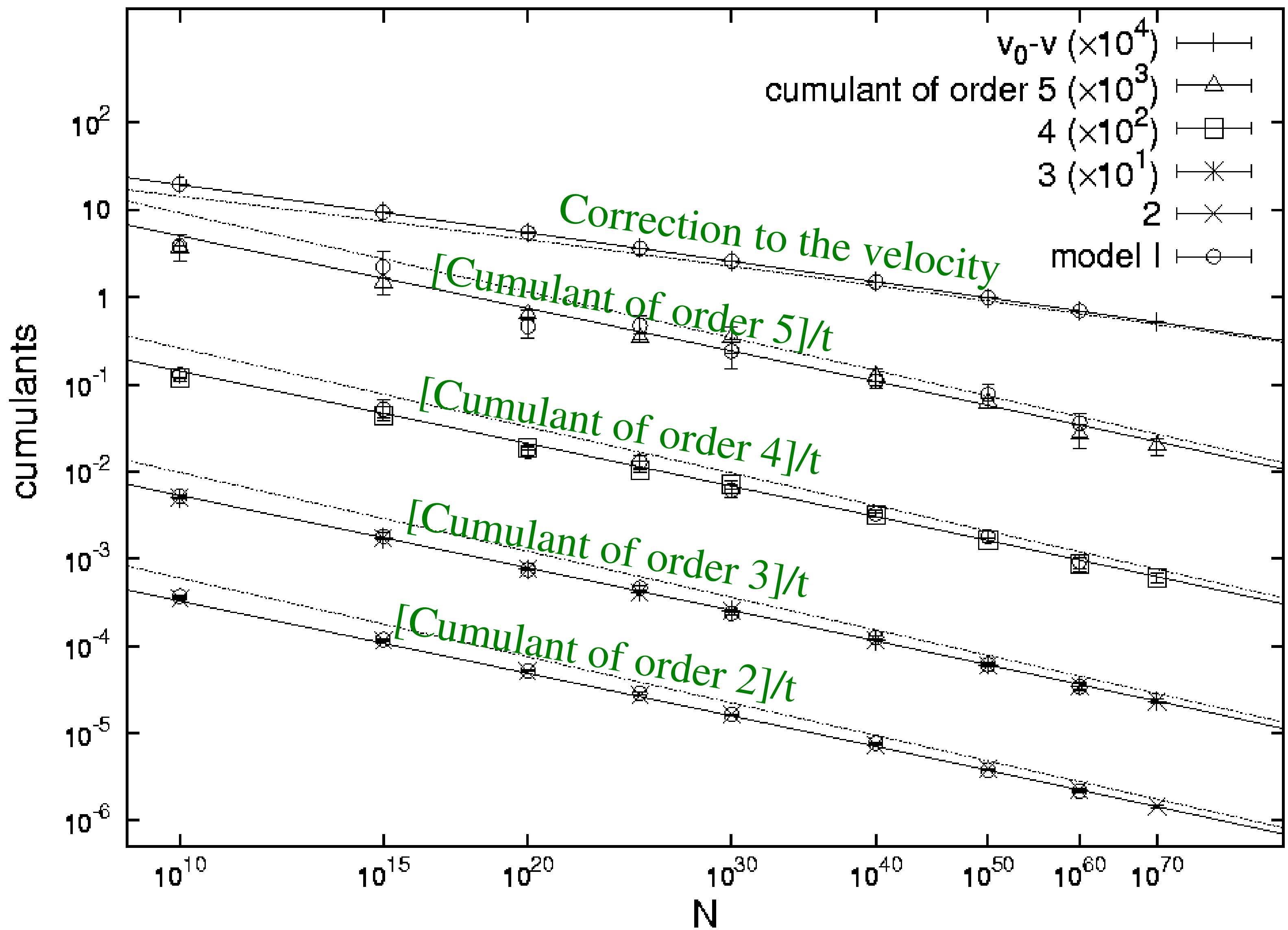
It leads to ***quantitative*** predictions for the position of the front:

$$V = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2\gamma_0\chi''(\gamma_0)}{2\ln^2 N} + \pi^2\gamma_0^2\chi''(\gamma_0)\frac{3\ln\ln N}{\gamma_0\ln^3 N}$$

$$\frac{[n\text{-th cumulant}]}{t} = \pi^2\gamma_0^2\chi''(\gamma_0)\frac{n!\,\zeta(n)}{\gamma_0^n\ln^3 N}$$

$\ln N \gg 1$

Numerical checks



Reaction-diffusion model, discrete in space and time

Use the dictionary...

Position x		$\ln(k^2/k_0^2)$
Time t		$\bar{\alpha}Y$
Particle density T	$\longleftrightarrow$	Partonic amplitude T
Maximum/equilibrium number of particles N		$\frac{1}{\alpha_s^2}$
Position of the wave front X		Saturation scale $\ln(Q_s^2/k_0^2)$

...to get predictions for QCD!

Shape of the *partonic* amplitude: $T \sim \left(r^2 Q_s^2(Y)\right)^{\gamma_0}$

Saturation scale: $\frac{d}{d(\bar{\alpha}Y)}\langle \ln Q_s^2 \rangle = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2\ln^2(1/\alpha_s^2)} + \pi^2 \gamma_0^2 \chi''(\gamma_0) \frac{\mathbf{3\ln\ln(1/\alpha_s^2)}}{\gamma_0 \mathbf{\ln^3(1/\alpha_s^2)}}$

$$\langle \mathbf{\ln^n Q_s^2} \rangle_{\mathbf{cumulant}} = \pi^2 \gamma_0^2 \chi''(\gamma_0) \frac{\mathbf{n!\zeta(n)}}{\gamma_0^{\mathbf{n}}} \left[\frac{\bar{\alpha}\mathbf{Y}}{\mathbf{\ln^3(1/\alpha_s^2)}} \right]$$

$$\Rightarrow A \sim A\left(\frac{r^2 Q_s^2(Y)}{\sqrt{\frac{\bar{\alpha}Y}{\ln^3(1/\alpha_s^2)}}} \right)$$

Validity

A priori, $Y \gg 1, \ln(1/\alpha_s^2) \gg 1$

In practice: analytical results reliable for $\alpha_s \ll 10^{-5}$

But we believe the picture itself for $\alpha_s < 0.1$

Summary

Instead of solving the full QCD evolution equations, we have identified, from the physics, the universality class of high energy QCD as the one of reaction-diffusion processes.

This lead us to study the shape and weight of individual Fock states, and a stochastic traveling wave equation of the F-KPP type:

$$\partial_{\bar{\alpha} Y} T = \chi(-\partial_{\ln k^2}) T - T^2 + \alpha_s \sqrt{T} \quad \nu$$

The properties of these QCD traveling waves (shape and position, i.e. form of the amplitude and rapidity dependence of the saturation scale) may be obtained directly by solving simpler equations in the universality class of the sFKPP equation.

Outlook

Understand the limits of the statistical approach

- how well does it reproduce QCD? What is beyond?
- can one derive more universal analytical results?
- can one get close to phenomenology from numerics?

...

"Statistical" approach

Replica approach

Field-theoretical approach

Itakura, in progress

$$V=\frac{\chi(\gamma_0)}{\gamma_0}-\frac{\pi^2\gamma_0\chi''(\gamma_0)}{2\ln^2 N}+\pi^2\gamma_0^2\chi''(\gamma_0)\frac{3\ln\ln N}{\gamma_0\ln^3 N}$$

$$\frac{[\mathrm{n-th\,cumulant}]}{t}=\pi^2\gamma_0^2\chi''(\gamma_0)\frac{n!\,\zeta(n)}{\gamma_0^n\ln^3 N}$$